Non-local modelling on the buckling of a weakened nanobeam

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In the framework of the theory of non-local elasticity, a model on the buckling of a weakened nanobeam subjected to constant axial load is presented. Based on this model, the characteristic equation for a simply supported weakened nanobeam is obtained, through which the value of buckling load can be determined. The influences of small length scale and weakened junction on the buckling load are discussed. It is shown that the buckling load is dependent upon the scale coefficient, junction stiffness and junction location.

1. Introduction: In the last two decades, structures with nanosizes have received tremendous attention from various branches of science. By the use of varieties of experimental, theoretical and computer simulation approaches, extensive research studies of properties of nanostructures have been carried out [1—13]. Structural beams fabricated from nanomaterials and of nanometre dimensions are referred to as nanobeams (viz. nanowires, nanorods, nanotubes etc.). In nanoelectromechanical systems, nanobeams play key roles of mechanical components. Consequently, a thorough understanding of the mechanical responses of individual nanobeams is of great importance for their potential applications.

Owing to the difficulties of experimental operation at nanoscale and being time consuming for atomistic simulations, classic (local) continuum models have been widely employed to study the mechanical behaviour of nanobeams [14—20]. However, it has been indicated from experimental and atomistic simulations that there exist significant size effects in the physical and mechanical properties for structures with nanosizes. Although the classic continuum models are efficient in mechanical analysis of nanostructures to some extent, the length scales associated with nanotechnology are often sufficiently small to call the applicability of classical local continuum models into question as they cannot consider the size effects. This has raised a major challenge to the classic continuum mechanics. It is a possible solution to extend the classic continuum approach to smaller length scales by incorporating information regarding the behaviour of material microstructures. It is accomplished quite easily by the use of the theory of non-local continuum mechanics [21—23]. Peddieson *et al.* [24] pointed out that nanoscale devices would exhibit small-scale effects and the non-local continuum mechanics could potentially play a useful role in analysis related to nanotechnology applications.

In recent years, small-scale effects on the buckling behaviour of nanostructures have been extensively investigated on the basis of the theory of non-local continuum mechanics [25—39]. For the buckling of nanobeams, various non-local models have been presented for the consideration of small-scale effects. Aydogdu [40] proposed a generalised non-local beam theory to study the buckling of nanobeams. The effects of non-locality and length of beams were investigated in detail for each considered problem. Niu *et al.* [41] developed a third-order non-local beam theory for the buckling of nanobeams. The buckling of nanobeams with a simply supported boundary condition was analysed. The effect of the non-local scale parameter on the buckling loads of nanobeams was discussed. Murmu and Adhikari [42] studied the axial instability of double-nanobeam-systems on the basis of Eringen's non-local elasticity. The small-scale effects arising at the nanoscale were considered. Sahmani and Ansari [43] established non-local elastic beam models by incorporating Eringen's equations of non-local elasticity into the classical beam theories for the buckling of nanobeams with a rectangular cross-section. The critical buckling loads including size effects can be predicted by the use of these developed beam models. Roque *et al.* [44] utilised the non-local elasticity theory to investigate the buckling properties of Timoshenko nanobeams, and the numerical solution was obtained based on a meshless method. Thai [45] proposed a non-local shear deformation beam theory for the bending, buckling and vibration of nanobeams using the non-local differential constitutive relations of Eringen. The theory accounts for both small-scale effects and the quadratic variation of shear strains and shear stresses through the thickness of the beam.

It should be pointed out that nanobeams are not always defect free. They could have vacancies and defects that are introduced in the synthesis and fabrication process, and thus their mechanical properties may be weakened. Wang *et al.* [46] studied the buckling problem of a weakened column with macrosize. The present Letter presents a model which investigates the elastic buckling of a weakened nanobeam under constant axial load on the basis of the Bernoulli-Euler beam theory and non-local elasticity. Exact buckling load values are obtained for the weakened nanobeam with simply supported ends.

2. Non-local elastic beam model for a weakened nanobeam: The treatment of beam flexure developed here is on the basis of the Bernoulli-Euler theory. Using the Bernoulli-Euler beam theory, the deflection curve of an elastic beam under constant axial compressive load is expressed by

$$EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4} + N\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = 0 \tag{1}$$

where x is the axial coordinate, w is the deflection of the beam, EI denotes the bending stiffness of the beam and N represents the constant axial compressive force.

The fundamental assumption behind the Bernoulli-Euler beam model is that the beam consists of fibres parallel to the x-axis, each in a state of uniaxial tension or compression. Adopting the Eringen non-local elasticity model, the classic Hooke's law for a uniaxial stress state is replaced by [24, 25]

$$\sigma - (e_0a)^2\frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2} = E\varepsilon \tag{2}$$

where σ is the axial stress, ε is the axial strain, E is Young's modulus, e_0 is a non-local parameter incorporating the small-scale

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effect and a is an internal characteristic length (e.g. lattice parameter, C—C bond length and granular distance).

