

## Research Article

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# Existence and concentration of positive solutions for a critical $p$ & $q$ equation

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**Abstract:** We show existence and concentration results for a class of  $p$  &  $q$  critical problems given by

$$-\operatorname{div} (a(\epsilon^p |\nabla u|^p) \epsilon^p |\nabla u|^{p-2} \nabla u) + V(z)b(|u|^p) |u|^{p-2} u = f(u) + |u|^{q^*-2} u \text{ in } \mathbb{R}^N,$$

where  $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ ,  $\epsilon > 0$  is a small parameter,  $1 < p \leq q < N$ ,  $N \geq 2$  and  $q^* = Nq/(N - q)$ . The potential  $V$  is positive and  $f$  is a superlinear function of  $C^1$  class. We use Mountain Pass Theorem and the penalization arguments introduced by Del Pino & Felmer's associated to Lions' Concentration and Compactness Principle in order to overcome the lack of compactness.

**Keywords:** Critical exponent,  $p$  &  $q$  Laplacian operator, Variational methods

**MSC:** Primary 35J60; Secondary 35J60, 35J10, 35J20.

## 1 Introduction

In this paper we are concerned with a class of problems, named  $p$  &  $q$  problems type. In the last years the main interest in this general class of problems has been due to the fact that they arise from applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction, as it can be seen for example in [20], [23] and [35]. In addition, such a class of problems encompasses a large class of problems, as can be seen in [4], [15] and [17].

More precisely, we show existence and concentration results of positive solutions for the critical problem given by

$$\begin{cases} -\operatorname{div} (a(\epsilon^p |\nabla u|^p) \epsilon^p |\nabla u|^{p-2} \nabla u) + V(z)b(|u|^p) |u|^{p-2} u = f(u) + |u|^{q^*-2} u \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \quad (P_\epsilon)$$

where  $\epsilon > 0$ ,  $N \geq 2$ ,  $1 < p \leq q < N$  and  $q^* = Nq/(N - q)$ . The hypotheses on the function  $a$  are the following:  $(a_1)$  the function  $a$  is of class  $C^1$  and there exist constants  $k_1, k_2 \geq 0$  such that

$$k_1 t^p + t^q \leq a(t^p) t^p \leq k_2 t^p + t^q, \quad \text{for all } t > 0;$$

$(a_2)$  the mapping  $t \mapsto \frac{a(t^p)}{t^{q-p}}$  is nonincreasing for  $t > 0$ ;

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(a<sub>3</sub>) if  $1 < p < 2 \leq N$  the mapping  $t \mapsto a(t)$  is nondecreasing for  $t > 0$ . If  $2 \leq p < N$  the mapping  $t \mapsto a(t^p)t^{p-2}$  is nondecreasing for  $t > 0$ .

As a direct consequence of (a<sub>2</sub>) we obtain that the map  $a$  and its derivative  $a'$  satisfy

$$a'(t)t \leq \frac{(q-p)}{p}a(t) \text{ for all } t > 0. \quad (1.1)$$

Now if we define the function  $h(t) = a(t)t - \frac{q}{p}A(t)$ , using (1.1) we can prove that the function  $h$  is nonincreasing. Then, there exists a positive real constant  $\gamma \geq \frac{q}{p}$  such that

$$\frac{1}{\gamma}a(t)t \leq A(t), \quad \text{for all } t \geq 0. \quad (1.2)$$

The hypotheses on the function  $b$  are the following:

(b<sub>1</sub>) The function  $b$  is of class  $C^1$  and there exist constants  $k_3, k_4 \geq 0$  such that

$$k_3t^p + t^q \leq b(t^p)t^p \leq k_4t^p + t^q, \quad \text{for all } t > 0;$$

(b<sub>2</sub>) the mapping  $t \mapsto \frac{b(t^p)}{t^{q-p}}$  is nonincreasing for  $t > 0$ .

(b<sub>3</sub>) if  $1 < p < 2 \leq N$  the mapping  $t \mapsto b(t)$  is nondecreasing for  $t > 0$ . If  $2 \leq p < N$  the mapping  $t \mapsto b(t^p)t^{p-2}$  is nondecreasing for  $t > 0$ .

Using the hypothesis (b<sub>2</sub>) and arguing as (1.1) and (1.2) we prove that there exists  $\gamma \geq \frac{q}{p}$  such that

$$\frac{1}{\gamma}b(t)t \leq B(t), \quad \text{for all } t \geq 0. \quad (1.3)$$

The nonlinearity  $f$  is assumed to be a  $C^1$  function with the following hypotheses:

(f<sub>1</sub>)

$$\lim_{|s| \rightarrow 0} \frac{f(s)}{|s|^{q-1}} = 0.$$

(f<sub>2</sub>) There exists  $q < r < q^* = \frac{qN}{N-q}$  such that

$$\lim_{|s| \rightarrow \infty} \frac{f(s)}{|s|^{r-1}} = 0.$$

(f<sub>3</sub>) There exists  $\theta \in (\gamma p, q^*)$  such that

$$0 < \theta F(s) \leq f(s)s \quad \text{for } s > 0,$$

where  $F(s) = \int_0^s f(t)dt$  and  $\gamma > 0$  was given in (1.2);

(f<sub>4</sub>)  $s \mapsto \frac{f(s)}{s^{q-1}}$  is nondecreasing for  $s > 0$ .

(f<sub>5</sub>) There exist  $\tau \in (q, q^*)$  and  $\lambda > 1$

$$f(s) \geq \lambda s^{\tau-1} \quad \forall s > 0.$$

We need to put some hypotheses on the potential  $V \in C(\mathbb{R}^N)$ .

(V<sub>1</sub>) There is  $V_0 > 0$ , such that

$$0 < V_0 \leq V(z) \text{ for all } z \in \mathbb{R}^N.$$

(V<sub>2</sub>) There exists a bounded domain  $\Omega \subset \mathbb{R}^N$  such that

$$0 < V_0 = \inf_{z \in \Omega} V(z) < \inf_{z \in \partial\Omega} V(z).$$

In order to illustrate the degree of generality of the kind of problems studied here, with adequate hypotheses on the functions  $a$  and  $b$ , in the following we present more some examples of problems which are also interesting from the mathematical point of view and have a wide range of applications in physics and related sciences.

**Problem 1:** Let  $a(t) = 1 + t^{\frac{q-p}{p}}$  and  $b(t) = 1 + t^{\frac{q-p}{p}}$ . In this case we are studying problem

$$-\Delta_p u - \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N.$$

The Problem 1 comes from a general reaction–diffusion system:  $u_t = \operatorname{div}(Du \nabla u) + g(x, u)$ , where  $Du := [|\nabla u|^{p-2} + |\nabla u|^{q-2}]$ . In such applications, the function  $u$  describes a concentration, the term  $\operatorname{div}(Du \nabla u)$  corresponds to the diffusion with a diffusion coefficient  $Du$  and  $g(\cdot, u)$  is the reaction and relates to source and loss processes. Usually, in chemical and biological applications, the reaction term  $g(\cdot, u)$  is a polynomial of  $u$  with variable coefficients.

**Problem 2:** Let  $a(t) = t^{\frac{q-p}{p}}$  and  $b(t) = t^{\frac{q-p}{p}}$ . In this case we are studying problem

$$-\epsilon^q \Delta_q u + V(x)|u|^{q-2}u = f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N$$

and it is related to the main result showed in [3] in the case  $q = 2$ . In [19] the author have studied the case  $1 < q < N$ .

**Problem 3:** Let  $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$  and  $b(t) = 1$ . In this case we are studying problem

$$-\epsilon^p \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \operatorname{div} \left( \frac{\epsilon^p |\nabla u|^{p-2} \nabla u}{(1 + \epsilon^p |\nabla u|^p)^{\frac{p-2}{p}}} \right) + V(x)|u|^{p-2}u = f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N.$$

**Problem 4:** Let  $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$  and  $b(t) = 1 + t^{\frac{q-p}{p}}$ . In this case we are studying problem

$$-\epsilon^p \Delta_p u - \epsilon^q \Delta_q u - \operatorname{div} \left( \frac{\epsilon^p |\nabla u|^{p-2} \nabla u}{(1 + \epsilon^p |\nabla u|^p)^{\frac{p-2}{p}}} \right) + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N.$$

The main result is the following:

**Theorem 1.1.** Suppose that  $a, b, f$  and  $V$  satisfy  $(a_1) - (a_3)$ ,  $(b_1) - (b_3)$ ,  $(f_1) - (f_5)$  and  $(V_1) - (V_2)$  respectively. Then there are  $\epsilon_0 > 0$  and  $\lambda^* > 1$  such that  $(P_\epsilon)$  has a positive solution  $w_\epsilon \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , for every  $\epsilon \in (0, \epsilon_0)$  and for every  $\lambda > \lambda^*$ . In addition, if  $P_\epsilon$  is the maximum point of  $w_\epsilon$ , then

$$\lim_{\epsilon \rightarrow 0} V(P_\epsilon) = V_0.$$

Moreover, there are positive constants  $C$  and  $\alpha$  such that

$$|w_\epsilon(z)| \leq C \exp \left( -\alpha \left| \frac{z - P_\epsilon}{\epsilon} \right| \right),$$

for all  $\epsilon \in (0, \epsilon_0)$  and for all  $z \in \mathbb{R}^N$ .

In a seminal paper [31], Rabinowitz used his famous Mountain Pass Theorem (joint with Ambrosetti) [5] and showed the existence of solution for a Nonlinear Schrödinger Equation given by

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (R)$$

where  $V$  is a continuous potential satisfying  $(V_1)$  and

$(R_1) \liminf_{|x| \rightarrow \infty} V(x) = V_\infty$ , where  $V_\infty < \infty$  or  $V_\infty = \infty$ .

In [31], Rabinowitz used the force of the parameter  $\epsilon$  and the geometry of the potential  $V$  in order to overcome the lack of compactness of Sobolev's embedding to obtain the positive solution. In [33], Wang showed that the solution found by Rabinowitz concentrates around a local minimum of the potential  $V$ , when  $\epsilon$  converges to zero. Wang also noted that the concentration of any family of solutions with energy uniformly bounded can only occur in a critical point of  $V$ . In [12], Del Pino and Felmer weakened the hypothesis  $(R_1)$  of Rabinowitz and created a method that is known as Del Pino and Felmer's penalization method.

As can be seen in [4], [15] and [17],  $p \& q$  problems are generalizations of  $(R)$ . However, as can be seen below, we show that the arguments found in [12], [31] and [33] cannot be used directly. But before that, we are going to report some results on  $p \& q$  problems type. There are interesting papers on such class of problems. We start with some problems in a bounded domain. For example, in [15] the author shows the existence and multiplicity of solutions for a critical  $p \& q$  problem considering nonlinearity of type concave and convex. The critical case with discontinuous nonlinearities has studied in [16].

Now we comment some results in  $\mathbb{R}^N$ . Existence results was studied in [11] and [17]. In [2] the authors studied concentration results in Orlicz-Sobolev spaces with subcritical nonlinearity and the potential satisfying the local condition introduced by Del pino and Felmer [12]. In [4], it was showed the existence and concentration results with subcritical nonlinearity and the potential satisfying the global condition introduced by Rabinowitz [31] (see also [33]).

The present work is strongly influenced by the articles above. Below we list what we believe that are the main contributions of our paper.

