Research Article

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Convergence Results for Elliptic Variational-Hemivariational Inequalities

https://doi.org/10.1515/anona-2020-0107
Received January 28, 2020; accepted April 7, 2020.

Abstract: We consider an elliptic variational-hemivariational inequality $\mathcal{P}$ in a reflexive Banach space, governed by a set of constraints K, a nonlinear operator A, and an element f. We associate to this inequality a sequence $\{\mathcal{P}_n\}$ of variational-hemivariational inequalities such that, for each $n \in \mathbb{N}$, inequality $\mathcal{P}_n$ is obtained by perturbing the data K and A and, moreover, it contains an additional term governed by a small parameter ε_n. The unique solvability of $\mathcal{P}$ and, for each $n \in \mathbb{N}$, the solvability of its perturbed version $\mathcal{P}_n$, are guaranteed by an existence and uniqueness result obtained in literature. Denote by u the solution of Problem $\mathcal{P}$ and, for each $n \in \mathbb{N}$, let u_n be a solution of Problem $\mathcal{P}_n$. The main result of this paper states the strong convergence of $u_n \to u$ in X, as $n \to \infty$. We show that the main result extends a number of results previously obtained in the study of Problem $\mathcal{P}$. Finally, we illustrate the use of our abstract results in the study of a mathematical model which describes the contact of an elastic body with a rigid-deformable foundation and provide the corresponding mechanical interpretations.

Keywords: variational-hemivariational inequality, penalty operator, Mosco convergence, internal approximation, Tykhonov well-posedness, contact problem

MSC: 47J20; 49J40; 49J45; 35M86; 74M10; 74M15

1 Introduction

Variational-hemivariational inequalities represent a special class of inequalities which arise in the study of nonsmooth boundary value problems. They are governed by both convex functions and locally Lipschitz functions, which could be nonconvex. For this reason, their study requires prerequisites on both convex and nonsmooth analysis. Variational-hemivariational inequalities have been introduced by Panagiotopoulos [24] in the context of applications in engineering problems. Later, they have been studied in a large number of papers, including the books [21, 23]. The mathematical literature concerning variational-hemivariational inequalities grew up rapidly in the last decade, motivated by important applications in Physics, Mechanics and Engineering Sciences. A recent reference is the book [27] which provides the state of the art in the field, together with relevant applications in Contact Mechanics.

Recently, a considerable effort was done to the study of variational-hemivariational inequalities in the functional framework that we describe below and we assume everywhere in this paper. Consider a real reflexive Banach space X and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and its dual X^*. Let $K \subset X$, $A\colon X \to X^*$,

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$\varphi\colon X \times X \to \mathbb{R}$, $j\colon X \to \mathbb{R}$ and $f \in X^*$. We assume that j is a locally Lipschitz function and we denote by $j^0(u; v)$ the generalized directional derivative of j at the point u in the direction v. Then, the inequality problem taken into consideration is the following.

Problem $\mathcal{P}$. *Find an element $u \in K$ such that*

$$\langle Au, v - u\rangle + \varphi(u, v) - \varphi(u, u) + j^0(u; v - u) \geq \langle f, v - u\rangle \qquad \forall\, v \in K. \tag{1.1}$$

A short survey of some results concerning Problem $\mathcal{P}$ is the following. First, its unique solvability was proved in [19] and, under slightly weaker assumptions, in [27]. The dependence of the solution with respect to the data, including the set K and the element f, was proved in [34, 37], under different assumptions. These results have been completed in [34] and [15] by considering an associate optimal control problem and an evolution inequality problem, respectively. Results on the well-posedness of Problem $\mathcal{P}$ in the sense of Tykhonov have been obtained in [31]. There, given a sequence of positive numbers $\{\varepsilon_n\}$, the following perturbation of Problem $\mathcal{P}$ was considered, for each $n \in \mathbb{N}$.

Problem $\mathcal{P}_{\varepsilon_n}$. *Find an element $u_n \in K$ such that*

$$\langle Au_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) + \varepsilon_n\|v - u_n\|_X \geq \langle f, v - u_n\rangle \qquad \forall\, v \in K. \tag{1.2}$$

Note that the solution of Problem $\mathcal{P}$ is solution of the perturbed problem $\mathcal{P}_{\varepsilon_n}$. Nevertheless, the solution of Problem $\mathcal{P}_{\varepsilon_n}$ could fail to be unique. Denote by u_n a solution of Problem $\mathcal{P}_{\varepsilon_n}$, for each $n \in \mathbb{N}$. Then, under appropriate assumptions, it was proved in [31] that the sequence $\{u_n\}$, called approximating sequence, converges to u in X. This property represents the main ingredient for the well-posedness of Problem $\mathcal{P}$ in the sense of Tykhonov, introduced in the study of variational inequalities in [17, 18] and extended to a particular class of hemivariational inequalities in [8]. References in the field include [1, 13, 14, 16, 29, 30, 35].

Other convergence results concerning Problem $\mathcal{P}$ are related to the penalty method. Given a sequence of positive numbers $\{\lambda_n\}$ and a penalty operator $G : X \to X^*$, the classical penalty method consists to replace Problem $\mathcal{P}$ by a sequence of problems $\{\mathcal{P}_{\lambda_n}\}$ which, for every $n \in \mathbb{N}$, can be formulated as follows.

Problem $\mathcal{P}_{\lambda_n}$. *Find an element $u_n \in X$ such that*

$$\langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) \geq \langle f, v - u_n\rangle \qquad \forall\, v \in X. \tag{1.3}$$

Note that Problem $\mathcal{P}_{\lambda_n}$ is formally obtained from Problem $\mathcal{P}$ by removing the constraint $u \in K$ and including a penalty term governed by a parameter $\lambda_n > 0$ and an operator $G : X \to X^*$. Penalty methods have been used in [6, 7, 26] and [19, 27, 28] as an approximation tool to treat constraints in variational inequalities and variational-hemivariational inequalities, respectively. In particular, the existence of a unique solution to Problem $\mathcal{P}_{\lambda_n}$ together with its convergence to the solution of Problem $\mathcal{P}$ as $\lambda_n \to 0$ was proved in [19, 27]. An extension of this convergence result was obtained in [33] where the operator G in (1.3) was replaced by an operator $G_n : X \to X^*$, which depends on n.

Another type of convergence results for the variational-hemivariational inequality (1.1) arise from its numerical analysis and, more precisely, from its numerical approximation. Given a sequence $\{K_n\}$, an approximation of Problem $\mathcal{P}$ is stated as follows.

Problem $\mathcal{P}_{K_n}$. *Find an element $u_n \in K_n$ such that*

$$\langle Au_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) \geq \langle f, v - u_n\rangle \qquad \forall\, v \in K_n. \tag{1.4}$$

Note that in various applications $K_n = X_n \cap K$ where X_n is a finite-dimensional space constructed with the finite element method. We refer the reader to [11, 12] for convergence results related to internal numerical approximations, and [9] for both internal and external numerical approximations of such inequalities. A comprehensive reference on the numerical analysis of Problem $\mathcal{P}$ can be found in the survey paper [10].

The aim of this paper is threefold. The first one is to construct a sequence of Problems $\{\mathcal{P}_n\}$ and to show that for each $n \in \mathbb{N}$, Problem $\mathcal{P}_n$ has at least a solution u_n which convergences to the solution u of Problem

$\mathcal{P}$, as $n \to \infty$. Our main result on this matter is Theorem 2 which states an existence and convergence result. Our second aim is to show that Theorem 2 can be used to recover various convergence results in the study of Problems $\mathcal{P}_{\varepsilon_n}$, $\mathcal{P}_{\lambda_n}$, $\mathcal{P}_{K_n}$ described above. To this end, we use the theorem with a particular choice of sets, operators and parameters. Finally, our third aim is to illustrate the use of our abstract result in the study of a frictional contact problem and to provide the corresponding mechanical interpretations. The novelty of our paper arises from the generality of our main result which unifies various convergence results in the study of Problem $\mathcal{P}$ and provides a new and nonstandard mathematical tool in the variational analysis of frictional contact problems with elastic materials.

The rest of the manuscript is structured as follows. In Section 2 we introduce some preliminary material, then we recall the existence and uniqueness result obtained in [19, 27]. In Section 3 we state and prove our main result, Theorem 2. Its proof is based on arguments of compactness, monotonicity, pseudomonotonicity, lower semicontinuity, combined with the properties of the Clarke subdifferential. In Section 4 we deduce some consequences of Theorem 2 that we present in a form of relevant particular cases. Finally, in Section 5 we illustrate the use of our abstract results in the analysis of a mathematical model of contact.

2 Preliminaries

We start with some notation and preliminaries and send the reader to [2, 3, 21, 22, 32] for more details on the material presented below in this section. We use $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ for the norm on the spaces X and X^*, and 0_X, 0_{X^*} for the zero element of X and X^*, respectively. We also use the notation X^*_w for the space X^* endowed with the weak* topology. All the limits, upper and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly. The symbols "$\rightharpoonup$" and "$\to$" denote the weak and the strong convergence in various spaces which will be specified.

For multivalued and singlevalued operators defined on X we recall the following definitions.

Definition 1. *A multivalued operator $T\colon X \to 2^{X^*}$ is said to be pseudomonotone if:*

(a) *For every $u \in X$, the set $Tu \subset X^*$ is nonempty, closed and convex.*

(b) *T is upper semicontinuous (u.s.c.) from each finite dimensional subspace of X into X^*_w.*

(c) *For any sequences $\{u_n\} \subset X$ and $\{u^*_n\} \subset X^*$ such that $u_n \to u$ weakly in X, $u^*_n \in Tu_n$ for all $n \in \mathbb{N}$ and* $\limsup \langle u^*_n, u_n - u\rangle \le 0$, *we have that for every $v \in X$ there exists $u^*(v) \in Tu$ such that*

$$\langle u^*(v), u - v\rangle \le \liminf \langle u^*_n, u_n - v\rangle.$$

Definition 2. *A multivalued operator $T\colon X \to 2^{X^*}$ is said to be generalized pseudomonotone if for any sequences $\{u_n\} \subset X$ and $\{u^*_n\} \subset X^*$ such that $u_n \to u$ weakly in X, $u^*_n \in Tu_n$ for all $n \in \mathbb{N}$, $u^*_n \to u^*$ in X^*_w and* $\limsup \langle u^*_n, u_n - u\rangle \le 0$, *we have $u^* \in Tu$ and*

$$\lim \langle u^*_n, u_n\rangle = \langle u^*, u\rangle.$$

Definition 3. *An singlevalued operator $A\colon X \to X^*$ is said to be:*

(a) *monotone, if for all $u, v \in X$, we have $\langle Au - Av, u - v\rangle \ge 0$;*

(b) *strongly monotone, if there exists $m_A > 0$ such that*

$$\langle Au - Av, u - v\rangle \ge m_A\|u - v\|^2_X \quad \text{for all } u,\ v \in X;$$

(c) *bounded, if A maps bounded sets of X into bounded sets of X^*;*

(d) *pseudomonotone, if it is bounded and $u_n \to u$ weakly in X with*

$$\limsup \langle Au_n, u_n - u\rangle \le 0$$

imply $\liminf \langle Au_n, u_n - v\rangle \ge \langle Au, u - v\rangle$ *for all $v \in X$;*

(e) *demicontinuous, if* $u_n \to u$ *in* X *implies* $Au_n \to Au$ *weakly in* X^*.