The equilibrium of forces in the vertical direction and the moment on a free body diagram of an infinitesimal element of a beam structure is given by [47]

$$\frac{\mathrm{d}S}{\mathrm{d}x} = 0 \tag{3}$$

$$S = \frac{\mathrm{d}M}{\mathrm{d}x} - N\frac{\mathrm{d}w}{\mathrm{d}x} \tag{4}$$

where S is the resultant shear force, and M is the resultant bending moment, which can be obtained by

$$M = \int_A y\sigma \,\mathrm{d}A \tag{5}$$

where y is the transverse coordinate measured positive in the direction of deflection and A is the cross-sectional area. In addition, for small deflections we have

$$\varepsilon = -y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{6}$$

Combination of (2), (5) and (6) results in

$$M - (e_0 a)^2 \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{7}$$

It follows from (3), (4) and (7) that

$$\left[EI - (e_0 a)^2 N\right]\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \tag{8}$$

which is the expression for the deflection curve of an elastic nanobeam under constant axial compressive load on the basis of nonlocal elasticity. It is noted that when the small-scale parameter e_0 vanishes, the above equation reduces to the classical Bernoulli-Euler expression (1).

For generality, dimensionless formulation is often adopted for the deflection curve of a beam.

Introducing dimensionless quantities into (8), we have

$$\frac{\mathrm{d}^4 \bar{w}}{\mathrm{d}\bar{x}^4} + \eta\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{x}^2} = 0 \tag{9}$$

where $\bar{w} = w/L$ is the normalised lateral deflection of the beam with length L, $\bar{x} = x/L$ and

$$\eta = \frac{\bar{N}}{1 - \alpha^2 \bar{N}} \tag{10}$$

where $\bar{N} = NL^2/EI$ is the non-dimensional axial compressive load, and $\alpha = e_0 a/L$ is the non-dimensional scale coefficient.

From (3), (4) and (7), the bending moment can be expressed as

$$M = \left[(e_0 a)^2 N - EI\right]\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{11}$$

Using the dimensionless quantities, the above equation becomes

$$\bar{M} = \left[\alpha^2 \bar{N} - 1\right]\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{x}^2} \tag{12}$$

where $\bar{M} = ML/EI$ is the normalised bending moment. Substituting (11) into (4), we obtain

$$S = \left[(e_0 a)^2 N - EI\right]\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - N\frac{\mathrm{d}w}{\mathrm{d}x} \tag{13}$$

With the introduction of the dimensionless quantity, the above equation can be rewritten as

$$\bar{S} = \left[\alpha^2 \bar{N} - 1\right]\frac{\mathrm{d}^3 \bar{w}}{\mathrm{d}\bar{x}^3} - \bar{N}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} \tag{14}$$

where $\bar{S} = SL^2/EI$ is the normalised shear force.

In what follows, a nanobeam weakened at an interior location is considered. The weakness can be modelled by a rotationally restrained junction [46, 48—50]. The beam is separated by the junction into two segments. To analyse the mechanical behaviour of the weakened beam, it can be regarded as two segments connected at the weakened section by a rotational spring. The value of the spring stiffness ranges from zero to infinite. When the value of the spring stiffness is zero, the two segments are connected by a frictionless free hinge. When the value of the spring stiffness is infinite, the beam is completely continuous. In addition, the continuity of displacement, bending moment and shear force is required at the junction. The restraining moment of the rotational spring is to be proportional to the difference of the junction slopes of the two segments.

Assume that the junction is located at $\bar{x} = l$ and the subscripts 1 and 2 denote the segments $0 \le \bar{x} \le l$ and $l \le \bar{x} \le 1$, respectively. Thus, $\bar{w}_1$ should match the boundary conditions at the end $\bar{x} = 0$, and $\bar{w}_2$ should match the boundary conditions at the end $\bar{x} = 1$. At the junction, the following conditions should be satisfied. It is required by the continuity of deflection that

$$\bar{w}_1(l) = \bar{w}_2(l) \tag{15}$$

It follows from the continuity of bending moment and (12) that

$$\frac{\mathrm{d}^2 \bar{w}_1(l)}{\mathrm{d}\bar{x}^2} = \frac{\mathrm{d}^2 \bar{w}_2(l)}{\mathrm{d}\bar{x}^2} \tag{16}$$

Combination of (14) and shear continuity yields

$$\frac{\mathrm{d}^3 \bar{w}_1(l)}{\mathrm{d}\bar{x}^3} + \eta\frac{\mathrm{d}\bar{w}_1(l)}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}^3 \bar{w}_2(l)}{\mathrm{d}\bar{x}^3} + \eta\frac{\mathrm{d}\bar{w}_2(l)}{\mathrm{d}\bar{x}} \tag{17}$$

As the moment at the junction is proportional to the difference in the slope of the deflection, we have

$$\frac{\mathrm{d}^2 \bar{w}_1(l)}{\mathrm{d}\bar{x}^2} = \frac{k}{1 - \alpha^2 \bar{N}}\left[\frac{\mathrm{d}\bar{w}_2(l)}{\mathrm{d}\bar{x}} - \frac{\mathrm{d}\bar{w}_1(l)}{\mathrm{d}\bar{x}}\right] \tag{18}$$

where $k = \gamma L/EI$ is the junction stiffness parameter, and γ is the rotational spring constant.