- (1) Unlike [4], [11] and [17], we show existence and concentration results considering the local hypothesis on potential introduced by Del Pino and Felmer [12].
- (2) Unlike [2], we are considering the critical nonlinearity.
- (3) Since the operator is not homogeneous, some estimates are different and more delicate than some estimates that can be found in [12] and [31]. For example, see Lemma 3.4, Proposition 5.1, Lemma 5.7 and all the Lemmas of Section 7.
- (4) In order to overcome the lack of compactness provoked by the critical growth, it is very common to use the Talenti's function (see [32]) to have some control on the minimax level, as can be seen in [10, Lemma 1.1]. The lack of homogeneity of the  $p \& q$  operator does not allow to use this argument. We overcome this difficulty using the solution of a problem in a bounded domain, as can be seen in Lemma 3.5.

The interest in the study of nonlinear partial differential equations with  $p \& q$  operator or fractional  $p \& q$  operator has increased because many applications arising in mathematical physics may be stated with an operator in this form. We cite the papers [6], [7], [8], [9], [18], [21], [22], [26], [27], [28], [29], [30] and their references. Several techniques have been developed or applied in their study, such as variational methods, fixed point theory, lower and upper solutions, global branching, and the theory of multivalued mappings.

This paper is organized as follows. In Section 2, we define an auxiliary problem using the penalization argument introduced by Del Pino and Felmer [12]. The existence of solution for the auxiliary problem was showed in Section 3. In order to show the concentration result, in Section 4 we studied the autonomous problem. The concentration result was showed in Section 5. In Section 6 we showed that the solutions of the auxiliary problem are solutions of the original problem. In Section 7 we showed the exponential decay of these solutions. To conclude the paper, we showed in an appendix the existence of a solution to a problem in a bounded domain that was important to overcome the lack of compactness.

## 2 Variational framework and an auxiliary problem

To prove Theorem 1.1, we will work with the problem below, which is equivalent to  $(P_\epsilon)$  by change variable  $z = \epsilon x$ , which is given by

$$\begin{cases} -\operatorname{div}(\epsilon a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V(\epsilon x)b(|u|^p)|u|^{p-2}u = f(u) + |u|^{q^*-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \quad (\tilde{P}_\epsilon)$$

where  $\epsilon > 0$ ,  $N \geq 2$  and  $1 < p \leq q < N$ .

In order to obtain solutions of  $(\tilde{P}_\epsilon)$ , consider the following subspace of  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  given by

$$W_\epsilon := \left\{ v \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)b(|v|^p)|v|^p dx < +\infty \right\},$$

which is a Banach space when endowed with the norm

$$\|u\| = \|u\|_{1,p} + \|u\|_{1,q},$$

where

$$\|u\|_{1,m} = \left( \int_{\mathbb{R}^N} |\nabla u|^m dx + \int_{\mathbb{R}^N} V(\epsilon x)|u|^m dx \right)^{\frac{1}{m}}, \text{ for } m \geq 1.$$

Since the approach is variational, consider the energy functional associated  $J_\epsilon : W_\epsilon \rightarrow \mathbb{R}$  given by

$$J_\epsilon(v) = \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla v|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon x)B(|v|^p) dx - \int_{\mathbb{R}^N} F(v) dx - \frac{1}{q^*} \int_{\mathbb{R}^N} v_+^{q^*} dx,$$

where  $u_+ = \max\{u, 0\}$ . By standard arguments, one can prove that  $J_\epsilon \in C^1(W_\epsilon, \mathbb{R})$ . As we are interested in nonnegative solutions we can assume that  $f(s) = 0$  for  $s \leq 0$ .

Let  $\beta$  be a positive number satisfying  $\beta > \max \left\{ \frac{p\gamma\theta}{q(\theta - p\gamma)}, \frac{V_0 p\gamma}{q}, 1 \right\}$ , where  $\theta$  was given in  $(f_3)$  and  $V_0$  appeared in  $(V_1)$ . From  $(f_4)$ , there exists  $\eta > 0$  such that  $\frac{f(\eta) + \eta^{q^*-1}}{\eta^{q-1}} = \frac{V_0}{\beta}$ . Then, using the above numbers, we define the function of  $C^1$  class given by

$$\tilde{f}(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ f(s) + s^{q^*-1} & \text{if } 0 < s \leq \frac{\eta}{2}, \\ \frac{V_0}{\beta} |s|^{q-2}s & \text{if } s > \eta. \end{cases}$$

We now define the function

$$g(z, s) := \chi_\Omega(z)[f(s) + (s^+)^{q^*-1}] + (1 - \chi_\Omega(z))\tilde{f}(s),$$

and the auxiliary problem

$$\begin{cases} -\operatorname{div}(\epsilon a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V(\epsilon x)b(|u|^p)|u|^{p-2}u = g(\epsilon x, u) & \text{in } \mathbb{R}^N, \\ u \in W_\epsilon, \end{cases} \quad (P_{\epsilon_{aux}})$$

where  $\chi_\Omega$  is the characteristic function of the set  $\Omega$ . It is easy to check that  $(f_1) - (f_4)$  imply that  $g$  is a Carathéodory function and for  $x \in \mathbb{R}^N$ , the function  $s \rightarrow g(\epsilon x, s)$  is of class  $C^1$  and satisfies the following conditions, uniformly for  $x \in \mathbb{R}^N$ :

$$\lim_{|s| \rightarrow 0} \frac{g(\epsilon x, s)}{|s|^{q-1}} = 0 \quad (g_1)$$

$$g(\epsilon x, s) \leq f(s) + s^{q^*-1}, \quad \forall s > 0 \text{ and } x \in \mathbb{R}^N \quad (g_2)$$

$$0 < \theta G(\epsilon x, s) \leq g(\epsilon x, s)s, \quad \forall \epsilon x \in \Omega \text{ and } \forall s > 0 \quad (g_3)_i$$

and

$$0 < qG(\epsilon x, s) \leq g(\epsilon x, s)s \leq \frac{1}{\beta} V(\epsilon x)|s|^q, \quad \forall \epsilon x \notin \Omega \text{ and } \forall s > 0, \quad (g_3)_{ii}$$

where  $G(\epsilon x, s) = \int_0^s g(\epsilon x, t) dt$ .

The function

$$s \rightarrow \frac{g(\epsilon x, s)}{|s|^{q-1}} \text{ is nondecreasing.} \quad (g_4)$$

**Remark 1.** Note that, for  $z = \epsilon x$ , if  $u_\epsilon$  is a positive solution of  $(P_{\epsilon_{aux}})$  with  $|u_\epsilon(z)| \leq \frac{\eta}{2}$  for every  $\epsilon x \in \mathbb{R}^N \setminus \Omega$ , then  $u_\epsilon(x)$  is also a positive solution of  $(P_\epsilon)$ .

### 3 Existence of ground state for problem $(P_{\epsilon_{aux}})$

Hereafter, let us denote by  $I_\epsilon : W_\epsilon \rightarrow \mathbb{R}$  the functional given by

$$I_\epsilon(v) = \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla v|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon x) B(|v|^p) dx - \int_{\mathbb{R}^N} G(\epsilon x, v) dx.$$

We denote by  $\mathcal{N}_\epsilon$  the Nehari manifold of  $I_\epsilon$ , that is,

$$\mathcal{N}_\epsilon := \{u \in W_\epsilon \setminus \{0\} : \langle I'_\epsilon(u), u \rangle = 0\}$$

and define the number  $b_\epsilon$  by setting

$$b_\epsilon := \inf_{u \in \mathcal{N}_\epsilon} I_\epsilon(u). \quad (3.1)$$

Using  $(f_1)$ ,  $(f_2)$  and  $(g_2)$  we have: for every  $\xi > 0$  there exists  $C_\xi$  such that

$$|g(\epsilon x, s)| \leq \xi |s|^{q-1} + C_\xi |s|^{r-1} + |s|^{q^*-1} \text{ for all } x \in \mathbb{R}^N, s \in \mathbb{R}. \quad (3.2)$$

Then, by definition of  $g$  and (3.2), there is  $r_\epsilon > 0$  such that

$$\|u\| \geq r_\epsilon > 0 \text{ for all } u \in \mathcal{N}_\epsilon. \quad (3.3)$$

The main result in this section is:

**Theorem 3.1.** Let  $a$  satisfying  $(a_1)$ – $(a_3)$ ,  $b$  satisfying  $(b_1)$ – $(b_3)$ ,  $f$  satisfying  $(f_1)$ – $(f_5)$  and  $V$  such that  $(V_1)$ – $(V_2)$  hold. Then, there is  $\lambda^* > 1$  such that  $(P_{\epsilon_{aux}})$  has positive solution  $u_\epsilon \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , for every  $\lambda > \lambda^*$ .

Moreover, we would like to highlight that in section 5, more precisely in Lemma 5.5, we are going to show that if  $\frac{P_\epsilon}{\epsilon}$  is the maximum point of  $u_\epsilon$  then

$$\lim_{\epsilon \rightarrow 0} V(P_\epsilon) = V_0.$$

In order to use the Mountain Pass Theorem [5], we define the Palais-Smale compactness condition. We say that a sequence  $(u_n) \subset W_\epsilon$  is a Palais-Smale sequence at level  $c$  for the functional  $I_\epsilon$  if

$$I_\epsilon(u_n) \rightarrow c \quad \text{and} \quad \|I'_\epsilon(u_n)\| \rightarrow 0 \text{ in } (W_\epsilon)',$$

where

$$c := \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I_\epsilon(\eta(t)) > 0 \quad \text{and} \quad \Gamma := \{\eta \in C([0,1], X) : \eta(0) = 0, I_\epsilon(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of  $I_\epsilon$  has a strong convergent subsequence, then one says that  $I_\epsilon$  satisfies the Palais-Smale condition ((PS) for short).

**Lemma 3.2.** *The functional  $I_\epsilon : W_\epsilon \rightarrow \mathbb{R}$  satisfies the following conditions*

(i) *There are  $\alpha, \rho > 0$  such that*

$$I_\epsilon(u) \geq \alpha, \quad \text{if } \|u\| = \rho.$$

(ii) *For any  $u \in C_0^\infty(\Omega_\epsilon, [0, \infty))$ , we have*

$$\lim_{t \rightarrow \infty} I_\epsilon(tu) = -\infty.$$

*Proof.* Using  $(a_1)$ ,  $(b_1)$  and (3.2) we obtain

$$I_\epsilon(u) \geq \frac{\min\{k_1, k_3\}}{p} \|u\|_{1,p}^p + \frac{1}{q} \|u\|_{1,q}^q - \frac{\xi}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{C_\xi}{r} \int_{\mathbb{R}^N} |u|^r dx - \frac{1}{q^*} \int_{\mathbb{R}^N} |u|^{q^*} dx.$$

By Sobolev embeddings, choosing  $\xi > 0$  appropriate and taking  $\|u\| < 1$  there are positive constants  $C_1, C_2, C_3$ , such that

$$I_\epsilon(u) \geq C_1 \left[ \|u\|_{1,p}^p + \|u\|_{1,q}^q \right] - C_2 \|u\|^r - C_3 \|u\|^{q^*} \geq C_4 \|u\|^q - C_2 \|u\|^r - C_3 \|u\|^{q^*}.$$

Then the item (i) follows.