It is well known that if $T\colon X \to 2^{X^*}$ is a pseudomonotone operator then T is generalized pseudomonotone. Moreover, it can be proved that if $A\colon X \to X^*$ is a pseudomonotone operator in the sense of Definition 3(d) then its multivalued extension defined as $X \ni u \to \{Au\} \in 2^{X^*}$ is pseudomonotone in the sense of Definition 1. In addition, the following results hold.

Proposition 1. a) *If the operator* $A\colon X \to X^*$ *is bounded, demicontinuous and monotone, then* A *is pseudomonotone.*

b) *If* $A, B\colon X \to X^*$ *are pseudomonotone operators, then the sum* $A + B\colon X \to X^*$ *is pseudomonotone.*

For real valued functions defined on X we recall the following definitions.

Definition 4. *A function* $j\colon X \to \mathbb{R}$ *is said to be locally Lipschitz if for every* $x \in X$*, there exists* U_x *a neighborhood of* x *and a constant* $L_x > 0$ *such that* $|j(y) - j(z)| \le L_x\|y - z\|_X$ *for all* $y, z \in U_x$*. For such functions the generalized (Clarke) directional derivative of* j *at the point* $x \in X$ *in the direction* $v \in X$ *is defined by*

$$j^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The generalized gradient (Clarke subdifferential) of j *at* x *is a subset of the dual space* X^* *given by*

$$\partial j(x) = \{\, \zeta \in X^* \ :\ j^0(x; v) \ge \langle \zeta, v\rangle \quad \forall\, v \in X \,\}.$$

The function j *is said to be regular (in the sense of Clarke) at the point* $x \in X$ *if for all* $v \in X$ *the one-sided directional derivative* $j'(x; v)$ *exists and* $j^0(x; v) = j'(x; v)$.

We shall use the following properties of the generalized directional derivative and the generalized gradient.

Proposition 2. *Assume that* $j\colon X \to \mathbb{R}$ *is a locally Lipschitz function. Then the following hold:*

a) *For every* $x \in X$*, the function* $X \ni v \mapsto j^0(x; v) \in \mathbb{R}$ *is positively homogeneous and subadditive, i.e.,* $j^0(x; \lambda v) = \lambda j^0(x; v)$ *for all* $\lambda \ge 0$, $v \in X$ *and* $j^0(x; v_1 + v_2) \le j^0(x; v_1) + j^0(x; v_2)$ *for all* $v_1, v_2 \in X$*, respectively.*

b) *For every* $v \in X$*, we have* $j^0(x; v) = \max\{\, \langle \xi, v\rangle \ :\ \xi \in \partial j(x) \,\}$.

c) *For every* $x \in X$*, the gradient* $\partial j(x)$ *is a nonempty, convex, and compact subset of* X^*_w *which is bounded by the Lipschitz constant* $L_x > 0$ *of* j *near* x.

We proceed with some miscellaneous definitions and results.

Definition 5. *Let* $\{K_n\}$ *be a sequence of nonempty subsets of* V *and* $\widetilde{K}$ *a nonempty subset of* X*. We say that the sequence* $\{K_n\}$ *converges to* $\widetilde{K}$ *in the sense of Mosco if the following conditions hold.*

(a) *For every* $v \in \widetilde{K}$*, there exists a sequence* $\{v_n\} \subset X$ *such that* $v_n \in K_n$ *for each* $n \in \mathbb{N}$ *and* $v_n \to v$ *in* X.

(b) *For each sequence* $\{v_n\}$ *such that* $v_n \in K_n$ *for each* $n \in \mathbb{N}$ *and* $v_n \rightharpoonup v$ *in* X*, we have* $v \in \widetilde{K}$.

Below in this paper we shall use the notation $K_n \xrightarrow{M} \widetilde{K}$ for the convergence in the sense of Mosco defined above.

Proposition 3. *Let* C *be a nonempty closed convex subset of* X*,* C^* *a nonempty closed convex and bounded subset of* X^*_w*,* $\varphi : X \to \mathbb{R}$ *a proper, convex lower semicontinuous function and let* y *be arbitrary element of* C*. Assume that, for each* $x \in C$*, there exists* $x^*(x) \in C^*$ *such that*

$$\langle x^*(x), x - y\rangle \ge \varphi(y) - \varphi(x).$$

Then, there exists $y^* \in C^*$ *such that*

$$\langle y^*, x - y\rangle \geq \varphi(y) - \varphi(x) \qquad \forall\, x \in C.$$

For the proof of Proposition 3 we refer to [5].

Definition 6. *An operator* $P\colon X \to X^*$ *is said to be a penalty operator of the set* $K \subset X$ *if* P *is bounded, demicontinuous, monotone and* $K = \{x \in X \mid Px = 0_{X^*}\}$.

Note that the penalty operator always exists. Indeed, we recall that any reflexive Banach space X can be always considered as equivalently renormed strictly convex space and, therefore, the duality map $J\colon X \to 2^{X^*}$, defined by

$$Jx = \{\, x^* \in X^* \ :\ \langle x^*, x\rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2 \,\} \quad \text{for all } \ x \in X$$

is a single-valued operator. Then, as proved in [4, 36], the following result holds.

Proposition 4. *Let* K *be a nonempty closed and convex subset of* X, $J\colon X \to X^*$ *the duality map,* $I\colon X \to X$ *the identity map on* X, *and* $\widetilde{P}_K\colon X \to K$ *the projection operator on* K. *Then* $P_K = J(I - \widetilde{P}_K) : X \to X^*$ *is a penalty operator of* K.

We end this section with an existence and uniqueness result concerning the variational-hemivariational inequality (1.1) and, to this end, we consider the following assumptions on the data.

$$K \text{ is nonempty, closed and convex subset of } X. \tag{2.1}$$

$$\begin{cases} A\colon X \to X^* \text{ is pseudomonotone and} \\ \text{strongly monotone with constant } m_A > 0. \end{cases} \tag{2.2}$$

$$\begin{cases} \varphi\colon X \times X \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } \varphi(\eta, \cdot)\colon X \to \mathbb{R} \text{ is convex and lower semicontinuous,} \\ \qquad\quad \text{for all } \eta \in X. \\ \quad \text{(b) there exists } \alpha_\varphi \geq 0 \text{ such that} \\ \qquad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \qquad\quad \leq \alpha_\varphi \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \ \text{ for all } \eta_1, \eta_2, v_1, v_2 \in X. \end{cases} \tag{2.3}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j \text{ is locally Lipschitz.} \\ \quad \text{(b) } \|\xi\|_{X^*} \leq \overline{c}_0 + \overline{c}_1 \|v\|_X \text{ for all } v \in X,\ \xi \in \partial j(v), \\ \qquad \text{with } \overline{c}_0,\ \overline{c}_1 \geq 0. \\ \quad \text{(c) there exists } \alpha_j \geq 0 \text{ such that} \\ \qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \\ \qquad \text{for all } v_1, v_2 \in X. \end{cases} \tag{2.4}$$

$$\alpha_\varphi + \alpha_j < m_A. \tag{2.5}$$

$$f \in X^*. \tag{2.6}$$

It can be proved that for a locally Lipschitz function $j\colon X \to \mathbb{R}$, hypothesis (2.4)(c) is equivalent to the so-called relaxed monotonicity condition see, e.g., [20]. Note also that if $j\colon X \to \mathbb{R}$ is a convex function, then

(2.4)(c) holds with $\alpha_j = 0$, since it reduces to the monotonicity of the (convex) subdifferential. Examples of functions which satisfy condition (2.4)(c) have been provided in [10, 19, 20], for instance.

The unique solvability of the variational-hemivariational inequality (1.1) is given by the following result.

Theorem 1. *Assume* (2.1)–(2.6). *Then, inequality* (1.1) *has a unique solution* $u \in K$.

For the Proof of Theorem 1 we refer the reader to Theorem 18 in [19] and Remark 13 in [27].

3 An existence and convergence result

In this section we state and prove our main existence and convergence result, Theorem 2. To this end, we consider a family of subsets $\{K_n\}$ of X, a family of operators $\{G_n\}$ defined on X with values in X^* and two sequences $\{\lambda_n\}$, $\{\varepsilon_n\} \subset \mathbb{R}$. Then, for each $n \in \mathbb{N}$, we consider the following problem.

Problem $\mathcal{P}_n$. Find $u_n \in K_n$ such that

$$\langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle G_n u_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) + \varepsilon_n\|v - u_n\|_X \geq \langle f, v - u_n\rangle \ \forall v \in K_n. \tag{3.1}$$

In the study of Problem $\mathcal{P}_n$ we assume that for each $n \in \mathbb{N}$, the following hold.

$$K_n \text{ is a nonempty closed convex subset of } X. \tag{3.2}$$

$$G_n : X \to X^* \text{ is a bounded demicontinuous monotone operator.} \tag{3.3}$$

$$\lambda_n > 0. \tag{3.4}$$

$$\varepsilon_n \geq 0. \tag{3.5}$$

$$K \subset K_n. \tag{3.6}$$

$$\langle G_n u, v - u\rangle \leq 0 \qquad \forall u \in K_n,\ v \in K. \tag{3.7}$$

Moreover, we assume that the following conditions are satisfied.

$$\text{There exists a set } \widetilde{K} \subset X \text{ such that } K_n \xrightarrow{M} \widetilde{K} \text{ as } n \to \infty. \tag{3.8}$$

$$\left\{\begin{array}{l} \text{There exists an operator } G : X \to X^* \text{ and} \\ \text{a sequence } \{c_n\} \subset \mathbb{R} \text{ such that} \\ \text{(a) } \|G_n u - Gu\|_{X^*} \leq c_n(1 + \|u\|_X) \quad \forall u \in K_n,\ n \in \mathbb{N}. \\ \text{(b) } c_n \to 0 \quad \text{as} \quad n \to \infty. \\ \text{(c) } G \text{ is a bounded demicontinuous monotone operator.} \\ \text{(d) } \langle Gu, v - u\rangle \leq 0 \quad \forall u \in \widetilde{K},\ v \in K. \\ \text{(e) One of the two conditions below holds.} \\ \quad \text{(i) } \widetilde{K} = X \text{ and } u \in X,\ Gu = 0_{X^*} \Rightarrow u \in K. \\ \quad \text{(ii) } u \in \widetilde{K},\ \langle Gu, v - u\rangle = 0 \text{ for all } v \in K \Rightarrow u \in K. \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{l} \text{For each } u \in K \text{ there exists } c_\varphi(u) \geq 0 \text{ such that} \\ \varphi(u, v_1) - \varphi(u, v_2) \leq c_\varphi(u)\|v_1 - v_2\|_X \quad \forall v_1,\ v_2 \in X. \end{array}\right. \tag{3.10}$$

$$\lambda_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

$$\varepsilon_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.12}$$

Our main result in this section is the following.