3. Solution of the problem: The solution to (9) is given by

$$\bar{w} = C_1 \sin\left(\sqrt{\eta}\bar{x}\right) + C_2 \cos\left(\sqrt{\eta}\bar{x}\right) + C_3 \bar{x} + C_4 \tag{19}$$

where C_1, C_2, C_3 and C_4 are constants determined using boundary conditions. As a consequence, for the segment, $0 \le \bar{x} \le 1$, we have

$$\bar{w}_1 = C_1 \sin(\sqrt{\eta}\bar{x}) + C_2 \cos(\sqrt{\eta}\bar{x}) + C_3\bar{x} + C_4 \qquad (20)$$

For the segment, $l \le \bar{x} \le 1$, the solution can be expressed as

$$\bar{w}_2 = C_5 \sin(\sqrt{\eta}\bar{x}) + C_6 \cos(\sqrt{\eta}\bar{x}) + C_7\bar{x} + C_8 \qquad (21)$$

where C_5, C_6, C_7 and C_8 are constants, which can be determined by the boundary conditions.

Assuming that the two ends of the beam are simply supported, the boundary conditions are given by

$$\bar{w}_1(0) = \bar{w}_2(1) = 0 \qquad (22)$$

and

$$\bar{M}_1(0) = \bar{M}_2(1) = 0 \qquad (23)$$

It follows from (12) and (23) that

$$\frac{d^2\bar{w}_1(0)}{d\bar{x}^2} = \frac{d^2\bar{w}_2(1)}{d\bar{x}^2} = 0 \qquad (24)$$

Using (20) and (21) in the eight conditions described by (15)—(18) and (22) and (24), a homogeneous set of linear algebraic equations for the eight constants is obtained. To allow a non-trivial solution, the determinant of the coefficient matrix should vanish. Consequently, the corresponding characteristic equation can be obtained as

$$\cos\sqrt{\eta} - \cos\left[(1-2l)\sqrt{\eta}\right] + 2k\frac{\sqrt{\eta}}{\bar{N}}\sin\sqrt{\eta} = 0 \qquad (25)$$

Through this characteristic equation, the buckling load can be determined by the use of the bisection method for the smallest eigenvalue $\bar{N}$ for given values of l, k and α.

4. Discussion: When the scale coefficient $\alpha = e_0a/L$ is set to be zero, (25) reduces to

$$\cos\sqrt{\bar{N}} - \cos\left[(1-2l)\sqrt{\bar{N}}\right] + \frac{2k}{\sqrt{\bar{N}}}\sin\sqrt{\bar{N}} = 0 \qquad (26)$$

which is the classical (local) results ignoring the effect of small length scale [46].

When the value of the junction stiffness parameter k is infinite, the weakness of the nanobeam is absent, and thus (25) reduces to

$$\sin\sqrt{\eta} = 0 \qquad (27)$$

It follows from (10) and (27) that

$$\bar{N} = \frac{m^2\pi^2L^2}{L^2 + (e_0a)^2 m^2\pi^2} \qquad (28)$$

where m is a positive integer. This is the non-local result for a defect-free nanobeam [25].

Moreover, when the scale coefficient α vanishes and the junction stiffness parameter k is simultaneously infinite, the buckling load $\bar{N} = \pi^2$ for a classical Euler beam with simply supported ends can be recovered from (25).

It should be pointed out that the choice of value for e_0 or e_0a is of great importance to ensure the validity of non-local models. For a specific material, the magnitude of e_0 can be determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. For nanosized beams or tubes, $0 \le e_0a \le 2$ nm is often chosen to discuss the scale effects [30, 40].

With the junction stiffness parameter $k = 5$, Fig. 1 gives the variations of the non-dimensional buckling load $\bar{N}$ with the junction location l for various scale coefficients α. With the scale coefficient $\alpha = 0.05$, Fig. 2 shows the variations of the non-dimensional buckling load $\bar{N}$ with the junction location l for various junction stiffnesses. Owing to the symmetry of the beam, only the range $0 \le l \le 0.5$ is plotted. It can be seen from Figs. 1 and 2 that the buckling load is dependent upon the scale coefficient and junction stiffness, which shows the significance of applying non-local continuum mechanics in structures with nanosize.