Now we show that the item (ii) holds. Consider a positive function  $w \in C_0^\infty(\Omega_\epsilon)$ ,  $t > 0$  and using  $(a_1)$ ,  $(b_1)$ ,  $(f_3)$  and Sobolev embedding, we have

$$I_\epsilon(tw) \leq \frac{t^p}{p} \max\{k_2, k_4\} \|w\|_{1,p}^p + \frac{t^q}{q} \|w\|_{1,q}^q - \frac{t^{q^*}}{q^*} \int_{\Omega_\epsilon} |w|^{q^*} dx.$$

This proves the second item. □

Hence, there exists a Palais-Smale sequence  $(u_n) \subset W_\epsilon$  at level  $c_\epsilon$ . Using  $(a_2)$ ,  $(b_2)$  and  $(f_4)$ , it is possible to prove that

$$c_\epsilon = b_\epsilon = \inf_{u \in W_\epsilon \setminus \{0\}} \sup_{t \geq 0} I_\epsilon(tu),$$

where  $b_\epsilon$  was defined in (3.1).

In order to prove the Palais-Smale condition, we need to prove the next lemma.

**Lemma 3.3.** *Let  $(u_n)$  be a  $(PS)_d$  sequence for  $I_\epsilon$ , then the sequence  $(u_n)$  is bounded  $W_\epsilon$ . Moreover, for each  $\xi > 0$  there exists  $R = R(\xi) > 0$  such that*

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R(0)} [a(|\nabla u_n|^p) |\nabla u_n|^p + V(\epsilon x) b(|u_n|^p) |u_n|^p] dx < \xi.$$

*Proof.* Since  $(u_n)$  is a  $(PS)_d$  sequence for functional  $I_d$ , then using (1.1), (1.3),  $(g_3)_i$  and  $(g_3)_{ii}$  we have that

$$\begin{aligned} o_n(1) + d + o_n(1) \|u_n\| &= I_\epsilon(u_n) - \frac{1}{\theta} I'_\epsilon(u_n) u_n \\ &\geq \left( \frac{1}{p\gamma} - \frac{1}{\theta} \right) \int_{\mathbb{R}^N} [a(|\nabla u_n|^p) |\nabla u_n|^p + [1 + \mu V(x)] b(|u_n|^p) |u_n|^p] dx \\ &\quad - \frac{1}{\beta} \int_{\mathbb{R}^N} [|\nabla u|^q + V(\epsilon x) |u|^q] dx \\ &\geq \left( \frac{1}{p\gamma} - \frac{1}{\theta} \right) \left( \min\{k_1, k_3\} \|u_n\|_{1,p}^p + \left( 1 - \frac{1}{\beta} \right) \|u_n\|_{1,q}^q \right). \end{aligned}$$

Then, arguing as the [4, Lemma 2.3], we can conclude that  $(u_n)$  is bounded in  $W_\epsilon$ .

Let  $\eta_R \in C^\infty(\mathbb{R}^N)$  be such that  $\eta_R(x) = 0$  if  $x \in B_{R/2}(0)$  and  $\eta_R(x) = 1$  if  $x \notin B_R(0)$ , with  $0 \leq \eta_R(x) \leq 1$  and  $|\nabla \eta_R| \leq \frac{C}{R}$ , where  $C$  is a constant independent of  $R$ . Since the sequence  $(\eta_R u_n)$  is bounded in  $W_\epsilon$ , and fixing  $R > 0$  such that  $\Omega_\epsilon \subset B_{R/2}(0)$  we obtain, by definition of the functional  $I_\epsilon$ ,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} \left[ a(|\nabla u_n|^p) |\nabla u_n|^p + V(\epsilon x) b(|u_n|^p) |u_n|^p \right] dx &= I_\epsilon(u_n) u_n \eta_R + \int_{\mathbb{R}^N} g(\epsilon x, u_n) u_n \eta_R dx \\ &\quad - \int_{\mathbb{R}^N} u_n a(|\nabla u_n|^p) |\nabla u_n|^{p-2} \nabla u_n \nabla \eta_R dx + o_n(1). \end{aligned}$$

Using  $(g_3)_{ii}$  we estimate

$$\begin{aligned} &\left(1 - \frac{1}{\beta}\right) \int_{\mathbb{R}^N \setminus B_R(0)} \left[ a(|\nabla u_n|^p) |\nabla u_n|^p + V(\epsilon x) b(|u_n|^p) |u_n|^p \right] dx \\ &\leq \int_{\mathbb{R}^N} |u_n| a(|\nabla u_n|^p) |\nabla u_n|^{p-1} |\nabla \eta_R| dx + o_n(1). \end{aligned}$$

As  $(u_n)$  is bounded in  $W_\epsilon$  and  $|\nabla \eta_R| \leq \frac{C}{R}$ . Passing to the limit in the last estimate, we get

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R} [a(|\nabla u_n|^p) |\nabla u_n|^p + V(\epsilon x) b(|u_n|^p) |u_n|^p] dx < \xi.$$

for some  $R$  sufficiently large and for some fixed  $\xi > 0$ .  $\square$

In the next result we show that the functional  $I_\epsilon$  satisfies the Palais-Smale condition for some levels. For this work we are denoting by  $S$  the best Sobolev constant for the embedding of  $D^{1,q}(\mathbb{R}^N)$  into  $L^{q^*}(\mathbb{R}^N)$ , that is, the largest positive constant  $S$  such that

$$S \left( \int_{\mathbb{R}^N} |u|^{q^*} dx \right)^{\frac{q}{q^*}} \leq \int_{\mathbb{R}^N} |\nabla u|^q dx \quad \text{for every } u \in D^{1,q}(\mathbb{R}^N). \quad (3.4)$$

**Lemma 3.4.** *The functional  $I_\epsilon$  satisfies the Palais-Smale condition at any level*

$$d < \left( \frac{1}{\theta} - \frac{1}{q^*} \right) S^{N/q}.$$

*Proof.* Let  $(u_n) \subset W_\epsilon$  be a Palais-Smale sequence at level  $d < \left( \frac{1}{\theta} - \frac{1}{q^*} \right) S^{N/q}$  for the functional  $I_\epsilon$ . Arguing as Lemma [4, Lemma 2.3] we have that  $(u_n)$  is bounded in  $W_\epsilon$ . Then by Sobolev embeddings we deduce, up to a subsequence, that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_\epsilon, \\ \nabla u_n(x) \rightarrow \nabla u(x) \text{ q.t.p in } \mathbb{R}^N, \\ u_n \rightarrow u \text{ strongly in } L^s_{loc}(\mathbb{R}^N) \text{ for any } p \leq s < q^*, \\ u_n(x) \rightarrow u(x) \text{ for a.e } x \in \mathbb{R}^N. \end{cases} \quad (3.5)$$

Using the same kind of ideas contained [4, Lemma 2.3], we may conclude that  $u$  is a critical point of  $I_\epsilon$ . From Lemma 3.3 and for each  $\xi > 0$  given there exists  $R > 0$  such that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R(0)} [a(|\nabla u_n|^p) |\nabla u_n|^p + V(\epsilon x) b(|u_n|^p) |u_n|^p] dx < \xi.$$



This inequality,  $(a_1)$ ,  $(b_1)$ ,  $(f_1)$ ,  $(f_2)$ ,  $(g_2)$  and the Sobolev embeddings imply, for  $n$  large enough, there exists a positive constant  $C_1$  such that

$$\left| \int_{\mathbb{R}^N \setminus B_R(0)} g(\epsilon x, u_n) u_n dx \right| \leq C_1 \left( \xi + \xi^{r/q} + \xi^{q^*/q} \right). \quad (3.6)$$

On the other hand, taking  $R$  large enough, we suppose that

$$\left| \int_{\mathbb{R}^N \setminus B_R(0)} g(\epsilon x, u) u dx \right| < \xi. \quad (3.7)$$

Therefore, by (3.6) and (3.7),

$$\int_{\mathbb{R}^N \setminus B_R(0)} g(\epsilon x, u_n) u_n dx = \int_{\mathbb{R}^N \setminus B_R(0)} g(\epsilon x, u) u dx + o_n(1). \quad (3.8)$$

We claim that

$$\int_{B_R(0) \cap (\mathbb{R}^N \setminus \Omega_\epsilon)} g(\epsilon x, u_n) u_n dx = \int_{B_R(0) \cap (\mathbb{R}^N \setminus \Omega_\epsilon)} g(\epsilon x, u) u dx + o_n(1). \quad (3.9)$$

Indeed, we have, in view of the definition of  $g$ ,

$$g(\epsilon x, u_n) u_n \leq f(u_n) u_n + \left( \frac{\eta}{2} \right)^{q^*} + \frac{V_0}{\beta} |u_n|^q \text{ for any } x \in \mathbb{R}^N \setminus \Omega_\epsilon.$$

Since the set  $B_R(0) \cap (\mathbb{R}^N \setminus \Omega_\epsilon)$  is bounded we can use the above estimate,  $(f_1)$ ,  $(f_2)$ , (3.5) and Lebesgue's Theorem to conclude that the convergence (3.9) holds.

Finally, we now prove the following convergence

$$\int_{\Omega_\epsilon} |u_n|^{q^*} dx = \int_{\Omega_\epsilon} |u|^{q^*} dx + o_n(1). \quad (3.10)$$

Since  $(u_n)$  is bounded in  $W_\epsilon$  and using the Lions's Concentration Compactness Principle [25], we may suppose that

$$|\nabla u_n|^q \rightharpoonup \mu \quad \text{and} \quad |u_n|^{q^*} \rightharpoonup \nu.$$

Then we obtain an at most countable index set  $\Gamma$ , sequences  $(x_i) \subset \mathbb{R}^N$  and  $(\mu_i)$ ,  $(\nu_i) \subset (0, \infty)$ , such that

$$\mu \geq |\nabla u|^q + \sum_{i \in \Gamma} \mu_i \delta_{x_i}, \quad \nu = |u|^{q^*} + \sum_{i \in \Gamma} \nu_i \delta_{x_i} \text{ and } S \nu_i^{q/q^*} \leq \mu_i, \quad (3.11)$$

for all  $i \in \Gamma$ , where  $\delta_{x_i}$  is the Dirac mass at  $x_i \in \mathbb{R}^N$ . Thus it is sufficient to show that  $\{x_i\}_{i \in \Gamma} \cap \Omega_\epsilon = \emptyset$ . Then, we suppose by contradiction that  $x_i \in \Omega_\epsilon$  for some  $i \in \Gamma$ . Consider  $R > 0$  and the function  $\psi_R := \psi(x_i - x)$ , where  $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$  is such that  $\psi \equiv 1$  in  $B_R(x_i)$ ,  $\psi \equiv 0$  in  $\mathbb{R}^N \setminus B_{2R}(x_i)$ ,  $|\nabla \psi|_\infty \leq 2$ , where  $R > 0$  will be chosen in such way that the support of  $\psi$  is contained in  $\Omega_\epsilon$ . Then, as  $(\psi_R u_n)$  is bounded and  $I'_\epsilon(u_n) \psi_R u_n = o_n(1)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^N} u_n a(|\nabla u_n|^p) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \psi_R dx + \int_{\mathbb{R}^N} \psi_R a(|\nabla u_n|^p) |\nabla u_n|^p dx \\ & + \int_{\mathbb{R}^N} \psi_R V(\epsilon x) b(|u_n|^p) |u_n|^p dx = \int_{\mathbb{R}^N} f(x, u_n) \psi_R u_n dx + \int_{\mathbb{R}^N} \psi_R |u_n|^{q^*} dx + o_n(1). \end{aligned}$$