Theorem 2. *Assume* (2.2)–(2.6) *and* (3.2)–(3.5). *Then, the following statements hold.*

a) *For each $n \in \mathbb{N}$, there exists at least one solution $u_n \in K_n$ to Problem $\mathcal{P}_n$. Moreover, the solution is unique if $\varepsilon_n = 0$.*

b) *If, in addition,* (2.1) *and* (3.6)–(3.12) *hold, u is the solution of Problem $\mathcal{P}$ and $\{u_n\} \subset X$ is a sequence such that u_n is a solution of Problem $\mathcal{P}_n$, for each $n \in \mathbb{N}$, then $u_n \to u$ in X.*

Proof. a) Let $n \in \mathbb{N}$. Assumptions (3.3), (3.4) and Proposition 1 a) imply that the operator $\frac{1}{\lambda_n} G_n : X \to X^*$ is pseudomonotone. Therefore (2.2) and Proposition 1 b) show that the operator $A_n : X \to X^*$ defined by $A_n = A + \frac{1}{\lambda_n} G_n$ is pseudomonotone, too. Moreover, since G_n is monotone and $\lambda_n > 0$, using assumption (2.2), again, we deduce that A_n is strongly monotone with constant m_A. We conclude from above that the operator A_n satisfies condition (2.2). On the other hand, recall that the set K_n satisfies condition (3.2). It follows from above that we are in a position to use Theorem 1 with K_n and A_n instead of K and A, respectively. In this way we deduce the existence of a unique element $u_n \in K_n$ such that

$$\langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle G_n u_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) \ge \langle f, v - u_n\rangle \qquad \forall\, v \in K_n. \tag{3.13}$$

This proves the unique solvability of Problem $\mathcal{P}_n$ in the case when $\varepsilon_n = 0$. Next, for $\varepsilon_n > 0$, it follows that the solution u_n of (3.13) satisfies inequality (3.1), too. This proves the existence of at least one solution to Problem $\mathcal{P}_n$.

b) Let $n \in \mathbb{N}$. We start by considering the auxiliary problem of finding an element $\widetilde{u}_n \in K_n$ such that

$$\langle A\widetilde{u}_n, v - \widetilde{u}_n\rangle + \frac{1}{\lambda_n}\langle G_n\widetilde{u}_n, v - \widetilde{u}_n\rangle + \varphi(u, v) - \varphi(u, \widetilde{u}_n) + j^0(\widetilde{u}_n; v - \widetilde{u}_n) \ge \langle f, v - \widetilde{u}_n\rangle \qquad \forall\, v \in K_n. \tag{3.14}$$

Note that the variational-hemivariational inequality (3.14) is similar to the variational-hemivariational inequality (3.13), the difference arising in the fact that in (3.14) the first argument of φ is the solution u of Problem $\mathcal{P}$. The existence of a unique solution to inequality (3.14) follows from Theorem 1, by using the same arguments as those used in the proof of unique solvability of inequality (3.13). Next, we divide the rest of the proof into four steps.

i) *We claim that there is an element $\widetilde{u} \in \widetilde{K}$ and a subsequence of $\{\widetilde{u}_n\}$, still denoted by $\{\widetilde{u}_n\}$, such that $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in X, as $n \to \infty$.*

To prove the claim, we establish the boundedness of the sequence $\{\widetilde{u}_n\}$ in X. Let $n \in \mathbb{N}$ and let u_0 be a given element in K. We use assumption (3.6) to deduce that

$$\langle A\widetilde{u}_n, \widetilde{u}_n - u_0\rangle \le \frac{1}{\lambda_n}\langle G_n\widetilde{u}_n, u_0 - \widetilde{u}_n\rangle + \varphi(u, u_0) - \varphi(u, \widetilde{u}_n) + j^0(\widetilde{u}_n; u_0 - \widetilde{u}_n) + \langle f, \widetilde{u}_n - u_0\rangle.$$

Then, by the strong monotonicity of the operator A we obtain

$$m_A\,\|\widetilde{u}_n - u_0\|_X^2 \le \langle Au_0, u_0 - \widetilde{u}_n\rangle + \frac{1}{\lambda_n}\langle G_n\widetilde{u}_n, u_0 - \widetilde{u}_n\rangle + \varphi(u, u_0) - \varphi(u, \widetilde{u}_n) + j^0(\widetilde{u}_n; u_0 - \widetilde{u}_n) + \langle f, \widetilde{u}_n - u_0\rangle. \tag{3.15}$$

Next, assumptions (3.4) and (3.7) imply that

$$\frac{1}{\lambda_n}\langle G_n\widetilde{u}_n, u_0 - \widetilde{u}_n\rangle \le 0 \tag{3.16}$$

and assumption (3.10) yields

$$\varphi(u, u_0) - \varphi(u, \widetilde{u}_n) \le c_\varphi(u)\|\widetilde{u}_n - u_0\|_X. \tag{3.17}$$

On the other hand, we write

$$\begin{aligned} j^0(\widetilde{u}_n; u_0 - \widetilde{u}_n) &= j^0(\widetilde{u}_n; u_0 - \widetilde{u}_n) + j^0(u_0; \widetilde{u}_n - u_0) - j^0(u_0; \widetilde{u}_n - u_0) \\ &\le j^0(\widetilde{u}_n; u_0 - \widetilde{u}_n) + j^0(u_0; \widetilde{u}_n - u_0) + |j^0(u_0; \widetilde{u}_n - u_0)|, \end{aligned}$$

then we use assumption (2.4)(b) and Proposition 2 b) to see that

$$j^0(\widetilde{u}_n; u_0 - \widetilde{u}_n) \le \alpha_j \|\widetilde{u}_n - u_0\|_X^2 + (\overline{c}_0 + \overline{c}_1 \|u_0\|_X)\|\widetilde{u}_n - u_0\|_X. \tag{3.18}$$

And, obviously,

$$\langle Au_0, u_0 - \widetilde{u}_n\rangle + \langle f, \widetilde{u}_n - u_0\rangle \le \|f - Au_0\|_{X^*}\|\widetilde{u}_n - u_0\|_X. \tag{3.19}$$

Next, we combine inequalities (3.15)–(3.19) to find that

$$(m_A - \alpha_j)\|\widetilde{u}_n - u_0\|_X \le c_\varphi(u) + \overline{c}_0 + \overline{c}_1\|u_0\|_X + \|f - Au_0\|_{X^*}.$$

We now use condition (2.5) and the above inequality to deduce that $\{\widetilde{u}_n\}$ is a bounded sequence in X. Therefore, by the reflexivity of X, there exists an element $\widetilde{u} \in X$ and a subsequence of $\{\widetilde{u}_n\}$, still denoted by $\{\widetilde{u}_n\}$, such that $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in X. Recall that $\widetilde{u}_n \in K_n$ for each $n \in \mathbb{N}$. Then, assumption (3.8) and Definition 5 imply that $\widetilde{u} \in \widetilde{K}$.

ii) *Next, we claim that $\widetilde{u}$ is the solution to Problem $\mathcal{P}$, i.e., $\widetilde{u} = u$.*

To prove this claim we use assumption (3.8) and consider an element $v \in \widetilde{K}$ together with a sequence $\{v_n\} \subset X$ such that $v_n \in K_n$ for every $n \in \mathbb{N}$ and $v_n \to v$ in X as $n \to \infty$. We now use inequality (3.14) with $v = v_n$ and assumptions (2.2), (3.10), (2.4)(b) to see that

$$\begin{aligned} &\frac{1}{\lambda_n}\langle G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle \le \langle A\widetilde{u}_n - Av_n, v_n - \widetilde{u}_n\rangle + \varphi(u, v_n) - \varphi(u, \widetilde{u}_n) \\ &+ j^0(\widetilde{u}_n; v_n - \widetilde{u}_n) + \langle f, \widetilde{u}_n - v_n\rangle + \langle Av_n, v_n - \widetilde{u}_n\rangle \\ &\le \varphi(u, v_n) - \varphi(u, \widetilde{u}_n) + j^0(\widetilde{u}_n; v_n - \widetilde{u}_n) + \langle f - Av_n, \widetilde{u}_n - v_n\rangle \\ &\le c_\varphi(u)\|\widetilde{u}_n - v_n\|_X + (\overline{c}_0 + \overline{c}_1\|\widetilde{u}_n\|_X)\|\widetilde{u}_n - v_n\|_X + \|f - Av_n\|_{X^*}\|\widetilde{u}_n - v_n\|_X \\ &\le (c_\varphi(u) + \overline{c}_0 + \overline{c}_1\|\widetilde{u}_n\|_X + \|f - Av_n\|_{X^*})\|\widetilde{u}_n - v_n\|_X. \end{aligned}$$

Then, due to the convergence $v_n \to v$ in X, the boundedness of sequence $\{\widetilde{u}_n\}$ and the boundedness of the operator A, we deduce that there exists a constant $D > 0$ which does not depend on n such that

$$\langle G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle \le \lambda_n D.$$

Passing to the upper limit in above inequality and using assumption (3.11) we have

$$\limsup \langle G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle \le 0. \tag{3.20}$$

On the other hand, we write

$$\begin{aligned} \langle G\widetilde{u}_n, \widetilde{u}_n - v\rangle &= \langle G\widetilde{u}_n, \widetilde{u}_n - v_n\rangle + \langle G\widetilde{u}_n, v_n - v\rangle \\ &= \langle G\widetilde{u}_n - G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle + \langle G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle + \langle G\widetilde{u}_n, v_n - v\rangle \\ &\le \|G\widetilde{u}_n - G_n\widetilde{u}_n\|_{X^*}\|\widetilde{u}_n - v_n\|_X + \langle G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle + \langle G\widetilde{u}_n, v_n - v\rangle \end{aligned}$$

and, using asumption (3.9)(a) we deduce that

$$\begin{aligned} &\langle G\widetilde{u}_n, \widetilde{u}_n - v\rangle \\ &\quad \le c_n(1 + \|\widetilde{u}_n\|_X)\|\widetilde{u}_n - v_n\|_X + \langle G_n\widetilde{u}_n, \widetilde{u}_n - v_n\rangle + \langle G\widetilde{u}_n, v_n - v\rangle. \end{aligned} \tag{3.21}$$