When the junction location is constant, it is indicated from Fig. 1 that the value of the buckling load decreases with the increase of the scale coefficient. In other words, the classic solution of the buckling load ignoring the scale effect could overestimate the buckling load of the nanobeam. As a consequence, it is of great necessity to give an evaluation of the size of the structure on which the non-local theories of continuum mechanics are essential. It is found from Fig. 1 that the values of the buckling load with $\alpha = 0.05$ are very close to those with $\alpha = 0$. For example, when the junction location $l = 0.1$, the relative error between non-local results ($\alpha = 0.05$) and local results ($\alpha = 0$) for the buckling load is about 2.2%, and when the

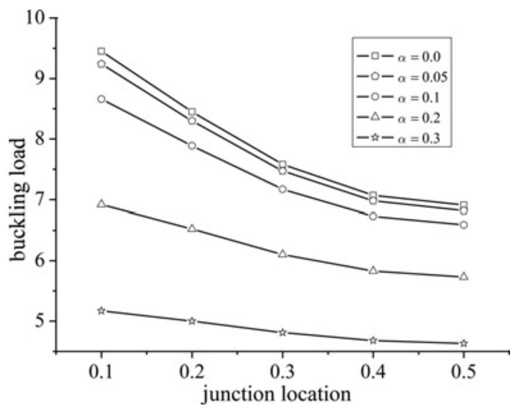

Figure 1 *Variations of the non-dimensional buckling load $\bar{N}$ with the junction location l for various scale coefficients α*

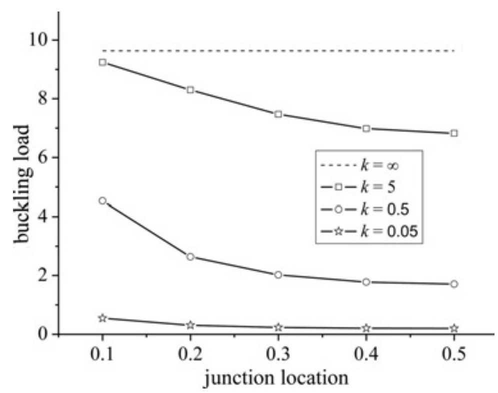

Figure 2 *Variations of the non-dimensional buckling load $\bar{N}$ with the junction location l for various junction stiffnesses k*

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junction location $l = 0.5$, the relative error is about 1.3%. If we suppose $e_0a = 2$ nm, the length of the nanobeam at $\alpha = 0.05$ is estimated to be 40 nm by a simple calculation. Consequently, it can be concluded that when the length of the nanobeam is over 40 nm, the scale effect is small enough to be ignored and the classic (local) beam models may be directly applied to study the properties of buckling of the weakened nanobeam with very small relative errors.

Furthermore, it can be observed from Fig. 2 that the value of the buckling load diminishes rapidly with decrease of junction stiffness. When the junction stiffness vanishes, the buckling load becomes zero. The reason is that when the value of the junction stiffness is equal to zero, there exists no resistance at the junction and a frictionless linkage is formed. In addition, it can be found from Figs. 1 and 2 that the buckling load becomes smaller until a minimum load is reached at $l = 0.5$ as the junction moves away from the beam end.

5. Conclusions: In the framework of the theory of non-local continuum mechanics, a model on the buckling of a weakened nanobeam subjected to constant axial load is presented. Based on this non-local model, the elastic buckling of a weakened nanobeam under constant axial load is analysed. The characteristic equation for a simply supported weakened nanobeam is obtained, through which the value of the buckling load can be determined. The influences of small length scale and weakened junction on the buckling load are investigated. It can be concluded that the buckling load is dependent upon the scale coefficient, the junction stiffness and the junction location. When the junction location is constant, the value of the buckling load decreases with the increase of the scale coefficient and increases with the increase of junction stiffness. When the junction moves away from the beam end, the buckling load diminishes until a minimum load is reached at $l = 0.5$.

This non-local beam model can be applied for column buckling (beam-like buckling) of both a non-hollow weakened structure of nanowires/rods and the hollow weakened structure of single-walled nanotubes. It is known that only the nanotubes with high aspect ratio (> 7.5 for carbon nanotubes) may buckle in a beam-like pattern, whereas the shell-like buckling (local buckling) will dominate the shorter nanotubes for their compressive instability [51]. Thus, this non-local beam model is suitable for weakened nanotubes with a high aspect ratio, whereas the local buckling of weakened nanotubes with a low aspect ratio cannot be simulated by it. In addition, the non-local multi-beam model for the buckling of weakened multi-walled nanotubes with a high aspect ratio needs to be developed as the non-local beam model in this present study does not consider the interaction between non-bonded atoms such as the van der Waal interaction pressure in multi-walled nanotubes.

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