Note that, using  $(a_1)$ ,  $(b_1)$  and that the function  $f$  has subcritical growth, we have

$$\lim_{R \rightarrow 0} \left[ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_n a(|\nabla u_n|^p) |\nabla u_n|^{p-2} \nabla u_n^p \cdot \nabla \psi_R dx \right] = 0,$$

$$\lim_{R \rightarrow 0} \left[ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\epsilon x) b(|u_n|^p) |u_n|^p \psi_R dx \right] = 0,$$

and

$$\lim_{R \rightarrow 0} \left[ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x, u_n) \psi_R u_n dx \right] = 0.$$

Therefore, by  $(a_1)$  again,

$$\int_{\mathbb{R}^N} \psi_R |\nabla u_n|^q dx \leq \int_{\mathbb{R}^N} |u_n|^{q^*} \psi_R dx + o_n(1).$$

Since  $\psi_R$  has compact support and letting  $n \rightarrow \infty$  in the above expression, we see that

$$\int_{\mathbb{R}^N} \psi_R d\mu \leq \int_{\mathbb{R}^N} \psi_R dv,$$

which implies

$$\mu_i \leq \nu_i.$$

From this inequality and (3.11) one easily sees that  $S^{N/q} \leq \nu_i$ . As  $\beta > \frac{p\gamma\theta}{q(\theta - p\gamma)}$  and  $S^{N/q} \leq \nu_i$  we have, by previous arguments,

$$\begin{aligned} d = I_\epsilon(u_n) - \frac{1}{\theta} I'_\epsilon(u_n) u_n + o_n(1) &\geq \left( \frac{\theta - p\gamma}{p\gamma\theta} - \frac{1}{q\beta} \right) \|u_n\|_{1,q}^q + \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \int_{\Omega_\epsilon} |u_n|^{q^*} dx + o_n(1) \\ &\geq \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \int_{\Omega_\epsilon} \psi_R |u_n|^{q^*} dx + o_n(1). \end{aligned}$$

Hence, taking the limit and using (3.11), we get

$$d \geq \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \sum_{i \in \Gamma} \psi_R(x_i) \nu_i = \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \nu_i \geq \left( \frac{1}{\theta} - \frac{1}{q^*} \right) S^{N/q}$$

which does not make sense. Thus we obtain the convergence (3.10).

Therefore

$$\int_{\mathbb{R}^N} g(\epsilon x, u_n) u_n dx = \int_{\mathbb{R}^N} g(\epsilon x, u) u dx + o_n(1). \quad (3.12)$$

Finally, we prove that, up to a subsequence,  $u_n \rightarrow u$  in  $W_\epsilon$ . Since  $I'_\epsilon(u_n) u_n = o_n(1)$ ,  $I'_\epsilon(u) = 0$ , (3.12) and Fatou's Lemma we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} [a(|\nabla u_n|^p) |\nabla u_n|^p - a(|\nabla u|^p) |\nabla u|^p] dx + \int_{\mathbb{R}^N} V(\epsilon x) [b(|u_n|^p) |u_n|^p - b(|u|^p) |u|^p] dx \\ &\quad + \int_{\mathbb{R}^N} [g(\epsilon x, u) u - g(\epsilon x, u_n) u_n] dx = o_n(1). \end{aligned}$$

Then, using  $(a_1)$  and  $(b_1)$ , we obtain  $\|u_n - u\| = o_n(1)$ , that is, the sequence  $(u_n)$  converges strongly to  $u$ .  $\square$

For each fixed  $\epsilon > 0$ , let us consider the following problem

$$\begin{cases} -k_2 \Delta_p u - \Delta_q u + \mathcal{V}(k_4 |u|^{p-2} u + |u|^{q-2} u) = |u|^{\tau-2} u & \text{in } \Omega_\epsilon, \\ u \in W_0^{1,q}(\Omega_\epsilon), \end{cases} \quad (P_\tau)$$

where  $\tau$  is the constant which appears in the hypothesis  $(f_5)$  and  $\mathcal{V} := \max_{x \in \overline{\Omega_\epsilon}} V(x)$  is a positive constant. We have associated to problem  $(P_\tau)$  the functional

$$I_\tau(u) = \frac{1}{p} \int_{\Omega_\epsilon} [k_2 |\nabla u|^p + \mathcal{V} k_4 |u|^p] dx + \frac{1}{q} \int_{\Omega_\epsilon} [|\nabla u|^q + \mathcal{V} |u|^q] dx - \frac{1}{\tau} \int_{\Omega_\epsilon} |u|^\tau dx$$

and the associated Nehari manifold

$$\mathcal{N}_\tau = \{u \in W_0^{1,q}(\Omega_\epsilon) : u \neq 0 \text{ and } I'_\tau(u)u = 0\}.$$

From Appendix there exists  $w_\tau \in W_0^{1,q}(\Omega_\epsilon)$  such that

$$I_\tau(w_\tau) = c_\tau := \inf_{u \in \mathcal{N}_\tau} I_\tau(u), \quad I'_\tau(w_\tau) = 0$$

and

$$c_\tau \geq \left( \frac{\tau - q}{\tau q} \right) \int_{\mathbb{R}^N} |w_\tau|^\tau dx. \quad (3.13)$$

Since  $\lambda$  is the parameter which appears in the hypothesis  $(f_5)$  we have the following result.

**Lemma 3.5.** *There exists  $\lambda^* > 1$ , such that if  $\lambda > \lambda^*$ , then  $c_\epsilon < \left( \frac{1}{\theta} - \frac{1}{q^*} \right) S^{N/q}$ .*

*Proof.* First of all, by the hypotheses  $(a_1)$ ,  $(b_1)$  and  $(f_5)$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} a(|\nabla w_\tau|^p) |\nabla w_\tau|^p dx + \int_{\mathbb{R}^N} V(\epsilon x) b(|w_\tau|^p) |w_\tau|^p dx &\leq \int_{\Omega_\epsilon} [k_2 |\nabla w_\tau|^p + \mathcal{V} k_4 |w_\tau|^p] dx \\ + \int_{\Omega_\epsilon} [|\nabla w_\tau|^q + \mathcal{V} |w_\tau|^q] dx &= \int_{\Omega_\epsilon} |w_\tau|^\tau dx \leq \int_{\Omega_\epsilon} f(w_\tau) w_\tau dx \leq \int_{\mathbb{R}^N} g(\epsilon x, w_\tau) w_\tau dx, \end{aligned}$$

where  $\mathcal{V} := \max_{x \in \overline{\Omega_\epsilon}} V(x)$ . This inequality implies that  $I'_\epsilon(w_\tau^\pm) w_\tau^\pm \leq 0$ , and then there exists  $t \in (0, 1)$  such that  $tw_\tau \in \mathcal{N}_\epsilon$ . Using  $(a_1)$ ,  $(b_1)$  and  $(f_5)$ , we obtain

$$\begin{aligned} c_\epsilon &\leq I_\epsilon(tw_\tau) \\ &\leq \frac{t^p}{p} \int_{\Omega_\epsilon} [k_2 |\nabla w_\tau|^p + \mathcal{V} k_4 |w_\tau|^p] dx + \frac{t^q}{q} \int_{\Omega_\epsilon} [|\nabla w_\tau|^q + \mathcal{V} |w_\tau|^q] dx - \frac{\lambda}{\tau} t^\tau \int_{\Omega_\epsilon} |w_\tau|^\tau dx. \end{aligned}$$

Since  $t \in (0, 1)$ ,  $p \leq q$  and  $I'_\epsilon(w_\tau)w_\tau = 0$ , we get

$$\begin{aligned} c_\epsilon &\leq I_\epsilon(tw_\tau) \\ &\leq \frac{t^p}{p} \int_{\Omega_\epsilon} [k_2 |\nabla w_\tau|^p + \mathcal{V} k_4 |w_\tau|^p] dx + \frac{t^p}{p} \int_{\Omega_\epsilon} [|\nabla w_\tau|^q + \mathcal{V} |w_\tau|^q] dx - \frac{\lambda}{\tau} t^\tau \int_{\Omega_\epsilon} |w_\tau|^\tau dx \\ &= \left[ \frac{t^p}{p} - \lambda \frac{t^\tau}{\tau} \right] \int_{\Omega_\epsilon} |w_\tau|^\tau dx \leq \max_{s \geq 0} \left[ \frac{s^p}{p} - \lambda \frac{s^\tau}{\tau} \right] \int_{\Omega_\epsilon} |w_\tau|^\tau dx. \end{aligned}$$

Using (3.13), we have

$$c_\epsilon \leq \max_{s \geq 0} \left[ \frac{s^p}{p} - \lambda \frac{s^\tau}{\tau} \right] \frac{c_\tau q \tau}{(\tau - q)} \leq \left[ \frac{\tau - p}{p \lambda^{p/(\tau-p)}} \right] \frac{c_\tau q}{(\tau - q)}$$

By some straightforward algebraic manipulations, we get

$$c_\epsilon \leq \left[ \frac{\tau - p}{p\lambda^{p/(\tau-p)}} \right] \frac{c_\tau q}{(\tau - q)}.$$

Then, if we choose  $\lambda > \lambda^* := \max \left\{ 1, \left[ \frac{(\tau-p)q}{(\tau-q)p} \frac{\theta q^*}{(q^*-\theta)} \frac{c_\tau}{S^{N/q}} \right]^{(\tau-p)/p} \right\}$  in the hypothesis  $(f_5)$ , the proof is complete.  $\square$

### 3.1 Proof of the Theorem 3.1

*Proof.* The proof is a consequence of Lemma 3.2, Lemma 3.4 and Lemma 3.5.  $\square$

## 4 The Autonomous Problem

In order to prove the concentration result, we consider the following problem

$$\begin{cases} -\operatorname{div} (a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V_0 b(|u|^p)|u|^{p-2}u = f(u) + |u|^{q^*-1} & \text{in } \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \end{cases} \quad (P_0)$$

which the functional associated  $I_0$  is given by

$$I_0(u) = \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)] dx - \int_{\mathbb{R}^N} F(u) dx - \frac{1}{q^*} \int_{\mathbb{R}^N} |u|^{q^*} dx,$$

and the corresponding Nehari manifold is given by

$$\mathcal{N}_0 = \{u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \setminus \{0\}; I'_0(u)u = 0\}.$$

We also define

$$c_0 = \inf_{\mathcal{N}_0} I_0.$$

Using the same arguments of the prove of Lemma 3.5, we conclude that

$$c_0 < \left( \frac{1}{\theta} - \frac{1}{q^*} \right) S^{N/q}. \quad (4.1)$$

The next result allows to show that problem  $(P_0)$  has a solution that reaches  $c_0$ .