We now use hypotheses (3.9)(b), (c), the boundedness of sequence $\{\widetilde{u}_n\}$ and the convergence $v_n \to v$ in X to see that

$$\lim \left[c_n(1 + \|\widetilde{u}_n\|_X)\|\widetilde{u}_n - v_n\|_X \right] = 0, \tag{3.22}$$

$$\lim \langle G\widetilde{u}_n, v_n - v \rangle = 0. \tag{3.23}$$

Next, we pass to upper limit in inequality (3.21) and use (3.20), (3.22) and (3.23) to find that

$$\limsup \langle G\widetilde{u}_n, \widetilde{u}_n - v \rangle \le 0. \tag{3.24}$$

Taking now $v = \widetilde{u} \in \widetilde{K}$ in (3.24) we deduce that $\limsup \langle G\widetilde{u}_n, \widetilde{u}_n - \widetilde{u} \rangle \le 0$. Recall assumption (3.9)(c) which guarantees that the operator $G : X \to X^*$ is pseudomonotone. Hence, using the pseudomonotonicity of G we deduce that

$$\langle G\widetilde{u}, \widetilde{u} - v \rangle \le \liminf \langle G\widetilde{u}_n, \widetilde{u}_n - v \rangle. \tag{3.25}$$

We now combine inequalities (3.24) and (3.25) to see that

$$\langle G\widetilde{u}, \widetilde{u} - v \rangle \le 0. \tag{3.26}$$

Recall that this inequality is valid for any $v \in \widetilde{K}$.

Assume that condition (3.9)(e)(i) is satisfied. Then, inequality (3.26) implies that $\langle G\widetilde{u}, \widetilde{u} - v \rangle \le 0$ for all $v \in X$, which yields $G\widetilde{u} = 0_{X^*}$ and, therefore, $\widetilde{u} \in K$. Assume now that condition (3.9)(e)(ii) is satisfied. Then, by assumptions (3.6) and (3.8) it is easy to see that $K \subset \widetilde{K}$ and, therefore, using (3.26) we obtain that

$$\langle G\widetilde{u}, \widetilde{u} - v \rangle \le 0 \qquad \forall\, v \in K.$$

On the other hand, from the assumption (3.9)(d) we have

$$\langle G\widetilde{u}, v - \widetilde{u} \rangle \le 0 \qquad \forall\, v \in K.$$

The last two inequalities imply that $\langle G\widetilde{u}, v - \widetilde{u} \rangle = 0$ for all $v \in K$ and, using (3.9) (e)(ii), we infer that $\widetilde{u} \in K$. We conclude from above that, either (3.9)(e)(i) or (3.9)(e)(ii) holds, we have

$$\widetilde{u} \in K. \tag{3.27}$$

Let $n \in \mathbb{N}$. Then, using (3.6) and inequality (3.14), we find that

$$\begin{aligned} &\langle A\widetilde{u}_n, v - \widetilde{u}_n \rangle + \frac{1}{\lambda_n}\langle G_n\widetilde{u}_n, v - \widetilde{u}_n \rangle + j^0(\widetilde{u}_n; v - \widetilde{u}_n) - \langle f, v - \widetilde{u}_n \rangle \\ &\qquad \ge \varphi(u, \widetilde{u}_n) - \varphi(u, v) \qquad \forall\, v \in K. \end{aligned} \tag{3.28}$$

Next, using Proposition 2 b) we deduce that for each $v \in K$ there exists an element $\omega_n(\widetilde{u}_n, v) \in \partial j(\widetilde{u}_n)$ such that $j^0(\widetilde{u}_n; v - \widetilde{u}_n) = \langle \omega_n(\widetilde{u}_n, v), v - \widetilde{u}_n \rangle$, and, therefore, inequality (3.28) yields

$$\langle A\widetilde{u}_n + \frac{1}{\lambda_n} G_n\widetilde{u}_n + \omega_n(\widetilde{u}_n, v) - f, v - \widetilde{u}_n \rangle \ge \varphi(u, \widetilde{u}_n) - \varphi(u, v) \tag{3.29}$$

for all $v \in K$. Recall that Proposition 2 c) guarantees the set

$$C^* = \{A\widetilde{u}_n + \frac{1}{\lambda_n} G_n\widetilde{u}_n + \xi_n - f \;:\; \xi_n \in \partial j(\widetilde{u}_n)\} \tag{3.30}$$

is nonempty closed convex and bounded in X^*_w. Then, assumption (2.3)(a) allows us to use Proposition 3 with $C = K$ and C^* defined by (3.30), $x = v$ and $y = \widetilde{u}_n$. In this way we find that there exists an element $\omega_n(\widetilde{u}_n) \in \partial j(\widetilde{u}_n)$ which does not depend on v such that

$$\langle A\widetilde{u}_n + \frac{1}{\lambda_n} G_n\widetilde{u}_n + \omega_n(\widetilde{u}_n) - f, v - \widetilde{u}_n \rangle \ge \varphi(u, \widetilde{u}_n) - \varphi(u, v) \quad \forall\, v \in K.$$

Therefore, assumptions (3.4) and (3.7) yield

$$\langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - v\rangle \le \varphi(u, v) - \varphi(u, \widetilde{u}_n) - \langle f, v - \widetilde{u}_n\rangle \quad \forall\, v \in K. \tag{3.31}$$

We now use the regularity (3.27) to take $v = \widetilde{u}$ in (3.31). Then we pass to the upper limit in the resulting inequality, use the convergence $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in X and the lower semicontinuity of φ with respect to its second argument to infer that

$$\limsup \langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}\rangle \le 0. \tag{3.32}$$

Due to the assumption (2.4)(b), the boundedness of the sequence $\{\widetilde{u}_n\}$ and the boundedness of the operator A, guaranteed by assumption (2.2), it follows that the sequence $\{A\widetilde{u}_n + \omega_n(\widetilde{u}_n)\}$ is bounded in X^*. This implies that there exists a subsequence of the sequence $\{A\widetilde{u}_n + \omega_n(\widetilde{u}_n)\}$, still denoted by$\{A\widetilde{u}_n + \omega_n(\widetilde{u}_n)\}$, and an element $\theta \in X^*$ such that

$$A\widetilde{u}_n + \omega_n(\widetilde{u}_n) \rightharpoonup \theta \quad \text{in} \quad X^*_w. \tag{3.33}$$

Moreover, as proved in [19, Lemma 20], we know that the multivalued operator $A + \partial j\colon X \to 2^{X^*}$ is generalized pseudomonotone. Exploiting now Definition 2 and the ingredients $\{\widetilde{u}_n\} \subset X$, $\{A\widetilde{u}_n + \xi_n(\widetilde{u}_n)\} \subset X^*$, $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in X, $A\widetilde{u}_n + \omega_n(\widetilde{u}_n) \in A\widetilde{u}_n + \partial j(\widetilde{u}_n)$, (3.33) and (3.32), we deduce that $\theta \in A\widetilde{u} + \partial j(\widetilde{u})$ and

$$\langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n\rangle \to \langle \theta, \widetilde{u}\rangle. \tag{3.34}$$

On the other hand, (3.33) implies that

$$\langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}\rangle \to \langle \theta, \widetilde{u}\rangle. \tag{3.35}$$

We now combine the convergences (3.34) and (3.35) to find that

$$\langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}\rangle \to 0. \tag{3.36}$$

Note that the inclusion $\theta \in A\widetilde{u} + \partial j(\widetilde{u})$ implies that there exists $\omega(\widetilde{u}) \in \partial j(\widetilde{u})$ such that

$$\theta = A\widetilde{u} + \omega(\widetilde{u}). \tag{3.37}$$

Consider now an element $v \in K$. We write

$$\langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - v\rangle = \langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}\rangle + \langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u} - v\rangle,$$

then we use (3.36), (3.34) and (3.37) to see that

$$\lim \langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - v\rangle = \langle A\widetilde{u} + \omega(\widetilde{u}), \widetilde{u} - v\rangle.$$

Then, by passing to upper limit in (3.31) and using assumption (2.3)(a) we have

$$\langle A\widetilde{u} + \omega(\widetilde{u}), \widetilde{u} - v\rangle \le \varphi(u, v) - \varphi(u, \widetilde{u}) - \langle f, v - \widetilde{u}\rangle$$

or, equivalently,

$$\langle f, v - \widetilde{u}\rangle \le \langle A\widetilde{u}, v - \widetilde{u}\rangle + \varphi(u, v) - \varphi(u, \widetilde{u}) + \langle \omega(\widetilde{u}), v - \widetilde{u}\rangle. \tag{3.38}$$

On the other hand, the definition of the Clarke subdifferential yields

$$\langle \omega(\widetilde{u}), v - \widetilde{u}\rangle \le j^0(\widetilde{u}; v - \widetilde{u}). \tag{3.39}$$

Then, combining (3.38) and (3.39) we deduce that

$$\langle f, v - \widetilde{u}\rangle \le \langle A\widetilde{u}, v - \widetilde{u}\rangle + \varphi(u, v) - \varphi(u, \widetilde{u}) + j^0(\widetilde{u}; v - \widetilde{u}). \tag{3.40}$$

Finally, we use (3.27) and (3.40) to see that $\widetilde{u}$ is a solution to Problem $\mathcal{P}$ and, by the uniqueness of the solution we have that $\widetilde{u} = u$, as claimed.

iii) *We now prove the convergence of the whole sequence* $\{\widetilde{u}_n\}$ *to* u.

