On the axial vibration of carbon nanotubes with different boundary conditions

Mustafa Özgür Yayli

Department of Civil Engineering, Bilecik Şeyh Edebali University, Gülümbe Kampüsü, Bilecik 11210, Turkey
E-mail: mozgur.yayli@bilecik.edu.tr

Published in Micro & Nano Letters; Received on 23rd July 2014; Revised on 26th September 2014; Accepted on 6th October 2014

In this reported work, the free axial vibration response of carbon nanotubes (CNTs) with arbitrary boundary conditions is studied based on the non-local elasticity theory. Using Fourier sine series together with Stokes'transformation, the general frequency determinant of CNTs is obtained. The main advantage of this method is its capability of dealing with rigid or restrained boundary conditions. Comparisons between the results of the presented method and previous works in the literature have been performed. Good agreement is obtained when enough terms are included in the Fourier series expansion. The effects of spring parameters on the vibration frequencies are discussed in detail. The proposed analytical method can be utilised for dynamic analyses of nanorods (CNTs) with arbitrary boundary conditions.

1. Introduction: Carbon nanotubes (CNTs) are found to have exceptional physical, chemical, mechanical, thermal, electrical and electronic properties, which leads to a variety of engineering applications in nanomechanical systems, nanobiological devices and nanoelectronics. Experimental studies related to CNTs [1–6] have shown that CNTs have an extremely high aspect ratio, low weight, high stiffness and are extremely sensitive to their environment changes. These superior features of CNTs make them promising candidates for new sensors, ultra-capacitors, microbial detection, ultra-high strength composite materials and for diagnostic devices [7, 8].

Eringen [9] presented a new size-dependent theory known as the non-local elasticity theory. This theory states that stresses at a reference point are a function not only of the strains at that point but also a function of the strains at every point in the domain. This non-classical approach is in accordance with predictions from atomic lattice dynamics. The non-local elasticity theory is a popular technique for modelling the mechanical behaviour of CNTs [10–13]. Some of the researchers have investigated the static behaviours of single-walled, double- and multi-walled CNTs, e.g. in [14, 15]. Reddy and Pang [16] presented Timoshenko beam and the Euler-Bernoulli theories using the non-classical constitutive relations of Eringen and Edelen [17]. Pradhan and Murmu [18] presented a single non-local beam model to investigate the static and dynamic characteristics of a nanocantilever beam. The vibration behaviours of CNTs embedded in an elastic medium have been examined by some researchers, e.g. [19, 20]. Also the free vibration behaviours of CNTs have been considered by some researchers [21–23]. The free axial vibration of nanorods was considered by Aydogdu [24]. The small size effects on free axial vibrations of heterojunction CNTs based on the classical and non-classical rod theories were investigated by Filiz and Aydogdu [25].

A short review of the literature reveals that the conducted experimental and theoretical works on nanorods are based on the assumptions that the boundary conditions are rigid. A very limited amount of literature is available for nanorods with elastical restraints. The study reported in this Letter is concerned with the derivation of the general formulation for the free axial vibration analysis of CNTs (nanorods) modelled as Eringen's non-local elasticity theory [9]. The general frequency determinant is obtained by a combination of the basic equations and Stokes' transformation. Its utility lies in the ability to solve any possible combination of boundary conditions. For the nanorods with rigid or restrained boundary conditions, the influence of the non-locality parameter and axial springs on the natural frequencies is examined in some numerical examples. There are very good agreements between this Letter and the previous results, indicating the accuracy and validity of the presented method. The purpose of this Letter is mainly to present a new analytical method for the axial vibration analysis of nanorods with rigid or restrained boundary conditions rather than to investigate a specific problem. The presented formulation can be helpful in the design of nanorods.