**Lemma 4.1.** *Let  $(u_n) \subset \mathcal{N}_0$  be a sequence such that  $I_0(u_n) \rightarrow c_0$ . Then there are a sequence  $(y_n) \subset \mathbb{R}^N$  and constants  $R, \eta > 0$  such that*

$$\limsup_{n \rightarrow \infty} \int_{B_R(y_n)} |u_n|^q dx \geq \eta. \quad (4.2)$$

*Proof.* Suppose that (4.2) is not satisfied. Since  $(u_n)$  is bounded in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  we have, by in [24, Lemma 2.1],

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0 \text{ for all } s \in (q, q^*).$$

Hence, from  $(f_1) - (f_3)$ ,

$$\int_{\mathbb{R}^N} f(u_n) u_n dx = o_n(1).$$

Since we also have  $(g_3)$  and that  $I'_{\varepsilon_n}(u_n)u_n = o_n(1)$ , we get

$$\int_{\mathbb{R}^N} |u_n|^{q^*} dx = \int_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p dx + V_0 b(|u_n|^p)|u_n|^p] dx + o_n(1) := l$$

We claim that  $l > 0$ . Indeed, if the claim is not true then, by  $(a_1)$  and  $(b_1)$ , we have  $c_0 = 0$  which is a contradiction. Therefore

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q^*} dx = l > 0. \quad (4.3)$$

By definition of the constant  $S$ , we have

$$S \leq \frac{\int_{\mathbb{R}^N} |\nabla u_n|^q dx}{\left( \int_{\mathbb{R}^N} |u_n|^{q^*} dx \right)^{q/q^*}} \leq l^{q/N}. \quad (4.4)$$

Thus, using (1.2), (1.3) and  $(f_3)$ , we deduce that

$$c_0 + o_n(1) = I_0(u_n) - \frac{1}{\theta} I_0(u_n)u_n \geq \left( \frac{1}{\theta} - \frac{1}{q^*} \right) \int_{\mathbb{R}^N} |u_n|^{q^*} dx + o_n(1).$$

Using (4.3), (4.4) and that  $c_0 > 0$ , we obtain  $c_0 \geq \left( \frac{1}{\theta} - \frac{1}{q^*} \right) S^{N/q}$  which is a contradiction with (4.1).  $\square$

We are going to show that the problem  $(P_0)$  has a solution that reaches the level  $c_0$ .

**Lemma 4.2.** (A Compactness Lemma) Let  $(u_n) \subset \mathcal{N}_0$  be a sequence satisfying  $I_0(u_n) \rightarrow c_0$ . Then there exists a sequence  $(\tilde{y}_n) \subset \mathbb{R}^N$  such that, up to a subsequence,  $v_n(x) = u_n(x + \tilde{y}_n)$  converges strongly in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ . In particular, there exists a minimizer for  $c_0$ .

*Proof.* Applying Ekeland's Variational Principle (see Theorem 8.5 in [34]), we may suppose that  $(u_n)$  is a  $(PS)_{c_0}$  for  $I_0$ . Since  $(u_n)$  is bounded in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  we can assume, up to subsequences, that  $u_n \rightharpoonup u$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ .

Using arguments found in [4, Lemma 2.3], we have that

$$\nabla u_n(x) \rightarrow \nabla u(x) \text{ a.e. in } \mathbb{R}^N \text{ and } I'_0(u) = 0. \quad (4.5)$$

Then, by (1.2), (1.3) and the Fatou's Lemma,

$$\begin{aligned} 0 &\leq \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)] dx - \frac{1}{\theta} \int_{\mathbb{R}^N} [a(|\nabla u|^p)|\nabla u|^p + V_0 b(|u|^p)|u|^p] dx \\ &\leq \liminf_{n \rightarrow +\infty} \left\{ \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla u_n|^p) + V_0 B(|u_n|^p)] dx - \frac{1}{\theta} \int_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p + V_0 b(|u_n|^p)|u_n|^p] dx \right\} \end{aligned}$$

Hence, if  $u \in \mathcal{N}_0$ ,

$$c_0 \leq I_0(u) - \frac{1}{\theta} I'_0(u)u \leq \liminf_{n \rightarrow +\infty} \left[ I_0(u_n) - \frac{1}{\theta} I'_0(u_n)u_n \right] = \lim_{n \rightarrow +\infty} I_0(u_n) = c_0.$$

By (4.5),  $(a_1)$ ,  $(b_1)$  and Lebesgue's theorem we conclude that  $u_n \rightarrow u$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ . Consequently,  $I_0(u) = c_0$  and the sequence  $(\tilde{y}_n)$  is the null sequence.

If  $u \equiv 0$ , then in that case we cannot have  $u_n \rightarrow u$  strongly in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  because  $c_{V_0} > 0$ . Hence, using Lemma 4.1, there exists a sequence  $\{\tilde{y}_n\} \subset \mathbb{R}^N$  such that

$$v_n \rightharpoonup v \text{ in } W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),$$

where  $v_n := u_n(x + \tilde{y}_n)$ . Therefore,  $(v_n)$  is also a  $(PS)_{c_0}$  sequence for  $I_0$  and  $v \not\equiv 0$ . It follows from the above arguments that, up to a subsequence,  $(v_n)$  converges strongly in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  and the proof is complete.  $\square$

## 5 Concentration results

In this section we prove some technical results in order to show the concentration result.

**Proposition 5.1.** *Let  $\epsilon_n \rightarrow 0$  and  $(u_n) \subset \mathcal{N}_{\epsilon_n}$  be such that  $I_{\epsilon_n}(u_n) \rightarrow c_0$ . Then there exists a sequence  $(\tilde{y}_n) \subset \mathbb{R}^N$  such that  $v_n(x) := u_n(x + \tilde{y}_n)$  has a convergent subsequence in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ . Moreover, up to a subsequence,  $y_n \rightarrow y \in \Omega$ , where  $y_n = \epsilon_n \tilde{y}_n$ .*

*Proof.* Since  $V$  satisfies  $(V_1)$  and  $c_0 > 0$ , we repeat the same arguments in Lemma 4.1 to conclude that there exist positive constants  $R$  and  $\tilde{\beta}$  and a sequence  $(\tilde{y}_n) \subset \mathbb{R}^N$  such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(\tilde{y}_n)} |u_n|^q \geq \tilde{\beta} > 0.$$

Since the sequence  $(u_n)$  is bounded in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  we immediately obtain, up to a subsequence,  $v_n \rightharpoonup v \not\equiv 0$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , where  $v_n(x) := u_n(x + \tilde{y}_n)$ . Let  $t_n > 0$  be such that

$$\tilde{v}_n = t_n v_n \in \mathcal{N}_0. \quad (5.1)$$

Then, since  $u_n \in \mathcal{N}_{\epsilon_n}$ , we have

$$c_0 \leq I_0(\tilde{v}_n) \leq I_{\epsilon_n}(\tilde{v}_n) \leq I_{\epsilon_n}(v_n) = I_{\epsilon_n}(u_n) = c_0 + o_n(1), \quad (5.2)$$

which implies that  $I_0(\tilde{v}_n) \rightarrow c_0$ , as  $n \rightarrow +\infty$ .

From boundedness of  $(v_n)$  and (5.2), we obtain that  $(t_n)$  is bounded. As a consequence, the sequence  $(\tilde{v}_n)$  is also bounded in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  which implies, up to a subsequence,  $\tilde{v}_n \rightharpoonup \tilde{v}$  weakly in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ .

Note that we can assume that  $t_n \rightarrow t_0 > 0$ . Then, this limit implies that  $\tilde{v} \not\equiv 0$ . From Lemma 4.2, we conclude that  $\tilde{v}_n \rightarrow \tilde{v}$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  and this implies that  $v_n \rightarrow v$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ .

To conclude the proof of this proposition, we consider  $y_n := \epsilon_n \tilde{y}_n$ . Our goal is to show that  $(y_n)$  has a subsequence, still denoted by  $(y_n)$ , satisfying  $y_n \rightarrow y$  for  $y \in \Omega$ . First of all, we claim that  $(y_n)$  is bounded. Indeed, suppose that there exists a subsequence, still denote by  $(y_n)$ , verifying  $|y_n| \rightarrow \infty$ . From  $(a_1)$ ,  $(b_1)$  and  $(V_1)$  we have

$$\int_{\mathbb{R}^N} [k_1 |\nabla v_n|^p + |\nabla v_n|^q] dx + V_0 \int_{\mathbb{R}^N} [k_3 |v_n|^p + |v_n|^q] dx \leq \int_{\mathbb{R}^N} g(\epsilon_n x + y_n, v_n) v_n dx.$$

Fix  $R > 0$  such that  $B_R(0) \supset \Omega$  and let  $\chi_{B_R(0)}$  be the characteristic function of  $B_R(0)$ . Since  $\chi_{B_R(0)}(\epsilon x + y_n) = o_n(1)$  for all  $x \in B_R(0)$  and  $v_n \rightarrow v$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , then

$$\int_{\mathbb{R}^N} \chi_{B_R(0)}(\epsilon x + y_n) g(\epsilon x + y_n, v_n) v_n dx = o_n(1).$$

By definition of  $\tilde{f}$  we obtain that

$$\begin{aligned} \int_{\mathbb{R}^N} [k_1 |\nabla v_n|^p + |\nabla v_n|^q] dx + V_0 \int_{\mathbb{R}^N} [k_3 |v_n|^p + |v_n|^q] dx &\leq \int_{\mathbb{R}^N \setminus B_R(0)} \tilde{f}(v_n) v_n dx + o_n(1) \\ &\leq \frac{V_0}{\beta} \int_{\mathbb{R}^N} |v_n|^q dx + o_n(1). \end{aligned}$$

It follows that  $v_n \rightarrow 0$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , obtain this way a contradiction because  $c_0 > 0$ .

Hence  $(y_n)$  is bounded and, up to a subsequence,

$$y_n \rightarrow \bar{y} \in \mathbb{R}^N.$$

Arguing as above, if  $\bar{y} \notin \bar{\Omega}$  we will obtain again  $v_n \rightarrow 0$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , and then  $\bar{y} \in \bar{\Omega}$ . Now if  $V(\bar{y}) = V_0$ , we have  $\bar{y} \notin \partial\Omega$  and consequently  $\bar{y} \in \Omega$ . Suppose by contradiction that  $V(\bar{y}) > V_0$ . Then, we have

$$c_0 = I_0(\bar{v}) < \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \bar{v}|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\bar{y}) B(|\bar{v}|^p) dx - \int_{\mathbb{R}^N} F(\bar{v}) dx - \int_{\mathbb{R}^N} |\bar{v}|^{q^*} dx.$$

Using the fact that  $\tilde{v}_n \rightarrow \bar{v}$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , from Fatou's Lemma we obtain

$$c_0 < \liminf_{n \rightarrow \infty} \left[ \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}_n|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon_n z + y_n) B(|\tilde{v}_n|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n) dx - \int_{\mathbb{R}^N} |\tilde{v}_n|^{q^*} dx \right]$$

Since  $u_n \in \mathcal{N}_{\epsilon_n}$ , this implies that

$$c_0 < \liminf_{n \rightarrow \infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n \rightarrow \infty} I_{\epsilon_n}(u_n) = c_0,$$

obtaining a contradiction.  $\square$

**Lemma 5.2.** *Let  $(\epsilon_n)$  be a sequence such that  $\epsilon_n \rightarrow 0$  and  $(u_n) \subset \mathcal{N}_{\epsilon_n}$  a solution of problem  $(P_{\epsilon_{aux}})$ . Then  $(v_n)$  converges uniformly on compacts of  $\mathbb{R}^N$ , where  $v_n(x) := u_n(x + \tilde{y}_n)$ . Moreover, given  $\xi > 0$ , there exist  $R > 0$  and  $n_0 \in \mathbb{N}$  such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi \text{ for all } n \geq n_0,$$

where  $(\tilde{y}_n)$  is the sequence of Proposition 5.1.