A careful analysis of the proof in step ii) reveals that every subsequence of $\{\widetilde{u}_n\}$ which converges weakly in X has the same weak limit u. Moreover, we recall that the sequence $\{\widetilde{u}_n\}$ is bounded in X. Therefore, using a standard argument we deduce that the whole sequence $\{\widetilde{u}_n\}$ converges weakly in X to u, as $n \to \infty$. This shows that all the statements in step ii) are valid for the whole sequence $\{\widetilde{u}_n\}$. In particular, (3.36) combined with equality $\widetilde{u} = u$ shows that

$$\langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - u\rangle \to 0. \tag{3.41}$$

Let $n \in \mathbb{N}$ and let $\omega(u) \in \partial j(u)$. We have

$$\langle \omega(u), \widetilde{u}_n - u\rangle \le j^0(u; \widetilde{u}_n - u), \qquad \langle \omega_n(\widetilde{u}_n), u - \widetilde{u}_n\rangle \le j^0(\widetilde{u}_n; u - \widetilde{u}_n),$$

which imply that

$$\langle \omega(u), \widetilde{u}_n - u\rangle + \langle \omega_n(\widetilde{u}_n), u - \widetilde{u}_n\rangle \le j^0(u; \widetilde{u}_n - u) + j^0(\widetilde{u}_n; u - \widetilde{u}_n).$$

We now use assumption (2.4)(c) to see that

$$-\alpha_j\|\widetilde{u}_n - u\|_X^2 \le \langle \omega(u), u - \widetilde{u}_n\rangle + \langle \omega_n(\widetilde{u}_n), \widetilde{u}_n - u\rangle. \tag{3.42}$$

On the other hand, assumption (2.2) yields

$$m_A\|\widetilde{u}_n - u\|_X^2 \le \langle A\widetilde{u}_n - Au, \widetilde{u}_n - u\rangle. \tag{3.43}$$

We now add the inequalities (3.42) and (3.43) to deduce that

$$(m_A - \alpha_j)\|\widetilde{u}_n - u\|_X^2 \le \langle A\widetilde{u}_n + \omega_n(\widetilde{u}_n), \widetilde{u}_n - u\rangle + \langle Au + \omega(u), u - \widetilde{u}_n\rangle.$$

Next, we use the convergences (3.41), $\widetilde{u}_n \rightharpoonup u$ in X as well as the smallness assumption (2.5) to find that

$$\|\widetilde{u}_n - u\|_X \to 0, \tag{3.44}$$

which show that $\widetilde{u}_n \to u$ in X as $n \to \infty$, as claimed.

iv) *In the final step of the proof we prove that* $u_n \to u$ *in* X*, as* $n \to \infty$.

Let $n \in \mathbb{N}$. We test with $v = \widetilde{u}_n$ in (3.1) and $v = u_n$ in (3.14), then we add the resulting inequalities to see that

$$\begin{aligned}
&\langle Au_n - A\widetilde{u}_n, u_n - \widetilde{u}_n\rangle \\
&\le \frac{1}{\lambda_n}\langle G_n\widetilde{u}_n - G_n u_n, u_n - \widetilde{u}_n\rangle + \varphi(u_n, \widetilde{u}_n) - \varphi(u_n, u_n) + \varphi(u, u_n) - \varphi(u, \widetilde{u}_n) \\
&\quad + j^0(u_n; \widetilde{u}_n - u_n) + j^0(\widetilde{u}_n; u_n - \widetilde{u}_n) + \varepsilon_n\|\widetilde{u}_n - u_n\|_X.
\end{aligned}$$

Next, using assumptions (3.3), (2.3)(b), (2.4)(c), we deduce that

$$\begin{aligned}
\langle Au_n - A\widetilde{u}_n, u_n - \widetilde{u}_n\rangle &\le \alpha_\varphi\|u_n - u\|_X\|\widetilde{u}_n - u_n\|_X \\
&\quad + \alpha_j\|\widetilde{u}_n - u_n\|_X^2 + \varepsilon_n\|\widetilde{u}_n - u_n\|_X
\end{aligned}$$

and, therefore, the strong monotonicity of the opeator A yields

$$(m_A - \alpha_j)\|\widetilde{u}_n - u_n\|_X \le \alpha_\varphi\|u_n - u\|_X + \varepsilon_n. \tag{3.45}$$

We now write

$$\alpha_\varphi\|u_n - u\|_X \le \alpha_\varphi\|u_n - \widetilde{u}_n\|_X + \alpha_\varphi\|\widetilde{u}_n - u\|_X$$

and substitute this inequality in (3.45) to deduce that

$$(m_A - \alpha_\varphi - \alpha_j)\|\widetilde{u}_n - u_n\|_X \le \alpha_\varphi\|\widetilde{u}_n - u\|_X + \varepsilon_n.$$

Then, using the smallness assumption (2.5) we obtain that

$$\|\widetilde{u}_n - u_n\|_X \le \frac{\alpha_\varphi}{m_A - \alpha_\varphi - \alpha_j}\|\widetilde{u}_n - u\|_X + \frac{\varepsilon_n}{m_A - \alpha_\varphi - \alpha_j}.$$

This inequality, the convergence (3.44) and assumption (3.12) imply that

$$\|\widetilde{u}_n - u_n\|_X \to 0. \tag{3.46}$$

Finally, writing $\|u_n - u\|_X \le \|u_n - \widetilde{u}_n\|_X + \|\widetilde{u}_n - u\|_X$ and using the convergences (3.44), (3.46) we deduce that $u_n \to u$ in X which concludes the proof. □

4 Relevant particular cases

In this section we present some relevant particular cases in which Theorem 2 can be applied. In particular, we show that using a convenient choice of sets, operators and parameters, Problem $\mathcal{P}_n$ reduces successively to Problems $\mathcal{P}_{\varepsilon_n}$, $\mathcal{P}_{\lambda_n}$ $\mathcal{P}_{K_n}$ described in the Introduction. Then, we use Theorem 2 to recover various convergence results previously obtained in the study of these problems. Everywhere in this section we assume that (2.2)–(2.6) hold and we denote by u the solution of Problem $\mathcal{P}$ obtained in Theorem 1. We start by considering the following assumptions.

$$\widetilde{K} \text{ is a nonempty closed convex subset of } X. \tag{4.1}$$

$$K \subset \widetilde{K}. \tag{4.2}$$

$$K_n \subset \widetilde{K} \ \text{ for each } n \in \mathbb{N}. \tag{4.3}$$

$$K_n \xrightarrow{M} K \ \text{ as } \ n \to \infty. \tag{4.4}$$

$$K_n \xrightarrow{M} \widetilde{K} \ \text{ as } \ n \to \infty. \tag{4.5}$$

$$G : X \to X^* \ \text{ is a penalty operator for } K. \tag{4.6}$$

$$f_n \in X^* \ \text{ for each } n \in \mathbb{N}. \tag{4.7}$$

$$f_n \to f \ \text{ in } \ X^*. \tag{4.8}$$

We are now in a position to introduce some relevant consequences of Theorem 2.

a) A first penalty method. Our first particular case is when $K_n = X$ and $G_n = G$ for each $n \in \mathbb{N}$, G being a penalty operator of K. In this case Theorem 2 leads to the following result.

Corollary 1. *Assume* (2.1)–(2.6), (3.4), (3.5), (3.10)–(3.12) *and* (4.6). *Then, the following statements hold.*

a) *For each $n \in \mathbb{N}$, there exists $u_n \in X$ such that*

$$\begin{aligned} &\langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) \\ &\quad + j^0(u_n; v - u_n) + \varepsilon_n\|v - u_n\|_X \ge \langle f, v - u_n\rangle \qquad \forall\, v \in X. \end{aligned} \tag{4.9}$$

Moreover, the solution is unique if $\varepsilon_n = 0$.

b) *If $\{u_n\} \subset X$ is a sequence such that u_n is a solution of* (4.9), *for each $n \in \mathbb{N}$, then $u_n \to u$ in X.*

Proof. Since $K_n = X$ it follows that conditions (3.2), (3.6), (3.8) are satisfied with $\widetilde{K} = X$. Moreover, since $G_n = G$ and (4.6) holds, it follows that conditions (3.3), (3.7), (3.9) hold, too, with $\widetilde{K} = X$ and $c_n = 0$. Corollary 1 is now a direct consequence of Theorem 2. □

Note that in the case when $\varepsilon_n = 0$ inequality (4.9) reduces to inequality (1.3), used in the classical penalty method for variational-hemivariational inequalities. Therefore, Corollary 1 provides the unique solvability of Problem $\mathcal{P}_{\lambda_n}$, for each $n \in \mathbb{N}$, and the convergence of the sequence of solutions to the solution of Problem $\mathcal{P}$. This result was obtained in [19], in the particular case when $\varphi(u, v) = \varphi(u)$ and extented in [25] in the case when φ depends on both u and v.

b) A second penalty method. Our second particular case is when $K_n = \widetilde{K}$ where $\widetilde{K}$ is a given set which satisfies condition (4.1) and $G_n = G$ for each $n \in \mathbb{N}$, G being a penalty operator of K. In this case Theorem 2 leads to the following result.

Corollary 2. *Assume* (2.1)–(2.6), (3.4), (3.5), (3.9)(e)(ii), (3.10)–(3.12), (4.1), (4.2) *and* (4.6). *Then, the following statements hold.*

a) *For each $n \in \mathbb{N}$, there exists $u_n \in \widetilde{K}$ such that*

$$\langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) \qquad (4.10)$$

$$+ j^0(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_X \geq \langle f, v - u_n\rangle \qquad \forall\, v \in \widetilde{K}.$$

Moreover, the solution is unique if $\varepsilon_n = 0$.

b) *If $\{u_n\} \subset X$ is a sequence such that u_n is a solution of* (4.10), *for each $n \in \mathbb{N}$, then $u_n \to u$ in X.*

Proof. Since $K_n = \widetilde{K}$ and (4.1), (4.2) hold, it follows that conditions (3.2), (3.6), (3.8) are satisfied. Moreover, since $G_n = G$ and (3.9)(e)(ii), (4.6) hold, it follows that conditions (3.3), (3.7), (3.9) are satisfied with $c_n = 0$. Corollary 1 is now a direct consequence of Theorem 2. □

Note that in the case when $\varepsilon_n = 0$ inequality (4.10) reduces to inequality

$$u_n \in \widetilde{K}, \;\; \langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle \qquad (4.11)$$

$$+ \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) \geq \langle f, v - u_n\rangle \qquad \forall\, v \in \widetilde{K}.$$

A brief comparison between inequalities (1.1) and (4.11) shows that (4.11) is obtained from (1.1) by replacing the set K with the set $\widetilde{K}$ and the operator A with the operator $A + \frac{1}{\lambda_n} G$, in which λ_n is a penalty parameter. For this reason we refer to (4.11) as a penalty problem of (1.1). Corollary 2 establishes the link between the solutions of these problems and, at the best of our knowledge, it represents a new result. Roughly speaking, it shows that, in the limit when $n \to \infty$, a partial relaxation of the set of constraints can be compensated by a perturbation of the nonlinear operator which governs Problem $\mathcal{P}$.

c) A continuous dependence result. Our third particular case is when $K_n \xrightarrow{M} K$, G_n vanishes and f is replaced by f_n. In this case Theorem 2 leads to the following result.