2. Free axial vibration of nanorods with deformable boundary conditions: Consider a nanosized rod of diameter d and length L. The following non-local differential equation is often used [9]

$$\sigma_{ij}^{nl} - (e_0 a)^2 \nabla^2 \sigma_{ij}^{nl} = C{:}\epsilon \tag{1}$$

where ε and C are the fourth-order strain and elasticity tensors, respectively. a denotes the internal characteristic length and e_0 is a material constant. Equation (1) can be approximated to the following one-dimensional (1D) form

$$\sigma_{xx}^{nl} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}^{nl}}{\partial x^2} = E\epsilon_{xx} \tag{2}$$

where E is the elasticity modulus. The equation of motion for the free axial vibration can be obtained as

$$\frac{\partial N^l}{\partial x} = m\frac{\partial^2 u(x,t)}{\partial t^2} \tag{3}$$

where m is the mass per unit length, $u(x, t)$ denotes the axial displacement and N^l is the axial force for classical elasticity. N^l can be expressed as

$$N^l = \int_A \sigma_{xx}\, \mathrm{d}A \tag{4}$$

where A is the cross-sectional area of the nanorod. Using (2)–(4), the following equation can be found in terms of axial force

$$N^{nl} - (e_0 a)^2 \frac{\partial^2 N^{nl}}{\partial x^2} = N^l \tag{5}$$

By substituting (5) into (3), the equation of motion of the non-local nanorod model in terms of the axial displacement is as follows [24]

$$EA\frac{\partial^2 u(x,t)}{\partial x^2} - \left\{1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right\} m\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \tag{6}$$

where m denotes the mass per unit length. Equation (6) is the linear partial differential equation of the non-local rod model for the free axial vibration of nanorods.

3. Modal displacement function: In non-local elasticity, Fourier series expansion together with Stokes' transformation will be used to represent the axial vibration response of nanorods. This method gives more flexibility to treat various boundary conditions. Assuming harmonic vibrations, $u(x, t)$ can be represented as

$$u(x, t) = \phi(x)\cos(\omega t) \tag{7}$$

where $\phi(x)$ denotes the modal displacement function and ω is the natural frequency. The modal displacement function $\phi(x)$ is described in three separate regions, two for supporting points and the other for the intermediate places between the supporting points

$$\phi_0 \quad x = 0 \tag{8}$$

$$\phi_L \quad x = L \tag{9}$$

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \quad 0 < x < L \tag{10}$$

with

$$C_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \tag{11}$$

It is not necessary that the Fourier sine series satisfies any particular boundary conditions since (8) and (9) allow freedom in choosing the modal displacement function for the natural frequencies. Term-wise differentiation of (10) yields

$$\phi'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} C_n \cos\left(\frac{n\pi x}{L}\right) \tag{12}$$

If $\phi'(x)$ is piecewise smooth, then it can be represented by a Fourier cosine series

$$\phi'(x) = \frac{b_0}{L} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \tag{13}$$

The coefficients are given by

$$b_0 = \frac{2}{L}\int_0^L \phi'(x)\,\mathrm{d}x = \frac{2}{L}[\phi(L) - \phi(0)] \tag{14}$$

$$b_n = \frac{2}{L}\int_0^L \phi'(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad n = 1, 2, \ldots \tag{15}$$

Integration by parts

$$b_n = \frac{2}{L}\left[\phi(x)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{2}{L}\left[\frac{n\pi}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x\right] \tag{16}$$

$$b_n = \frac{2}{L}[(-1)^n\phi(L) - \phi(0)] + \frac{n\pi}{L}C_n \tag{17}$$

The second derivative can be computed using the above similar procedure. The derivatives of $\phi(x)$ can be separately determined by employing Stokes' transformation as follows

$$\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \frac{\phi_L - \phi_0}{L} + \sum_{n=1}^{\infty}\cos(\beta_n x)\left(\frac{2((-1)^n\phi_L - \phi_0)}{L} + \beta_n C_n\right) \tag{18}$$

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = -\sum_{n=1}^{\infty}\beta_n\sin(\beta_n x)\left(\frac{2((-1)^n\phi_L - \phi_0)}{L} + \beta_n C_n\right) \tag{19}$$

where

$$\beta_n = \frac{n\pi}{L} \tag{20}$$

Equation (19) is substituted into (6) to result in

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{L}(\cos(\omega t)\sin(\beta_n x)LC_m(\beta_n^2(\mu m\omega^2 - EA)\\ &\quad + m\omega^2) + 2\beta_n(\phi_0 - (-1)^n\phi_L)(EA - \mu m\omega^2) = 0\end{aligned} \tag{21}$$