*Proof.* Note that  $v_n$  is a solution of problem

$$\begin{cases} -\operatorname{div}(a(|\nabla v_n|^p) |\nabla v_n|^{p-2} \nabla v_n) + V(\epsilon x + y_n) b(|v_n|^p) |v_n|^{p-2} v_n = g(\epsilon x + y_n, v_n) & \text{in } \mathbb{R}^N, \\ v_n \in W_\epsilon, \end{cases}$$

where  $y_n = \epsilon_n \tilde{y}_n$ . Adapting some arguments explored in [4, Lemma 5.5], we have that the sequence  $(v_n)$  is bounded in  $L^\infty(\mathbb{R}^N)$  and there exist  $R > 0$  and  $n_0 \in \mathbb{N}$  such that

$$\|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi, \text{ for all } n \geq n_0.$$

Then, for any bounded domain  $\Omega' \subset \mathbb{R}^N$ , from  $(g_1) - (g_2)$  and continuity of  $V$  there exists  $C > 0$  such that

$$|V(\epsilon x + y_n) v_n^{p-1} - g(\epsilon x + y_n, v_n)| \leq C, \text{ for all } n \in \mathbb{N}.$$

Hence,

$$|V(\epsilon x + y_n) v_n^{p-1} - g(\epsilon x + y_n, v_n)| \leq C + |\nabla v_n|^p, \text{ for all } n \in \mathbb{N}.$$

Considering  $\Psi(x) = C$ , we get that  $\Psi \in L^t(\Omega')$  with  $t > \frac{p}{p-1}N$ . From [13, Theorem 1], we have

$$\|\nabla v_n\| \in L_{loc}^\infty(\mathbb{R}^N).$$

Therefore, for all compact  $K \subset \Omega'$  there exists a constant  $C_0 > 0$ , dependent only on  $C, N, p$  and  $\text{dist}(K, \partial\Omega')$ , such that

$$|\nabla v_n|_{\infty, K} \leq C_0.$$

Then,

$$|v_n|_{C_{loc}^{0,v}(\mathbb{R}^N)} \leq C, \text{ for all } n \in \mathbb{N} \text{ and } 0 < v < 1.$$

From Schauder's embedding,  $(v_n)$  has a subsequence convergent in  $C_{loc}^{0,v}(\mathbb{R}^N)$ .  $\square$

**Lemma 5.3.** *Given  $\epsilon > 0$ , the solution  $u_\epsilon$  of problem  $(P_{\epsilon_{aux}})$  satisfies*

$$\lim_{\epsilon \rightarrow 0} I_\epsilon(u_\epsilon) = c_{V_0}.$$

*Proof.* Consider  $z_0 \in \Omega$  such that  $V(z_0) = V_0$ . Let us now consider  $R > 0$  and set  $Q \in \partial B_R(z_0)$ . If necessary, take  $R$  small enough such that  $B(Q, R/4) \subset \Omega$ . Taking  $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\psi \equiv 1$  in  $B(Q, R/4)$  and  $\psi \equiv 0$  in  $\mathbb{R}^N \setminus B(Q, R/2)$ .

Let  $w_0 \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  be a ground-state positive solution of the problem  $(P_0)$  which satisfies  $c_0 = I_0(w_0)$  (see Lemma 4.2). Then, we consider the function  $w_\epsilon : \mathbb{R}^N \rightarrow \mathbb{R}$  be given by

$$w_\epsilon(x) := \psi_i(\epsilon x) w_0\left(x - \frac{z_0}{\epsilon}\right) \in W_\epsilon$$

and  $t_\epsilon > 0$ , such that  $t_\epsilon w_\epsilon \in \mathcal{N}_\epsilon$ . Then, with a direct computation, we have

$$I_\epsilon(u_\epsilon) \leq I_\epsilon(t_\epsilon w_\epsilon) = c_0 + o_\epsilon(1).$$

Finally, taking  $R \rightarrow 0$  in the last inequality and using the continuity of the minimax function (see [1], [31]) we get

$$\limsup_{\epsilon \rightarrow 0} I_\epsilon(u_\epsilon) \leq c_0.$$

Let  $t_{\epsilon,0} > 0$  be such that  $t_{\epsilon,0} u_\epsilon \in \mathcal{N}_0$ . Then,

$$c_0 \leq I_0(t_{\epsilon,0} u_\epsilon) \leq I_\epsilon(t_{\epsilon,0} u_\epsilon) \leq I_\epsilon(u_\epsilon)$$

and the proof is complete.  $\square$

**Lemma 5.4.** *Let  $(\epsilon_n)$  be a sequence such that  $\epsilon_n \rightarrow 0$  and for each  $n \in \mathbb{N}$ , let  $(u_n) \subset \mathcal{N}_{\epsilon_n}$  be a solution of problem  $(P_{\epsilon_{aux}})$ . Then, there are  $\delta^* > 0$  and  $n_0 \in \mathbb{N}$  such that, for  $v_n(x) = u_n(x + \tilde{y}_n)$ , we have*

$$v_n(x) \geq \delta^*, \text{ for all } x \in B_R(0) \text{ and } n \geq n_0,$$

where  $R > 0$  and  $(\tilde{y}_n)$  were given in Lemma 5.2.

*Proof.* Suppose, by contradiction, that  $\|u_n\|_{L^\infty(|x| < R)} = \|u_n\|_{L^\infty(|x - \tilde{y}_n| < R)} \rightarrow 0$ . By Lemma 5.2, we have  $\|v_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$ . It follows from  $(f_1)$  that

$$|f(v_n) + v_n^{q^*-1}| \leq \frac{V_0}{2} |v_n|^{q-1} \text{ for } n \text{ sufficient large.} \quad (5.3)$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} a(|\nabla v_n|^p) |\nabla v_n|^p dx &+ \int_{\mathbb{R}^N} V(\epsilon_n x + y_n) b(|v_n|^p) |v_n|^p dx \\ &= \int_{\mathbb{R}^N} f(v_n) v_n dx + o_n(1) \\ &\leq \frac{V_0}{2} \int_{\mathbb{R}^N} |v_n|^q dx + o_n(1), \end{aligned}$$



which implies from  $(a_1)$  and  $(b_1)$  that,

$$\|u_n\|_{W_{\epsilon_n}} \rightarrow 0,$$

which is a contradiction with Lemma 5.3.  $\square$

We are now ready to show the concentration of the ground state solution.

**Lemma 5.5.** *If  $\frac{P_\epsilon}{\epsilon}$  is the maximum point of  $u_\epsilon$ , then*

$$\lim_{\epsilon \rightarrow 0} V(P_\epsilon) = V_0.$$

*Proof.* We first notice that using Lemma 5.4 there exist  $\delta^* > 0$  and  $n_0 \in \mathbb{N}$  such that

$$v_n(q_n) := \max_{z \in \mathbb{R}^N} v_n(z) = u_n(q_n + \tilde{y}_n) \geq u_n(x) \geq \delta^*, \text{ for all } n \geq n_0, \text{ for all } x \in B_R(0).$$

We claim that  $(q_n)$  is bounded, otherwise using Lemma 5.2 and 5.4, there exists  $R^* > 0$  such that  $\|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_{R^*})} \leq \frac{\delta^*}{2}$ , which implies that  $|v_n(q_n)| \leq \frac{\delta^*}{2}$ , where we obtain a contradiction.

Then,  $P_{\epsilon_n} = \epsilon_n q_n + y_n$  which implies

$$\lim_{n \rightarrow +\infty} P_{\epsilon_n} = \lim_{n \rightarrow +\infty} y_n = \bar{y} \in \Omega.$$

Hence from continuity of  $V$  it follows that

$$\lim_{n \rightarrow +\infty} V(P_{\epsilon_n}) = V(\bar{y}) \geq V_0.$$

We claim that  $V(\bar{y}) = V_0$ . Indeed, suppose by contradiction that  $V(\bar{y}) > V_0$ . Then, we have

$$c_0 = I_0(\tilde{v}) < \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\bar{y}) B(|\tilde{v}|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}) - \frac{1}{q^*} \int_{\mathbb{R}^N} |\tilde{v}|^{q^*} dx.$$

Using that  $\tilde{v}_n \rightarrow \tilde{v}$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$  we obtain, from Fatou's Lemma,

$$c_0 < \liminf_{n \rightarrow \infty} \left[ \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}_n|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon_n z + y_n) B(|\tilde{v}_n|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n) - \frac{1}{q^*} \int_{\mathbb{R}^N} |\tilde{v}_n|^{q^*} dx \right],$$

and therefore

$$c_0 < \liminf_{n \rightarrow \infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n \rightarrow \infty} I_{\epsilon_n}(u_n) = c_0.$$

This contradiction shows that  $V(\bar{y}) = V_0$ .  $\square$

**Lemma 5.6.** *Let  $\{\epsilon_n\}$  be a sequence of positive numbers such that  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and let  $(x_n) \subset \overline{\Omega}_{\epsilon_n}$  be a sequence such that  $u_{\epsilon_n}(x_n) \geq Y > 0$  for some constant  $Y$ , where for each  $n \in \mathbb{N}$ ,  $u_{\epsilon_n}$  is a solution of  $(P_{\epsilon_{aux}})$ . Then,*

$$\lim_{n \rightarrow \infty} V(\bar{x}_n) = V_0$$

where  $\bar{x}_n = \epsilon_n x_n$ .

*Proof.* Up to a subsequence,

$$\bar{x}_n \rightarrow \bar{x} \in \overline{\Omega}.$$

From Lemma 5.3 we have that

$$I_{\epsilon_n}(u_{\epsilon_n}) \rightarrow c_0,$$

and there exists a positive constant  $C$  such that

$$\|u_{\epsilon_n}\| \leq C, \quad \forall n \in \mathbb{N}, \quad \text{for some } C > 0.$$

Setting  $v_n(z) := u_{\epsilon_n}(z + x_n)$ , we have  $\|v_n\| \leq C$  and  $v_n \rightharpoonup v$  in  $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ . Recalling that

$$v_n(0) = u_{\epsilon_n}(x_n) \geq Y > 0,$$

we conclude that  $v \not\equiv 0$ .

Fix  $t_n > 0$  verifying  $\tilde{v}_n = t_n v_n \in \mathcal{N}_0$ , for each  $n \in \mathbb{N}$ . Hence,

$$c_0 \leq I_0(\tilde{v}_n) \leq I_{\epsilon_n}(t_n v_n) \leq I_{\epsilon}(v_n) = I_{\epsilon}(u_n) = c_0 + o_n(1).$$

Thus,  $I_0(\tilde{v}_n) \rightarrow c_0$ , with  $\{\tilde{v}_n\} \subset \mathcal{N}_0$ . By Lemma 4.2, we have

$$\tilde{v}_n \rightarrow \tilde{v} \quad \text{in } W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \quad \text{and} \quad I_0(\tilde{v}) = c_0. \quad (5.4)$$

Since  $\tilde{v} \not\equiv 0$ , by Proposition 5.1 we have  $y_n = 0$ , for  $n \in \mathbb{N}$ . Moreover, recalling that  $V$  is continuous, we have

$$\lim_{n \rightarrow \infty} V(\bar{x}_n) = V(\bar{x}).$$

We claim that  $V(\bar{x}) = V_0$ . Indeed, Suppose by contradiction that  $V(\bar{x}) > V_0$ , then

$$c_0 = I_0(\tilde{v}) < \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\bar{x}) B(|\tilde{v}|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}) dx - \frac{1}{q^*} \int_{\mathbb{R}^N} |\tilde{v}|^{q^*} dx.$$

Thus, by (5.4) and Fatou's Lemma, we have

$$\begin{aligned} c_0 &< \liminf_{n \rightarrow \infty} \left[ \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}_n|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon_n z + \bar{x}_n) B(|\tilde{v}_n|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n) dx - \frac{1}{q^*} \int_{\mathbb{R}^N} |\tilde{v}|^{q^*} dx \right] \\ &\leq \liminf_{n \rightarrow \infty} \left[ \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla t_n v_n|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon_n z + \bar{x}_n) B(|t_n v_n|^p) dx - \int_{\mathbb{R}^N} F(t_n v_n) dx \right] \\ &= \liminf_{n \rightarrow \infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n \rightarrow \infty} I_{\epsilon_n}(u_n) = c_0, \end{aligned}$$

which leads a absurd. Consequently  $\lim_{n \rightarrow \infty} V(\bar{x}_n) = V_0$ . □