Corollary 3. *Assume* (2.1)–(2.6), (3.2), (3.6), (3.10), (4.4), (4.7) *and* (4.8). *Then, for each $n \in \mathbb{N}$, there exists a unique element $u_n \in K_n$ such that*

$$\langle Au_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) \qquad (4.12)$$

$$+ j^0(u_n; v - u_n) \geq \langle f_n, v - u_n\rangle \qquad \forall\, v \in K_n.$$

Moreover, $u_n \to u$ in X.

Proof. The existence of a unique solution to inequality (4.12) is a direct consequence of Theorem 1. Let $n \in \mathbb{N}$. Then, using (4.12) it is easy to see that

$$\langle Au_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n)$$

$$+ j^0(u_n; v - u_n) + \langle f - f_n, v - u_n\rangle \geq \langle f, v - u_n\rangle \qquad \forall\, v \in K_n.$$

and, denoting $\varepsilon_n = \|f - f_n\|_{X^*}$, it follows that

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) \\ &\quad + j^0(u_n; v - u_n) + \varepsilon_n\|v - u_n\|_X \geq \langle f, v - u_n\rangle \qquad \forall\, v \in K_n.\end{aligned} \tag{4.13}$$

On the other hand, since (4.4) holds it follows that condition (3.8) is satisfied with $\widetilde{K} = K$. Moreover, since G_n vanishes, it follows that conditions (3.3), (3.7), (3.9) hold, with $Gv = 0_{X^*}$ for all $v \in X$ and $c_n = 0$. In addition, assumption (4.8) implies that (3.12) holds, too. We are now in a position to use Theorem 2 b) with $\lambda_n = \frac{1}{n}$, for instance, to deduce the convergence $u_n \to u$ in X, which concludes the proof. □

Note that Corollary 3 represents a continuous dependence result of the solution to Problem $\mathcal{P}$ with respect to the set K and the element f. Similar convergence results have been obtained in [34, 37], under different assumptions on functions and operators.

d) A Tykhonov well-posedness result. Our fourth particular case is when $K_n = K$ and G_n vanishes. In this case Theorem 2 leads to the following result in the study of Problem $\mathcal{P}_{\varepsilon_n}$ described in the Introduction.

Corollary 4. *Assume* (2.1)–(2.6), (3.5), (3.10) *and* (3.12). *Then, the following statements hold.*

a) *For each $n \in \mathbb{N}$, there exists an element $u_n \in K$ such that* (1.2) *holds.*

b) *If $\{u_n\} \subset X$ is a sequence such that u_n is a solution of Problem $\mathcal{P}_n$, for each $n \in \mathbb{N}$, then $u_n \to u$ in X.*

The proof of Corollary 4 is based on arguments similar to those presented above and, therefore, we skip it. We restrict ourselves to note that an elementary proof can be used to obtain the convergence result in Corollary 4, without assumption (3.10). The details can be found in [31]. Finally, using the definitions in [29, 31] we remark that Theorem 1 combined with Corollary 4 provides the well-posedness of Problem $\mathcal{P}$ in the sense of Tykhonov.

e) An existence, uniqueness and convergence result. We end this section with an existence, uniqueness and convergence result which completes our analysis of Problem $\mathcal{P}$ and has some interest in its own. To this end we assume in what follows that (2.1), (3.2) and (4.1) hold. Let $J\colon X \to X^*$ be the duality map, $I\colon X \to X$ the identity map on X, $P_K\colon X \to K$ the projection operator on K, $P_{\widetilde{K}}\colon X \to \widetilde{K}$ the projection operator on $\widetilde{K}$ and, for each $n \in \mathbb{N}$ let $P_{K_n}\colon X \to K_n$ be the projection operator on K_n. Consider the operators P, $\widetilde{P}$, P_n, G, G_n, both defined on X with values in X^*, given by equalities

$$P = J(I - P_K), \quad \widetilde{P} = J(I - P_{\widetilde{K}}), \quad P_n = J(I - P_{K_n}), \tag{4.14}$$

$$G = P + \widetilde{P}, \quad G_n = P + P_n. \tag{4.15}$$

We use these notation to state and prove the following result.

Corollary 5. *Assume* (2.1)–(2.6), (3.2), (3.4), (3.6), (3.10), (3.11) *and* (4.1), (4.3) *and* (4.5). *Then, for each $n \in \mathbb{N}$, there exists a unique element $u_n \in K_n$ such that*

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\langle G_n u_n, v - u_n\rangle + \varphi(u_n, v) - \varphi(u_n, u_n) \\ &\quad + j^0(u_n; v - u_n) \geq \langle f, v - u_n\rangle \qquad \forall\, v \in K_n.\end{aligned} \tag{4.16}$$

Moreover, $u_n \to u$ in X.

Proof. Recall that Proposition 4 guarantees that P, $\widetilde{P}$ and P_n are penalty operators of K, $\widetilde{K}$ and K_n, respectively. This implies that these operators are bounded demicontinuous and monotone. Therefore, so are the operators G and G_n defined by (4.15). This shows that conditions (3.3) and (3.9)(c) are satisfied. The existence of a unique solution of inequality (4.16) results from Theorem 2 with $\varepsilon_n = 0$.

Assume now that $u \in K_n$, $v \in K$ and recall assumption (3.6) which states that $K \subset K_n$. This implies that $Pv = 0_{X^*}$, $P_n v = 0_{X^*}$ and, therefore, (4.15) yields

$$\begin{aligned}\langle G_n u, v-u\rangle &= \langle Pu, v-u\rangle + \langle P_n u, v-u\rangle \\ &= \langle Pu - Pv, v-u\rangle + \langle P_n u - P_n v, v-u\rangle.\end{aligned}$$

We now use the monotonicity of the operators P and P_n to see that $\langle G_n u, v-u\rangle \leq 0$ which implies that condition (3.7) is satisfied. A similar argument based on assumption (4.2), guaranteed by (3.6) and (4.3), shows that condition (3.9)(d) holds, too.

Let $n \in \mathbb{N}$ and $u \in K_n$. Then it follows that $P_{K_n} u = u$ and, since (4.3) guarantees that $K_n \subset \widetilde{K}$, we deduce that $P_{\widetilde{K}} u = u$, too. We now use (4.15) and (4.14) to see that

$$G_n u - Gu = P_n u - \widetilde{P} u = J(u - P_{K_n} u) - J(u - P_{\widetilde{K}} u) = J(0_X) - J(0_X) = 0_{X^*}.$$

It follows from here that condition (3.9)(a) holds with $c_n = 0$. This implies that condition (3.9)(b) is satisfied, too.

Assume now that $u \in \widetilde{K}$ is such that

$$\langle Gu, v-u\rangle = 0 \qquad \forall\, v \in K. \tag{4.17}$$

Then, since $K \subset K_n \subset \widetilde{K}$ we have that $Pv = \widetilde{P}v = 0_{X^*}$ for all $v \in K$ and, therefore, (4.15) yields

$$\langle Gu, v-u\rangle = \langle Pu - Pv, v-u\rangle + \langle \widetilde{P}u - \widetilde{P}v, v-u\rangle \qquad \forall\, v \in K. \tag{4.18}$$

We now combine (4.17) and (4.18) to deduce that

$$\langle Pu - Pv, v-u\rangle + \langle \widetilde{P}u - \widetilde{P}v, v-u\rangle = 0 \qquad \forall\, v \in K. \tag{4.19}$$

On the other hand, using the monotonicity of the operators P and $\widetilde{P}$ we have

$$\langle Pu - Pv, v-u\rangle \leq 0, \qquad \langle \widetilde{P}u - \widetilde{P}v, v-u\rangle \leq 0 \qquad \forall\, v \in K. \tag{4.20}$$

We now use (4.19), (4.20) and the elementary implication

$$a \leq 0, \quad b \leq 0, \quad a+b=0 \implies a=b=0 \tag{4.21}$$

to deduce that

$$\langle Pu - Pv, v-u\rangle = 0 \qquad \forall\, v \in K,$$

which implies that

$$\langle Pu, u-v\rangle = 0 \qquad \forall\, v \in K.$$

We now take $v = P_K u$ in the previous equality and use the definition (4.14) of the operator P and the properties of the duality mapping J to see that

$$\langle J(u - P_K u), u - P_K u\rangle = \|u - P_K u\|_X^2 = 0.$$

This implies that $u \in K$ and, therefore, condition (3.9)(e) holds.

We conclude from above that conditions (3.7), (3.9) are satisfied, the later with $c_n = 0$. Moreover, we recall assumption (4.5), which implies (3.8). Therefore, we are in a position to apply Theorem 2 with $\varepsilon_n = 0$ in order to conclude the proof. □

5 A frictional contact problem

In this section we apply our abstract results in Section 3 in the study of a frictional contact problem with normal compliance and unilateral constraint. To this end we consider a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary Γ composed of three sets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ with the mutually disjoint relatively open sets Γ_1, Γ_2 and Γ_3, such that $meas\,(\Gamma_1) > 0$. We use boldface letters for vectors and tensors, such as the outward unit normal on Γ, denoted by $\boldsymbol{\nu}$. A typical point in $\mathbb{R}^d$ is denoted by $\boldsymbol{x} = (x_i)$. The indices i, j run between 1 and d and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. Moreover, the indices ν and τ represent normal components and tangential parts of vectors and tensors. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The zero element, the canonical inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$, " $\cdot$ " and $\|\cdot\|$, respectively. Then, the classical formulation of the contact problem we consider in this section is the following.

Problem $\mathcal{Q}$. *Find a displacement field* $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, *a stress field* $\sigma\colon \Omega \to \mathbb{S}^{\mathbf{d}}$ *and an interface function* $\xi_\nu\colon \Gamma_3 \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{5.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0} \quad \text{in} \quad \Omega, \tag{5.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{5.4}$$

$$\left.\begin{aligned} &u_\nu \le k, \quad \sigma_\nu + \xi_\nu \le 0,\\ &(u_\nu - k)(\sigma_\nu + \xi_\nu) = 0,\\ &\xi_\nu \in \partial j_\nu(u_\nu)\end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{5.5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \ \text{if}\ \boldsymbol{u}_\tau \ne \mathbf{0} \quad \text{on} \quad \Gamma_3. \tag{5.6}$$

Problem $\mathcal{Q}$ describes the equilibrium of an elastic body acted upon by body forces and surface tractions, in frictional contact with a foundation made of a rigid body covered by a layer made of elastic material, say asperities. It was already considered in [27] and, therefore we skip the mechanical assumptions which lead to this model. We restrict ourselves to the following short description of the equations and boundary conditions. First, equation (5.1) represents the elastic constitutive law in which $\mathcal{F}$ is the elasticity operator, assumed to be nonlinear, and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearized strain tensor. Equation (5.2) is the equilibrium equation in which $\boldsymbol{f}_0$ denotes the density of body forces. Conditions (5.3) and (5.4) are the displacement and traction conditions, respectively, in which $\boldsymbol{f}_2$ represents the density of surface tractions. Condition (5.5) is the contact conditions in which $k \ge 0$ is a given bound and ∂j_ν is the Clarke subdifferential of a given function j_ν. Finally, (5.5) represents a version of the Coulomb's law of dry friction in which F_b denotes the friction bound.