The Fourier coefficient C_n can be written in terms of ϕ_0 and ϕ_L as follows

$$C_n = \frac{2}{L}\frac{(L^2 - \lambda^2(e_0a)^2)\beta_n(\phi_0 - (-1)^n\phi_L)}{-\lambda^2 + (L^2 - \lambda^2(e_0a)^2)\beta_n^2} \tag{22}$$

where

$$\lambda^2 = \frac{m\omega^2L^2}{EA} \tag{23}$$

The axial displacement function for the free vibration of a nanorod having no axial restraints at both ends becomes

$$\begin{aligned}u(x, t) = &\sum_{n=1}^{\infty}\frac{2}{L}\frac{(L^2 - \lambda^2(e_0a)^2)\beta_n(\phi_0 - (-1)^n\phi_L)}{-\lambda^2 + (L^2 - \lambda^2(e_0a)^2)\beta_n^2}\\ &\times\cos(\omega t)\sin(\beta_n x)\end{aligned} \tag{24}$$

The inclusion of the non-locality parameter $(e_0a)^2$ in the above equation takes into account the non-local effects.

4. General frequency determinant in non-local elasticity: It should be noted that the nanorods (CNTs) may be used as the atomic force microscope (see Fig. 1). During an experiment, nanosized samples can behave as masses attached by a linear spring to the testing device and this may affect the vibration behaviour of the atomic force microscope.

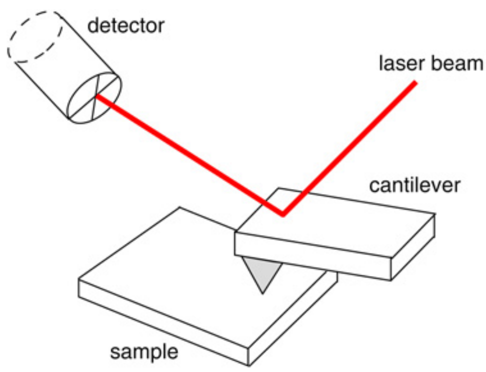


Figure 1 *Schematic of the atomic force microscope*

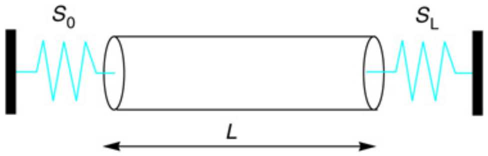


Figure 2 *Nanorod elastically restrained by means of axial springs*

To overcome this difficulty, it is assumed that the nanorod is elastically restrained by means of axial springs at the ends, as shown in Fig. 2. The non-local boundary conditions are mathematically written as

$$EA\frac{\partial u}{\partial x} + (e_0a)^2 m\frac{\partial^3 u}{\partial x \partial t^2} = s_0\phi_0, \quad x = 0 \tag{25}$$

$$EA\frac{\partial u}{\partial x} + (e_0a)^2 m\frac{\partial^3 u}{\partial x \partial t^2} = -s_L\phi_L, \quad x = L \tag{26}$$

where s_0 and s_L denote the stiffnesses of the springs at the ends of the nanorod. After some mathematical manipulations, the substitution of (18) and (22) into (25) and (26) leads to the two homogeneous simultaneous equations

$$\left(1 + \tilde{S}_0 - \gamma^2\lambda^2 + \sum_{n=1}^{\infty}\frac{2\lambda^2 - 2\gamma^2\lambda^4}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)}\right)\phi_0 + \left(-1 + \gamma^2\lambda^2 - \sum_{n=1}^{\infty}\frac{2\lambda^2(-1)^n - 2\gamma^2\lambda^4(-1)^n}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)}\right)\phi_L = 0 \tag{27}$$

$$\left(-1 + \gamma^2\lambda^2 - \sum_{n=1}^{\infty}\frac{2\lambda^2(-1)^n - 2\gamma^2\lambda^4(-1)^n}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)}\right)\phi_0 + \left(1 + \tilde{S}_L - \gamma^2\lambda^2 + \sum_{n=1}^{\infty}\frac{2\lambda^2 - 2\gamma^2\lambda^4}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)}\right)\phi_L = 0 \tag{28}$$

where

$$\gamma = \frac{e_0 a}{L} \tag{29}$$

$$\tilde{S}_0 = \frac{s_0 L}{EA} \tag{30}$$

$$\tilde{S}_L = \frac{s_L L}{EA} \tag{31}$$

and one can obtain the following system of equations in the matrix form to be solved for the constants (ϕ_0, ϕ_L)