**Lemma 5.7.** *If  $m_\epsilon$  is given by  $m_\epsilon = \sup\{\max_{\partial\Omega_\epsilon} u_\epsilon : u_\epsilon \text{ is a solution of } (P_{\epsilon_{aux}})\}$ , then there exists  $\bar{\epsilon} > 0$  such that the sequence  $(m_\epsilon)$  is bounded for all  $\epsilon \in (0, \bar{\epsilon})$ . Moreover, we have  $\lim_{\epsilon \rightarrow 0} m_\epsilon = 0$ .*

*Proof.* Suppose, by contradiction,  $\lim_{\epsilon \rightarrow 0} m_\epsilon = +\infty$ , then there exist  $u_\epsilon$  a solution of  $(P_{\epsilon_{aux}})$  in  $\mathcal{N}_\epsilon$  and  $Y > 0$  such that

$$\max_{\partial\Omega_\epsilon} u_\epsilon \geq Y > 0$$

Thus there exists  $\{\epsilon_n\} \subset \mathbb{R}^+$  with  $\epsilon_n \rightarrow 0$  and there exists a sequence  $\{x_n\} \subset \partial\Omega_{\epsilon_n}$  such that

$$u_{\epsilon_n}(x_n) \geq Y > 0.$$

Thus, by Lemma 5.6, we have

$$\lim_{n \rightarrow \infty} V(\bar{x}_n) = V_0,$$

where  $\bar{x}_n = \epsilon_n x_n$  and  $\{\bar{x}_n\} \subset \partial\Omega$ . Hence, up to a subsequence, we have  $\bar{x}_n \rightarrow \bar{x}$  in  $\partial\Omega$  and  $V(\bar{x}) = V_0$ , which does not make sense by  $(V_2)$ . Hence, there exists  $\bar{\epsilon} > 0$  such that  $(m_\epsilon)$  is bounded, for all  $\epsilon \in (0, \bar{\epsilon})$ .

Suppose by contradiction that there exists  $\delta > 0$  and a sequence  $\{\epsilon_n\} \subset \mathbb{R}^+$  satisfying

$$m_{\epsilon_n} \geq \delta > 0$$

Thus, there exists  $u_{\epsilon_n}$  a solution of  $(P_{\epsilon_{aux}})$  such that

$$m_{\epsilon_n} - \frac{\delta}{2} < \max_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n} \leq m_{\epsilon_n}.$$

Hence,

$$\frac{\delta}{2} = \delta - \frac{\delta}{2} \leq m_{\epsilon_n} - \frac{\delta}{2} < \max_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n},$$

and then there exists a sequence  $(x_n) \subset \partial\Omega_{\epsilon_n}$ , such that

$$u_{\epsilon_n}(x_n) \geq \frac{\delta}{2}.$$

Repeating the above arguments, we will get an absurd. Thus, the proof is finished.  $\square$

## 6 Proof of Theorem 1.1

*Proof.* Let  $u_\epsilon$  be a solution of  $(P_{\epsilon_{aux}})$ . By Lemma 5.7, there exists  $\bar{\epsilon} > 0$  such that  $m_\epsilon < \frac{\eta}{2}$  for all  $\epsilon \in (0, \bar{\epsilon})$ , then  $(u_\epsilon - \frac{\eta}{2})_+(x) \equiv 0$  for a neighborhood from  $\partial\Omega_\epsilon$ . Hence,  $(u_\epsilon - \frac{\eta}{2})_+ \in W_0^{1,p}(\mathbb{R}^N \setminus \Omega_\epsilon) \cap W_0^{1,q}(\mathbb{R}^N \setminus \Omega_\epsilon)$  and the function  $(u_\epsilon - \frac{\eta}{2})_+^* \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ , where

$$(u_\epsilon - \frac{\eta}{2})_+^*(x) := \begin{cases} 0 & \text{if } x \in \Omega_\epsilon, \\ (u_\epsilon - \frac{\eta}{2})_+(x) & \text{if } x \in \mathbb{R}^N \setminus \Omega_\epsilon. \end{cases}$$

Using  $(u_\epsilon - \frac{\eta}{2})_+^*$  as test function. Then, by  $(a_1)$ ,  $(b_1)$  and  $(g_3)_{ii}$ , we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N \setminus \Omega_\epsilon} a(|\nabla u_\epsilon|^p) |\nabla (u_\epsilon - \frac{\eta}{2})_+^*|^p dx \\ &+ \int_{\mathbb{R}^N \setminus \Omega_\epsilon} \left[ V_0 b(|u_\epsilon|^p) |u_\epsilon|^{p-2} - \frac{V_0}{\beta} |u_\epsilon|^{q-2} \right] ((u_\epsilon - \frac{\eta}{2})_+^*)^2 dx \\ &+ \int_{\mathbb{R}^N \setminus \Omega_\epsilon} \left[ V(\epsilon x) b(|u_\epsilon|^p) |u_\epsilon|^{p-2} - \frac{V_0}{\beta} |u_\epsilon|^{q-2} \right] \frac{\eta}{2} (u_\epsilon - \frac{\eta}{2})_+^* dx = 0 \end{aligned}$$

The last equality implies

$$(u_\epsilon - \frac{\eta}{2})_+^* = 0, \text{ a.e in } x \in \mathbb{R}^N \setminus \Omega_\epsilon.$$

This implies that  $|u_\epsilon| \leq \frac{\eta}{2}$  for  $z \in \mathbb{R}^N \setminus \Omega_\epsilon$ , and by Remark 1 the result follows.  $\square$

## 7 Exponential decay of the solution $u_\epsilon$

Finally, we are going to prove the exponential decay. First technical results

**Lemma 7.1.** Consider  $M, \alpha > 0$  and  $\psi(x) := M \exp(-\alpha|x|)$ . Then

$$\begin{aligned} i) & -\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \\ & = \alpha^{p-1} \left[ -p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1} \left( \frac{(N-1)}{|x|} - \alpha(p-1) \right) \right], \\ ii) & -\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \geq \left( \frac{(N-1)}{|x|} - \alpha(q-1) \right) a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}. \end{aligned}$$

*Proof.* Note that

$$\frac{\partial\psi}{\partial x_i}(x) = M \exp(-\alpha|x|) \frac{\partial}{\partial x_i}(-\alpha|x|) = M \exp(-\alpha|x|)(-\alpha) \frac{x_i}{|x|} = -\alpha \frac{x_i}{|x|} \psi(x),$$

which implies  $|\nabla\psi| = \alpha\psi$ . Then

$$\begin{aligned} -\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) & = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left[ a(|\nabla\psi|^p)|\nabla\psi|^{p-2} \frac{\partial\psi}{\partial x_i} \right] \\ & = \alpha^{p-1} \sum_{i=1}^N \frac{\partial}{\partial x_i} \left[ a(\alpha^p\psi^p)\psi^{p-1} \frac{x_i}{|x|} \right] \\ & = \alpha^{p-1} \sum_{i=1}^N \left[ a'(\alpha^p\psi^p) \frac{\partial}{\partial x_i}(\alpha^p\psi^p) \psi^{p-1} \frac{x_i}{|x|} + a(\alpha^p\psi^p) \frac{\partial}{\partial x_i} \left( \psi^{p-1} \frac{x_i}{|x|} \right) \right] \\ & = \alpha^{p-1} \sum_{i=1}^N \left[ a'(\alpha^p\psi^p) \alpha^p p \psi^{2p-2} \frac{\partial\psi}{\partial x_i} \frac{x_i}{|x|} + a(\alpha^p\psi^p) \left( \frac{|x|^2 - x_i^2}{|x|^3} \psi^{p-1} + (p-1)\psi^{p-2} \frac{\partial\psi}{\partial x_i} \frac{x_i}{|x|} \right) \right] \\ & = \alpha^{p-1} \left[ -p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1} \left( \frac{(N-1)}{|x|} - \alpha(p-1) \right) \right], \end{aligned}$$

this prove the first item.

To prove the item ii) we are going to use (1.2) and the item i). Hence we have

$$-a'(\alpha^p\psi^p)\alpha^p\psi^p \geq -\frac{(q-p)}{p}a(\alpha^p\psi^p),$$

and consequently

$$-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} \geq -\alpha\psi^{p-1}(q-p)a(\alpha^p\psi^p).$$

Therefore, by the item i),

$$\begin{aligned} & -\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \\ & \geq \alpha^{p-1} \left[ -\alpha(q-p)a(\alpha^p\psi^p)\psi^{p-1} + \left( \frac{(N-1)}{|x|} - \alpha(p-1) \right) a(\alpha^p\psi^p)\psi^{p-1} \right] \\ & = \left( \frac{(N-1)}{|x|} - \alpha(q-1) \right) a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}. \end{aligned}$$

□

**Corollary 7.2.** Since  $V(x) \geq V_0$  in  $\mathbb{R}^N$ , then for  $\alpha > 0$  small enough we have

$$-\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) + k_3 V_0 \psi^{p-1} + \frac{V_0}{4} \psi^{q-1} \geq 0 \text{ in } \mathbb{R}^N.$$

*Proof.* Using  $(a_1)$  and Lemma 7.1 we obtain that

$$\begin{aligned} -\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) & \geq -\alpha(q-1)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1} \\ & \geq -\alpha(q-1) \left( k_2 \alpha^{p-1}\psi^{p-1} + \alpha^{q-1}\psi^{q-1} \right) \end{aligned}$$

$$= -\alpha(q-1)k_2\alpha^{p-1}\psi^{p-1} - \alpha(q-1)\alpha^{q-1}\psi^{q-1}$$

Moreover, since  $V_0 > 0$  and  $\alpha > 0$  is small enough, we concluded that

$$k_3 V_0 - \alpha(q-1)k_2\alpha^{p-1} \geq 0$$

and

$$\frac{V_0}{4} - \alpha(q-1)\alpha^{q-1} \geq 0.$$

Consequently

$$-\operatorname{div}(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) + k_3 V_0\psi^{p-1} + \frac{V_0}{4}\psi^{q-1} \geq 0 \text{ in } \mathbb{R}^N.$$

□

Let us now relate the positive solution  $v_\epsilon$  to the exponential function  $\psi$  for small  $\epsilon$ .