In the study of Problem $\mathcal{Q}$ we consider the following assumptions on the data.

$$\left\{\begin{array}{l}\mathcal{F}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\ \text{(a) there exists } L_{\mathcal{F}}>0 \text{ such that}\\ \qquad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega,\\ \text{(b) there exists } m_{\mathcal{F}}>0 \text{ such that}\\ \qquad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega,\\ \text{(c) } \mathcal{F}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d,\\ \text{(d) } \mathcal{F}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{0}\ \text{ for a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{5.7}$$

$$\left\{\begin{array}{l} j_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R} \text{ is such that}\\ \text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R} \text{ and there}\\ \qquad \text{exists } \bar{e}\in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,\bar{e}(\cdot))\in L^1(\Gamma_3),\\ \text{(b) } j_\nu(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x}\in\Gamma_3,\\ \text{(c) } |\partial j_\nu(\boldsymbol{x},r)|\le \bar{c}_0+\bar{c}_1\,|r| \text{ for a.e. } \boldsymbol{x}\in\Gamma_3,\\ \qquad \text{for all } r\in\mathbb{R} \text{ with } \bar{c}_0,\ \bar{c}_1\ge 0,\\ \text{(d) } j_\nu^0(\boldsymbol{x},r_1;r_2-r_1)+j_\nu^0(\boldsymbol{x},r_2;r_1-r_2)\le \alpha_{j_\nu}\,|r_1-r_2|^2\\ \qquad \text{for a.e. } \boldsymbol{x}\in\Gamma_3,\ \text{for all } r_1,\ r_2\in\mathbb{R} \text{ with } \alpha_{j_\nu}\ge 0,\\ \text{(e) either } j_\nu(\boldsymbol{x},\cdot) \text{ or } -j_\nu(\boldsymbol{x},\cdot) \text{ is regular on } \mathbb{R} \text{ for a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{5.8}$$

$$\left\{\begin{array}{l} F_b\colon \Gamma_3\times\mathbb{R}\to\mathbb{R} \text{ is such that}\\ \text{(a) there exists } L_{F_b}>0 \text{ such that}\\ \qquad |F_b(\boldsymbol{x},r_1)-F_b(\boldsymbol{x},r_2)|\le L_{F_b}|r_1-r_2|\\ \qquad\quad \text{for all } \ r_1,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3,\\ \text{(b) } F_b(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R},\\ \text{(c) } F_b(\boldsymbol{x},r)=0\ \text{ for all } r\le 0,\ F_b(\boldsymbol{x},r)\ge 0\ \text{ for all } r>0,\\ \qquad \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{5.9}$$

$$\left(L_{F_b}+\alpha_{j_\nu}\right)\,\|\gamma\|^2<m_{\mathcal{F}}. \tag{5.10}$$

$$\boldsymbol{f}_0\in L^2(\Omega)^d,\quad \boldsymbol{f}_2\in L^2(\Gamma_2)^d. \tag{5.11}$$

$$k\ge 0. \tag{5.12}$$

Next, we use the space V defined by

$$V=\{\,\boldsymbol{v}\in H^1(\Omega)^d\ :\ \boldsymbol{v}=\boldsymbol{0}\ \text{ a.e. on } \Gamma_1\,\}, \tag{5.13}$$

which is real Hilbert space with the canonical inner product

$$(\boldsymbol{v},\boldsymbol{u})_V=\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,$$

and the associated norm $\|\cdot\|_V$. Here and below, for every $\boldsymbol{v}\in V$ we use the notation

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=(\varepsilon_{ij}(\boldsymbol{v})),\quad \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}\,(u_{i,j}+u_{j,i}),\quad v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu},\quad \boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}.$$

We also use V^* for the dual of V, $\langle\cdot,\cdot\rangle$ for the duality pairing between V and V^* and $\|y\|$ for the norm of the trace operator $y\colon V \to L^2(\Gamma_3)^d$. We denote by K the set of admissible displacement fields defined by

$$K = \{\, \boldsymbol{v} \in V \;:\; v_\nu \le k \text{ a.e. on } \Gamma_3 \,\} \tag{5.14}$$

and, finally, we introduce the following notation.

$$A\colon V \to V^*, \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\S, \tag{5.15}$$

$$\varphi\colon V\times V \to \mathbb{R}, \varphi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} F_b(u_\nu)\,\|\boldsymbol{v}_\tau\|\,d\Gamma, \tag{5.16}$$

$$j\colon V \to \mathbb{R}, j(\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(\boldsymbol{v}_\nu)\,d\Gamma, \tag{5.17}$$

$$\boldsymbol{f} \in V^*, \langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2\cdot\boldsymbol{v}\,d\Gamma \tag{5.18}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$. It can be proved that the function j is locally Lipschitz. Therefore, as usual, we shall use the notation $j^0(\boldsymbol{u}; \boldsymbol{v})$ for the generalized directional derivative of j at $\boldsymbol{u}$ in the direction $\boldsymbol{v}$.

The variational formulation of Problem $\mathcal{Q}$, obtained by a standard procedure, is as follows.

Problem $\mathcal{Q}^V$. *Find a displacement field* $\boldsymbol{u} \in K$ *such that*

$$\langle A\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle + \varphi(\boldsymbol{u}, \boldsymbol{v}) - \varphi(\boldsymbol{u}, \boldsymbol{u}) + j^0(\boldsymbol{u}; \boldsymbol{v}-\boldsymbol{u}) \ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle \qquad \forall\, \boldsymbol{v} \in K. \tag{5.19}$$

Next, for each $n \in \mathbb{N}$ we consider the following contact problem.

Problem $\mathcal{Q}_n$. *Find a displacement field* $\boldsymbol{u}_n\colon \Omega \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma}_n\colon \Omega \to \mathbb{S}^d$ *and an interface function* $\xi_{n\nu}\colon \Gamma_3 \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma}_n = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) \quad \text{in} \quad \Omega, \tag{5.20}$$

$$\operatorname{Div}\boldsymbol{\sigma}_n + \boldsymbol{f}_{0n} = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{5.21}$$

$$\boldsymbol{u}_n = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{5.22}$$

$$\boldsymbol{\sigma}_n\boldsymbol{\nu} = \boldsymbol{f}_{2n} \quad \text{on} \quad \Gamma_2, \tag{5.23}$$

$$\left.\begin{aligned} &u_{n\nu} \le k_n, \quad \sigma_{n\nu} + \tfrac{1}{\lambda_n}\, p_\nu(u_{n\nu} - g_n) + \xi_{n\nu} \le 0, \\ &(u_{n\nu} - k_n)(\sigma_{n\nu} + \tfrac{1}{\lambda_n}\, p_\nu(u_{n\nu} - g_n) + \xi_{n\nu}) = 0, \\ &\xi_{n\nu} \in \partial j_\nu(u_{n\nu}) \end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{5.24}$$

$$\|\boldsymbol{\sigma}_{n\tau}\| \le F_b(u_{n\nu}), \quad -\boldsymbol{\sigma}_{n\tau} = F_b(u_{n\nu})\frac{\boldsymbol{u}_{n\tau}}{\|\boldsymbol{u}_{n\tau}\|} \text{ if } \boldsymbol{u}_{n\tau} \ne \boldsymbol{0} \quad \text{on} \quad \Gamma_3. \tag{5.25}$$

The difference between Problems $\mathcal{Q}_n$ and $\mathcal{Q}$ is twofold. First, in Problem $\mathcal{Q}_n$ the densities of body forces $\boldsymbol{f}_0$ and surface tractions $\boldsymbol{f}_2$ as well as the bound k have been replaced by their perturbation $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$ and k_n, respectively. Second, the boundary contact condition (5.5) has been replaced by the contact boundary condition (5.24) in which $\lambda_n > 0$ is a deformability coefficient, p_ν is a normal compliance function and g_n is a given gap. This condition still models the contact with a rigid foundation covered by a layer of deformable material. Nevertheless, the thickness of this material changed (since k was replaced by k_n) as well as its elastic response (since the additional term $\frac{1}{\lambda_n}\, p_\nu(u_{n\nu} - g_n)$ was introduced in this condition).

In the study of Problem $\mathcal{Q}_n$ we consider the following assumptions.

$$\left\{\begin{array}{l} p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ is such that} \\ \quad \text{(a) there exists } L_{p_\nu} > 0 \text{ such that} \\ \qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_{p_\nu}|r_1 - r_2| \\ \qquad\quad \text{for all } r_1,\ r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ \quad \text{(b) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))\,(r_1 - r_2) \ge 0 \\ \qquad\quad \text{for all } r_1,\ r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ \quad \text{(c) } p_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}, \\ \quad \text{(d) } p_\nu(\boldsymbol{x}, r) = 0 \text{ if and only if } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{5.26}$$

$$\boldsymbol{f}_{0n} \in L^2(\Omega)^d, \quad \boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d. \tag{5.27}$$

$$\boldsymbol{f}_{0n} \to \boldsymbol{f} \quad \text{in} \quad L^2(\Omega)^d, \quad \boldsymbol{f}_{2n} \to \boldsymbol{f}_2 \quad \text{in} \quad L^2(\Gamma_2)^d. \tag{5.28}$$

$$k_n \ge g_n \ge k, \qquad \lambda_n > 0. \tag{5.29}$$

$$\widetilde{k} \in \mathbb{R}, \quad k_n \to \widetilde{k}, \quad g_n \to k, \quad \lambda_n \to 0. \tag{5.30}$$

Moreover, we introduce the notation

$$K_n = \{\, \boldsymbol{v} \in V \ :\ v_\nu \le k_n \ \text{ a.e. on } \Gamma_3 \,\}, \tag{5.31}$$

$$G_n : V \to V^*, \qquad \langle G_n \boldsymbol{u}, \boldsymbol{v}\rangle = \int_{\Gamma_3} p_\nu(u_\nu - g_n) v_\nu \, d\Gamma, \tag{5.32}$$

$$\boldsymbol{f}_n \in V^*, \qquad \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle = \int_{\Omega} \boldsymbol{f}_{0n} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_{2n} \cdot \boldsymbol{v}\, d\Gamma \tag{5.33}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

The variational formulation of Problem $\mathcal{Q}_n$ is as follows.