$$\begin{bmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{bmatrix}\begin{bmatrix}\phi_0\\ \phi_L\end{bmatrix} = 0 \tag{32}$$

where

$$\psi_{11} = 1 + \tilde{S}_0 - \gamma^2\lambda^2 + \sum_{n=1}^{\infty}\frac{2\lambda^2 - 2\gamma^2\lambda^4}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)} \tag{33}$$

$$\psi_{12} = -1 + \gamma^2\lambda^2 - \sum_{n=1}^{\infty}\frac{2\lambda^2(-1)^n - 2\gamma^2\lambda^4(-1)^n}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)} \tag{34}$$

$$\psi_{21} = -1 + \gamma^2\lambda^2 - \sum_{n=1}^{\infty}\frac{2\lambda^2(-1)^n - 2\gamma^2\lambda^4(-1)^n}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)} \tag{35}$$

Table 1 Comparison of the first three axial frequency parameter, of nanorods with clamped ends obtained using local elasticity theory

Mode	CC		$\tilde{S}_0 = \tilde{S}_L = 10\,000$
	Ref. [24] λ_i	Ref. [26] λ_i	Present λ_i
1	3.141	3.141	3.14096
2	6.284	6.283	6.28193
3	9.425	9.424	9.42289

$$\psi_{22} = 1 + \tilde{S}_L - \gamma^2\lambda^2 + \sum_{n=1}^{\infty}\frac{2\lambda^2 - 2\gamma^2\lambda^4}{\lambda^2 + \pi^2 n^2(\gamma^2\lambda^2 - 1)} \tag{36}$$

Equation (32) defines a standard eigenvalue problem. The eigenvalues (λ_n) can be computed by setting the following determinant in (32) to zero

$$\left|\psi_{ij}\right| = 0 \quad (i, j = 1, 2) \tag{37}$$

The characteristic equation of this determinant can be derived by assigning the different values of $\tilde{S}_0$ and $\tilde{S}_L$ corresponding to the end constraints.

5. Results and discussion: Here, analytical solutions for the axial vibration analysis of CNTs are presented, considering the effects of deformable boundary conditions and small-scale effects. However, before venturing into more complicated and time-consuming calculations with all the parameters, it is desirable to assess the accuracy of the Fourier expansion procedure when applied to some special cases of boundary conditions. The results using these boundary conditions are then compared with the available results reported in the literature.

Tables 1 and 2 show the comparison between the first three modes of the vibration frequency parameter computed using the present formulation and that obtained in [24, 26]. In this numerical validation, the frequency parameters are obtained by the present approach using the first 50 terms of the infinite series. The non-locality parameter (γ^2) of the nanorod is zero and the axial spring parameters are taken as $\left(\tilde{S}_0 = \tilde{S}_L = 10\,000\right)$ for the clamped–clamped (CC) case. It is observed from the Table that when the stiffnesses of springs are infinitely large $\left(\tilde{S}_0 = \tilde{S}_L = 10\,000\right)$, the results are very close to those calculated for the CC case. The cantilever solutions in [24, 26] are also compared with the present approach for $\tilde{S}_0 = 10\,000$, $\tilde{S}_L = 0$ values to show the validation. $\tilde{S}_L = 0$ means that there is no axial constraint at $x = L$, namely, the nanorod is free for axial displacement. As seen from Tables 1 and 2, good agreement is found between the three results. After validation of the present method, the effects of various axial spring parameters on the vibration response are discussed.

To investigate the effects of the deformable boundary conditions on the free vibration response of the nanorod, frequency parameters are listed in Table 3 for various values of spring coefficient and

Table 2 Comparison of the first three axial frequency parameters of nanorods with clamped-free ends obtained using local elasticity theory

Mode	Clamped-free		$\tilde{S}_0 = 10\,000$, $\tilde{S}_L = 0$
	Ref. [24] λ_i	Ref. [26] λ_i	Present λ_i
1	1.571	1.570	1.57697
2	4.712	4.712	4.73090
3	7.854	7.850	7.88480