**Lemma 7.3.** *Let  $u_\epsilon$  be the solution found in Theorem 3.1 and  $v_\epsilon(x) := u_\epsilon(x + \tilde{y}_\epsilon)$  given in Proposition 5.1. For  $\varphi_\epsilon = \max\{v_\epsilon - \psi, 0\}$  and  $\epsilon > 0$  sufficient small, we have*

$$\int_{\mathbb{R}^N} a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \nabla \varphi_\epsilon \, dx + k_3 V_0 \int_{\mathbb{R}^N} |v_\epsilon|^{p-1} \varphi_\epsilon \, dx + \frac{V_0}{4} \int_{\mathbb{R}^N} |v_\epsilon|^{q-1} \varphi_\epsilon \, dx \leq 0.$$

*Proof.* From Lemma 5.2, Lemma 5.3 and hypothesis  $(f_1)$ , there exist  $\rho_0 > 0$  such that  $\epsilon > 0$  small enough,

$$\frac{f(v_\epsilon) + v_\epsilon^{q-1}}{|v_\epsilon|^{q-1}} \leq \frac{3}{4} V_0, \quad \text{for all } |x| \geq \rho_0.$$

Since  $\psi(x) := M \exp(-\alpha|x|)$  for  $x \in \mathbb{R}^N$ , we can find  $\tilde{M} > 0$  such that if  $M \geq \tilde{M}$ , then  $\varphi_\epsilon := \max\{|v_{i,\epsilon}| - \psi, 0\} \equiv 0$  in  $B_{\rho_0}(0)$  and  $\varphi_\epsilon \in W^{1,p}(|x| \geq \rho_0) \cap W^{1,q}(|x| \geq \rho_0)$ . Therefore, the above inequality and  $(b_1)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^N} a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \nabla \varphi_\epsilon \, dx + V_0 \int_{\mathbb{R}^N} [k_3 |v_\epsilon|^{p-1} \varphi_\epsilon + |v_\epsilon|^{q-1} \varphi_\epsilon] \, dx \\ & \leq \int_{\mathbb{R}^N} a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \nabla \varphi_\epsilon \, dx + \int_{\mathbb{R}^N} V(\epsilon x + y_\epsilon) b(|v_\epsilon|^p) |v_\epsilon|^{p-2} v_\epsilon \varphi_\epsilon \, dx \\ & \leq \int_{\mathbb{R}^N} f(v_\epsilon) \varphi_\epsilon \, dx \leq \frac{3V_0}{4} \int_{\mathbb{R}^N} |v_\epsilon|^{q-1} \varphi_\epsilon \, dx \end{aligned}$$

and the lemma is proved. □

Finally we are going to show the exponential decay for the functions  $u_\epsilon$ .

**Proposition 7.4.** *There are  $\epsilon_0 > 0$  and  $C > 0$  such that*

$$|u_\epsilon(z)| \leq C \exp\left(-\alpha \left| \frac{z - P_\epsilon}{\epsilon} \right| \right), \quad \text{for all } z \in \mathbb{R}^N.$$

*Proof.* From [14, Lemma 2.4], we have that

$$\left\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x - y \right\rangle \geq 0, \quad \forall x, y \in \mathbb{R}^N.$$

Consider  $v_\epsilon(x) := u_\epsilon(x + \tilde{y}_\epsilon)$  the set

$$\Lambda := \{x \in \mathbb{R}^N : |x| \geq \rho_0 \text{ and } |v_\epsilon| - \psi \geq 0\},$$

where  $\psi$  is the function is given by Lemma 7.1,  $(\tilde{y}_n)$  is given by Proposition 5.1. Then, using Corollary 7.2 and Proposition 7.3, we obtain

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^N} \left\langle a(|\nabla v_\epsilon|^p) |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon - a(|\nabla \psi|^p) |\nabla \psi|^{p-2} \nabla \psi, \nabla \tilde{\varphi} \right\rangle dx \\ &\quad + V_0 k_3 \int_{\mathbb{R}^N} \left( |v_\epsilon|^{p-1} - |\psi|^{p-1} \right) \tilde{\varphi} dx + \frac{V_0}{4} \int_{\mathbb{R}^N} \left( |v_\epsilon|^{q-1} - |\psi|^{q-1} \right) \tilde{\varphi} dx \\ &\geq V_0 k_3 \int_{\mathbb{R}^N} \left( |v_\epsilon|^{p-1} - |\psi|^{p-1} \right) \tilde{\varphi} dx + \frac{V_0}{4} \int_{\mathbb{R}^N} \left( |v_\epsilon|^{q-1} - |\psi|^{q-1} \right) \tilde{\varphi} dx \\ &= V_0 k_3 \int_{\Lambda} \left( |v_\epsilon|^{p-1} - |\psi|^{p-1} \right) (v_\epsilon - \psi) dx \\ &\quad + \frac{V_0}{4} \int_{\Lambda} \left( |v_\epsilon|^{q-1} - |\psi|^{q-1} \right) (v_\epsilon - \psi) dx \geq 0. \end{aligned}$$

Then  $|\Lambda| = 0$  and consequently

$$v_\epsilon(x) \leq M \exp(-\alpha|x|), \quad \forall |x| \geq \rho_0.$$

Considering  $x = z - \tilde{y}_\epsilon$  and using Lemma 5.5 there exists a constant  $C > 0$  satisfying

$$\begin{aligned} |u_\epsilon(z)| &\leq M \exp\left(-\alpha \left| \frac{z - y_\epsilon}{\epsilon} \right| \right) = M \exp\left(-\alpha \left| \frac{z - P_\epsilon + \epsilon q_\epsilon}{\epsilon} \right| \right) \\ &\leq M \exp\left(-\alpha \left| \frac{z - P_\epsilon}{\epsilon} \right| \right) \exp(-\alpha |q_\epsilon|) \leq C \exp\left(-\alpha \left| \frac{z - P_\epsilon}{\epsilon} \right| \right), \end{aligned} \quad (7.1)$$

for all  $|z - \tilde{y}_\epsilon| \geq \rho_0$  and for  $\epsilon > 0$  small enough.

Now we are going to show the inequality (7.1) holds, for all  $z \in \mathbb{R}^N$ . Since  $(y_\epsilon)$  converges, it follows that

$$|z| \geq \rho_0 - |\tilde{y}_\epsilon| = \rho_0 - \frac{|y_\epsilon|}{\epsilon} > \rho_0 - \frac{1 + |y_\epsilon|}{\epsilon} \rightarrow -\infty \text{ as } \epsilon \rightarrow 0.$$

Then, there exists  $\epsilon_0 > 0$  such that

$$|u_\epsilon(z)| \leq C \exp\left(-\alpha \left| \frac{z - P_\epsilon}{\epsilon} \right| \right), \quad \forall z \in \mathbb{R}^N \text{ and } \forall \epsilon \in (0, \epsilon_0).$$

□

## Appendix

In this appendix we are going to show the existence of positive solution for a problem in a bounded domain with smooth boundary, denoted by  $\Omega$ . More precisely, we are going to study the following problem

$$\begin{cases} -k_2 \Delta_p u - \Delta_q u + \mathcal{V} k_4 |u|^{p-2} u + \mathcal{V} |u|^{q-2} u = |u|^{\tau-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_\tau)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  and  $k_2, k_4, \mathcal{V}$  are positive constants. We have associated to problem  $(P_\tau)$  the functional

$$I_\tau(u) = \frac{1}{p} \int_{\Omega} [k_2 |\nabla u|^p + \mathcal{V} k_4 |u|^p] dx + \frac{1}{q} \int_{\Omega} [|\nabla u|^q + \mathcal{V} |u|^q] dx - \frac{1}{\tau} \int_{\Omega} |u|^\tau dx$$

and the Nehari manifold

$$\mathcal{N}_\tau = \{u \in W_0^{1,q}(\Omega) : u \neq 0 \text{ and } I'_\tau(u)u = 0\}$$

**Lemma 7.5.** For all  $u \in W_0^{1,q}(\Omega) \setminus \{0\}$  there exists a unique  $t_u \in (0, +\infty)$ , such that  $tu \in \mathcal{N}_\tau$ .

*Proof.* Note that if  $u \in W_0^{1,q}(\Omega) \setminus \{0\}$  and  $t > 0$ , we have

$$I_\tau(tu) = t^\tau \left[ \frac{t^{p-\tau}}{p} \int_{\Omega} [k_2 |\nabla u|^p + \mathcal{V} k_4 |u|^p] dx + \frac{t^{q-\tau}}{q} \int_{\Omega} [|\nabla u|^q + \mathcal{V} |u|^q] dx - \frac{1}{\tau} \int_{\Omega} |u|^\tau dx \right].$$

Then,

$$\lim_{t \rightarrow 0} \frac{I_\tau(tu)}{t^\tau} = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{I_\tau(tu)}{t^\tau} = -\frac{1}{\tau} \int_{\Omega} |u|^\tau dx < 0.$$

Consequently, there exists  $t_u \in (0, +\infty)$  such that  $I_\tau(t_u u) = \sup_{t \geq 0} I_\tau(tu)$  and  $t_u u \in \mathcal{N}_\tau$ .

In order to show the unicity of  $t_u$ , consider  $f(t) = t^\tau$  and note that  $\frac{f(t)}{t^q}$  is increasing. □

**Lemma 7.6.** The following properties hold:

- (i) There exists  $\rho_\tau > 0$  such that  $\left[ \int_{\Omega} |\nabla u|^q dx \right]^{1/q} \geq \rho_\tau$ , for all  $u \in \mathcal{N}_\tau$ ;
- (ii) There exists a constant  $C_\tau > 0$  such that  $I_\tau(u) \geq C_\tau \int_{\Omega} |\nabla u|^q dx$ , for all  $u \in \mathcal{N}_\tau$ .

*Proof.* By Sobolev's embeddings, there exists  $C > 0$  such that

$$\begin{aligned} \int_{\Omega} |\nabla u|^q dx &\leq \int_{\Omega} [k_2 |\nabla u|^p + \mathcal{V} k_4 |u|^p] dx + \int_{\Omega} [|\nabla u|^q + \mathcal{V} |u|^q] dx = \int_{\Omega} |u|^\tau dx \\ &\leq C \left[ \int_{\Omega} |\nabla u|^q dx \right]^{\tau/q}. \end{aligned}$$

Since  $\tau > q$ , the item (i) follows.

To verify the second assertion observe that

$$\begin{aligned} I_\tau(u) &= I_\tau(u) - \frac{1}{\tau} I'_\tau(u)u \geq \left( \frac{1}{p} - \frac{1}{\tau} \right) \int_{\Omega} [k_2 |\nabla u|^p + \mathcal{V} k_4 |u|^p] dx \\ &\quad + \left( \frac{1}{q} - \frac{1}{\tau} \right) \int_{\Omega} [|\nabla u|^q + \mathcal{V} |u|^q] dx \geq \left( \frac{1}{q} - \frac{1}{\tau} \right) \int_{\Omega} |\nabla u|^q dx. \end{aligned}$$

□

**Proposition 7.7.** There exists  $w_\tau \in W_0^{1,q}(\Omega)$  such that  $w_\tau$  is a solution of  $(P_\tau)$  and  $I_\tau(w_\tau) = \inf_{\mathcal{N}_\tau} I_\tau$ .

*Proof.* Let  $(u_n)$  be a minimizing sequence for  $I_\tau$  in  $\mathcal{N}_\tau$ . By Lemma 7.6, we conclude that  $(u_n)$  is bounded in  $W_0^{1,q}(\Omega)$ . Then there exists  $u \in W_0^{1,q}(\Omega)$  such that, up to a subsequence,  $u_n \rightharpoonup u$  in  $W_0^{1,q}(\Omega)$  and

$$\begin{cases} u_n \rightarrow u \text{ strongly in } L^s(\Omega) \text{ for any } 1 \leq s < q^*, \\ u_n(x) \rightarrow u(x) \text{ for a.e. } x \in \Omega. \end{cases} \quad (7.2)$$

Since  $\tau \in (q, q^*)$  we have, by Lemma 7.6 again, that  $u \neq 0$ . Hence,

$$c_\tau \leq I_\tau(t_u u) \leq \liminf_{n \rightarrow \infty} I_\tau(t_u u_n) \leq \liminf_{n \rightarrow \infty} I_\tau(u_n) + o_n(1) = c_\tau.$$

Considering  $w_\tau := t_u u$  we have  $I_\tau(w_\tau) = c_\tau$  and using Implicit Theorem we conclude that  $I'_\tau(w_\tau) = 0$ . □

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