Problem $\mathcal{Q}_n^V$. *Find a displacement field* $\boldsymbol{u}_n \in K_n$ *such that*

$$\begin{aligned} &\langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \frac{1}{\lambda_n}\langle G_n\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n) + \varphi(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi(\boldsymbol{u}_n, \boldsymbol{u}_n) \\ &\qquad + j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n) \ge \langle \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle \quad \forall\, \boldsymbol{v} \in K_n. \end{aligned} \tag{5.34}$$

Our main resut in this section is the following existence, uniqueness and convergence result.

Theorem 3. *Assume* (5.7)–(5.12), (5.26)–(5.30). *Then, the following statements hold.*

a) *There exists a unique solution* $\boldsymbol{u} \in K$ *to Problem* $\mathcal{Q}^V$. *Moreover, for each* $n \in \mathbb{N}$, *there exists a unique solution* $\boldsymbol{u}_n \in K_n$ *to Problem* $\mathcal{Q}_n^V$.

b) *The solution* $\boldsymbol{u}_n$ *of Problem* $\mathcal{Q}_n^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{Q}^V$, *i.e.,* $\boldsymbol{u}_n \to \boldsymbol{u}$ *in* V, *as* $n \to \infty$.

Proof. a) The unique solvability of Problem $\mathcal{Q}^V$ corresponds to Theorem 109 in [27] and, for this reason, we do not provide its proof. We restrict ourselves to mention that it represents a direct consequence of Theorem 1. The unique solvability of Problem $\mathcal{Q}_n^V$ follows from Theorem 2 a). Indeed, Problem $\mathcal{Q}_n^V$ is a special case of Problem $\mathcal{P}_n$ in which $\varepsilon_n = 0$.

b) Let $n \in \mathbb{N}$. We use inequality (5.34) to see that

$$\begin{aligned} &\langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \frac{1}{\lambda_n}\langle G_n\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n) + \varphi(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi(\boldsymbol{u}_n, \boldsymbol{u}_n) \\ &\qquad + j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n) + \langle \boldsymbol{f} - \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n\rangle \quad \forall\, \boldsymbol{v} \in K_n. \end{aligned} \tag{5.35}$$

and, using the notation $\varepsilon_n = \|\boldsymbol{f}_n - \boldsymbol{f}\|_{V^*}$, we deduce that $\boldsymbol{u}_n$ is a solution of the following inequality problem.

Problem $\widetilde{\mathcal{Q}}_n^V$. *Find a displacement* $\boldsymbol{u}_n \in K_n$ *such that*

$$\langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \frac{1}{\lambda_n}\langle G_n\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \varphi(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi(\boldsymbol{u}_n, \boldsymbol{u}_n) \tag{5.36}$$
$$+ j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n) + \varepsilon_n\|\boldsymbol{v} - \boldsymbol{u}_n\|_V \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n\rangle \quad \forall\, \boldsymbol{v} \in K_n.$$

Our aim in what follows is to use Theorem 2 b) in the particular case when problems $\mathcal{P}$ and $\mathcal{P}_n$ are given by problems $\mathcal{Q}^V$ and $\widetilde{\mathcal{Q}}_n^V$, respectively. To this end, we need to check, point by point, the validity of the conditions in this theorem. Note that part of the conditions are obviously satisfied such as conditions (3.2), (3.4)–(3.6), for instance, and part of them have been verified in the proof of the first part of this theorem. The details can be found in [27, Ch. 8], as already mentioned. Therefore, in order to avoid repetition we focus in what follows on the conditions (3.7), (3.8), (3.9), (3.10) and, to this end, we introduce the following additional notations.

$$\widetilde{K} = \{\, \boldsymbol{v} \in V \;:\; v_\nu \leq \widetilde{k} \text{ a.e. on } \Gamma_3 \,\}, \tag{5.37}$$

$$G\colon V \to V^*, \qquad \langle G\boldsymbol{u}, \boldsymbol{v}\rangle = \int_{\Gamma_3} p_\nu(u_\nu - k)v_\nu\, d\Gamma \tag{5.38}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

Let $n \in \mathbb{N}$ and let $\boldsymbol{u} \in K_n$, $\boldsymbol{v} \in K$. We write

$$p_\nu(u_\nu - g_n)(v_\nu - u_\nu) = p_\nu(u_\nu - g_n)(v_\nu - k) + p_\nu(u_\nu - g_n)(k - u_\nu)$$

and, using the properties of the function p_ν combined with inequalities $k_n \geq g_n \geq k$ we deduce that

$$p_\nu(u_\nu - g_n)(v_\nu - u_\nu) \leq 0 \qquad \text{a.e. on } \Gamma_3.$$

This implies that $\langle G_n\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle \leq 0$ and, therefore condition (3.7) is satisfied.

Assume now that $\widetilde{k} > 0$. Then, using the definitions (5.31) and (5.37) we deduce that $K_n = \frac{k_n}{\widetilde{k}}\widetilde{K}$ which implies that $K_n \xrightarrow{M} \widetilde{K}$. Indeed, if $\boldsymbol{v} \in \widetilde{K}$ and $\boldsymbol{v}_n = \frac{k_n}{\widetilde{k}}\boldsymbol{v}$, then the sequence $\{\boldsymbol{v}_n\}$ satisfies condition (a) in Definition 5. Note also that condition (b) in Definition 5 follows from a standard measure theory argument. If $\widetilde{k} = 0$ we arrive to the same conclusions, by using the sequence $\{\boldsymbol{v}_n\}$ defined by $\boldsymbol{v}_n = \boldsymbol{v}$ for all $n \in \mathbb{N}$. This implies that, in any case, condition (3.8) is satisfied.

We now check the validity of condition (3.9) for the operators (5.32) and (5.38). Let $\boldsymbol{u}, \boldsymbol{v} \in V$. Using (5.26)(a), inequality $g_n \geq k$ and the properties of the trace operator we have

$$|\langle G_n\boldsymbol{u} - G\boldsymbol{u}, \boldsymbol{v}\rangle| \leq \int_{\Gamma_3} |(p_\nu(u_\nu - g_n) - p_\nu(u_\nu - k))v_\nu|\, d\Gamma$$
$$\leq L_{p_\nu}(g_n - k)\int_{\Gamma_3} |v_\nu|\, d\Gamma \leq L_0(g_n - k)\|\boldsymbol{v}\|_V$$

where L_0 is a positive constant. This proves that $\|G_n\boldsymbol{u} - G\boldsymbol{u}\|_{V^*} \leq L_0(g_n - k)$ and, therefore, condition (3.9)(a) holds with

$$c_n = L_0(g_n - k). \tag{5.39}$$

Using now the convergence $g_n \to k$ in (5.30) we find that (3.9)(b) holds, too. Next, condition (3.9)(c) follows from standard arguments, based on the properties of the function p_ν and the trace operator.

Consider now two elements $\boldsymbol{u} \in \widetilde{K}$ and $\boldsymbol{v} \in K$. We write

$$p_\nu(u_\nu - k)(v_\nu - u_\nu) = p_\nu(u_\nu - k)(v_\nu - k) + p_\nu(u_\nu - k)(k - u_\nu),$$

then we use (5.26) and inequality $\widetilde{k} \geq k$, guaranteed by assumptions (5.29) and (5.30), to deduce that

$$p_\nu(u_\nu - k)(v_\nu - u_\nu) \leq 0 \qquad \text{a.e. on } \Gamma_3.$$

This implies that $\langle G\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle \leq 0$ and, therefore condition (3.9)(d) is satisfied.

Assume now that $\langle G\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle = 0$. Then,

$$\int_{\Gamma_3} p_\nu(u_\nu - k)(v_\nu - k)\, d\Gamma + \int_{\Gamma_3} p_\nu(u_\nu - k)(k - u_\nu)\, d\Gamma = 0. \tag{5.40}$$

On the other hand, note that the properties of the function p_ν imply that

$$p_\nu(u_\nu - k)(v_\nu - k) \leq 0, \qquad p_\nu(u_\nu - k)(k - u_\nu) \leq 0 \qquad \text{a.e. on } \Gamma_3$$

and, therefore,

$$\int_{\Gamma_3} p_\nu(u_\nu - k)(v_\nu - k)\, d\Gamma \leq 0, \qquad \int_{\Gamma_3} p_\nu(u_\nu - k)(k - u_\nu)\, d\Gamma \leq 0. \tag{5.41}$$

We now use (5.40), (5.41) and implication (4.21) to see that

$$\int_{\Gamma_3} p_\nu(u_\nu - k)(k - u_\nu)\, d\Gamma = 0.$$

Therefore, since the integrand is negative, we deduce that

$$p_\nu(u_\nu - k)u_\nu = 0 \quad \text{a.e. on } \Gamma_3.$$

This equality combined with assumption (5.26)(d) implies that $u_\nu \leq k$ a.e. on Γ_3. Thus, $\boldsymbol{u} \in K$ and, therefore, condition (3.9)(e) is satisfied.

Finally, let $\boldsymbol{u}, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$. We use definition (5.16) and assumption (5.9) to see that

$$\varphi(\boldsymbol{u}, \boldsymbol{v}_1) - \varphi(\boldsymbol{u}, \boldsymbol{v}_2) \leq \int_{\Gamma_3} F_b(u_\nu)\, \|\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\|\, d\Gamma \leq L_{F_b}\|\gamma\|^2 \|\boldsymbol{u}\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V,$$

which shows that condition (3.10) holds with $c_\varphi(\boldsymbol{u}) = L_{F_b}\|\gamma\|^2\|\boldsymbol{u}\|_V$.

It follows from above that we are in a position to use Theorem 2. In this way we obtain that if $\{\widetilde{\boldsymbol{u}}_n\}$ is a sequence of elements of V such that $\widetilde{\boldsymbol{u}}_n$ is a solution of Problem $\widetilde{\mathcal{Q}}_n^V$, for each $n \in \mathbb{N}$, then $\widetilde{\boldsymbol{u}}_n \to \boldsymbol{u}$ in V. Recall now that for each $n \in \mathbb{N}$ the solution $\boldsymbol{u}_n$ of Problem $\mathcal{Q}_n^V$ is a solution of Problem $\widetilde{\mathcal{Q}}_n^V$. It follows from here that $\boldsymbol{u}_n \to \boldsymbol{u}$ in V which concludes the proof. □

In addition to the mathematical interest in the convergence result in Theorem 3 b), it is important from the mechanical point of view, since it establishes the link between the solutions of two different contact models. It also shows that the weak solution of the elastic frictional contact problem $\mathcal{Q}$ depends continuously on the densities of body forces and surface tractions and the thickness of the deformable layer.

Acknowledgements: This research was supported by the National Natural Science Foundation of China (11771067), the Applied Basic Project of Sichuan Province (2019YJ0204), the Fundamental Research Funds for the Central Universities (ZYGX2019J095), and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No 823731 CONMECH.

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