Table 3 Effect of spring constants on the first frequency parameter of nanorod

e_0a	$\tilde{S}_0 = \tilde{S}_L$					CC ends
	1	10	100	1000	10 000	Ref. [24]
0	1.310	2.628	3.080	3,135	3.140	3.141
0.1	1.310	2.627	3.078	3.133	3.139	3.140
0.2	1.309	2.625	3.074	3.129	3.134	3.135
0.3	1.309	2.620	3.066	3.121	3.127	3.127
0.4	1.308	2.614	3.056	3.110	3.116	3.117
0.5	1.307	2.606	3.044	3.097	3.102	3.103
0.6	1.306	2.596	3.028	3.081	3.086	3.087
0.7	1.304	2.585	3.010	3.062	3.067	3.068
0.8	1.303	2.572	2.990	3.041	3.046	3.046
0.9	1.301	2.558	2.968	3.017	3.022	3.023
1.0	1.299	2.542	2.943	2.991	2.997	2.997

different non-local parameters. In this case, the nanorod's length is taken as 10 nm. Based on the results in Table 3, small-scale effects are more significant for the nanorods with hard spring supports.

To illustrate the non-local effect on the axial vibration response, the frequency parameter ratio (FPR) is defined as

$$\text{FPR} = \frac{\text{frequency parameter (non-local theory)}}{\text{frequency parameter (local theory)}} \qquad (38)$$

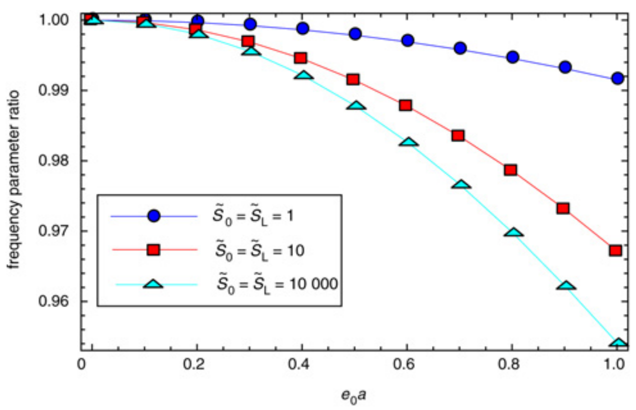


Figure 3 *Small-scale (non-local) effect on nanorods with elastically restrained ends at various spring parameters*

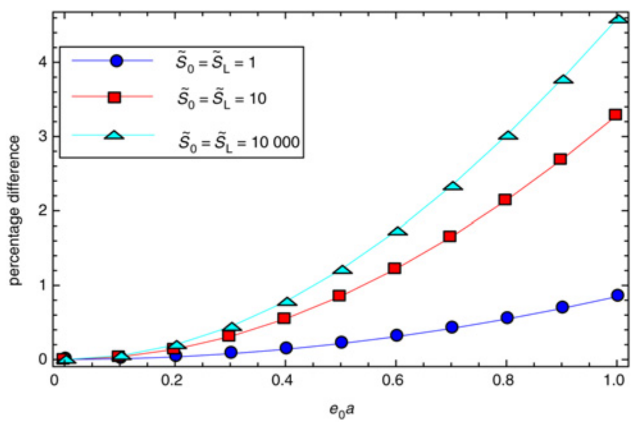
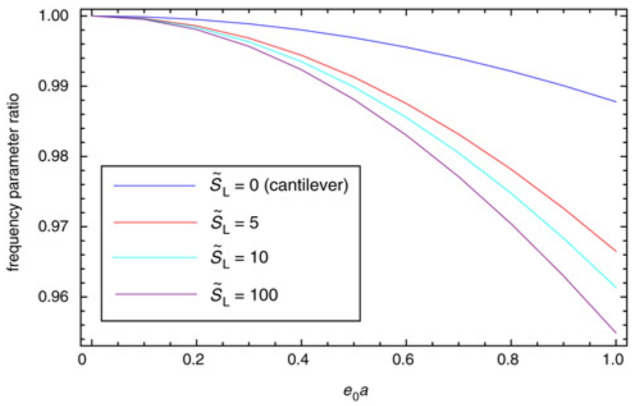


Figure 4 *PD against the non-locality parameter of the nanorod for different spring parameters*

A variation of the first frequency parameter ratio with the non-locality parameter is given for different spring parameters in Fig. 3. It is seen that axial frequency parameters calculated using the non-local elasticity theory are always smaller than when using classical ones. The frequency ratio decreases as the non-locality parameter increases.

To assess the small-scale effect on the axial vibration response of the nanorods for different spring constants, the percentage difference (PD) in the frequency parameter ratio is defined as follows

$$\text{PD} = \frac{\text{FPR (local theory)} - \text{FPR (nonlocal theory)}}{\text{FPR (local theory)}} \qquad (39)$$



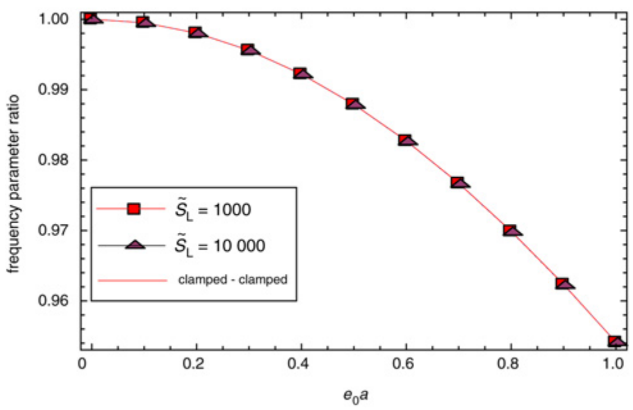


Figure 5 *Small-scale (non-local) effect on clamped-spring nanorods at various non-local parameters for different spring constants*

Table 4 Effect of axial spring and non-local parameter on the frequencies of nanorods for $\tilde{S} = 10\,000$

	$\tilde{S}_L$					C-F ends
e_0a	10 000	1000	100	10	0	Ref. [24]
0	3.140	3.138	3.110	2.863	1.576	1.570
0.1	3.139	3.136	3.108	2.862	1.576	1.570
0.2	3.134	3.131	3.104	2.858	1.576	1.570
0.3	3.127	3.124	3.096	2.852	1.575	1.569
0.4	3.116	3.113	3.086	2.844	1.573	1.567
0.5	3.102	3.100	3.073	2.834	1.572	1.565
0.6	3.086	3.083	3.057	2.821	1.569	1.563
0.7	3.067	3.065	3.039	2.807	1.567	1.561
0.8	3.046	3.043	3.018	2.791	1.564	1.558
0.9	3.022	3.020	2.995	2.772	1.561	1.555
1.0	2.997	2.994	2.969	2.752	1.557	1.551

doi: 10.1049/mnl.2014.0398

In Fig. 4, the variation of PD with the non-locality parameter is depicted. It is seen that the non-local effects are more significant for large spring parameters when compared with small ones.

In Table 4, variation of the first frequency parameter of the nanorod is given for constant values of $\tilde{S}_0 = 10000$ and different values of $\tilde{S}_L$, (e_0a). Increases are observed for the first frequency parameter with increasing $\tilde{S}_L$. Sensitivity of the first frequency parameter is more significant for larger (e_0a) values. It is concluded that the hard elastically restrained end is more affected from small-scale (non-locality) parameters.

The first frequency parameter ratio against the non-locality parameter of the nanorod is given in Fig. 5 for different spring parameters of $\tilde{S}_L$. According to this Figure, it is seen that approximately for $\tilde{S}_L \geq 1000$ all results converge to the CC case. It means that when $\tilde{S}_L$ is large, the vibrational results are very close to those calculated for the CC case.

6. Conclusion: On the basis of the non-local elasticity theory, the free axial vibrational response of nanorods under various boundary conditions has been investigated. A unified analytical method has been developed, which can be used for a nanorod with any types of boundary conditions. The general frequency determinant is obtained by a combination of the Fourier series expansion and Stokes' transformation. The main advantage of this determinant is the capability of considering each combination of boundary conditions. The validity of this method is established for CC and clamped-free boundary conditions. The influence of the non-locality parameter and axial springs on the natural frequencies is examined in some numerical examples. The non-local effects are significant when the nanorod length to the length scale parameter ratio is a small quantity. On the basis of the results of this study, small-scale effects are more significant for the CNTs (nanorods) with hard elastic restraints. It is suggested that by controlling the spring parameters and natural frequencies, the structure of nanotubes can be produced for nanosized devices. The non-local (small scale) parameter should also be considered in the free axial vibration analysis of nanorods with restrained boundary conditions.

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