

# Longitudinal vibration of nanorods embedded in an elastic medium with elastic restraints at both ends

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Longitudinal vibration of nanorods embedded in an elastic medium with elastic restraints at both ends is studied based on the non-local elasticity theory. Using Fourier sine series and Stokes' transformation, a coefficient matrix is obtained. It is very useful for calculating the vibrational frequencies of a nanorod with any type of boundary condition (rigid or restrained). Finally, carrying out some numerical computations, the effects of the elastic medium, non-local parameters and elastic restraints at both ends on the values of vibrational frequencies have been determined. The numerical results are validated through comparison of calculated values with those in the literature.

**1. Introduction:** Nanomaterials are materials possessing grain sizes on the order of a billionth of a metre. They reveal extremely fascinating and attractive properties which can be exploited for a variety of structural and non-structural applications. Since nanomaterials own distinctive, advantageous physical, chemical and mechanical properties, they can be used for a wide variety of applications [1].

Invention of carbon nanotubes initiated a new era in the nanoworld [2, 3]. They are the one of the most important research topics in engineering and material sciences in the last two decades [4]. Nanostructures such as nanotubes, nanowires, nanorods and nanobelts have extraordinary mechanical, thermal, electrical and magnetic properties. One-dimensional nanostructures are characterised by their aspect ratio (generally the aspect ratio of a nanorod is smaller than 20 and the aspect ratio of a nanowire is larger than 20) [5]. In particular, one-dimensional nanostructures, such as nanowires or nanorods, have attracted remarkable attentions due to their excellent optical or electrical properties for potential applications in nanoelectronic, nanooptical and nanomechanical devices [6–8].

In the past decades, many physical problems have been formulated into non-local continuum mechanics [9–13]. Longitudinal vibration response of nanorods with rigid non-local boundary conditions has been determined by Aydogdu [14, 15]. Although dynamic analysis of nanorods with rigid boundary conditions is a widely studied topic, there are only few papers that exist in the literature pertaining to the analysis of nanorods with elastically restrained ends. In this Letter, longitudinal vibration of nanorods embedded in an elastic medium with elastic restraints at both ends is studied by using the non-local elasticity theory. It is desired to present a more general procedure for arbitrary boundary conditions rather than investigate a specific problem.

## 2. Formulations

2.1. Nanorod embedded in elastic medium: The governing differential equation of motion for a nanorod embedded in elastic medium [15] is

$$EA \frac{\partial^2 u(x, t)}{\partial x^2} - \left\{ 1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2} \right\} m \frac{\partial^2 u(x, t)}{\partial t^2} - k_w u(x, t) + k_w (e_0 a)^2 \frac{\partial^2 u(x, t)}{\partial x^2} = 0, \quad (1)$$

where  $E$  is the modulus of elasticity,  $A$  is the cross-sectional area of the nanorod,  $u(x, t)$  is the axial displacement,  $m$  is the mass per unit

length,  $k_w$  is foundation modulus,  $a$  is an internal characteristic length and  $e_0$  is a constant. Assuming harmonic vibrations,  $u(x, t)$  can be represented as

$$u(x, t) = \Gamma(x) \cos(\omega t), \quad (2)$$

where  $\omega$  is the natural frequency and  $\Gamma(x)$  denotes the modal displacement function. The modal displacement function is described in three separate regions as follows

$$\Gamma_0, \quad x = 0, \quad (3)$$

$$\Gamma_L, \quad x = L, \quad (4)$$

$$\Gamma(x) = \sum_{j=1}^{\infty} A_j \sin\left(\frac{j\pi x}{L}\right), \quad 0 < x < L \quad (5)$$

with

$$A_j = \frac{2}{L} \int_0^L \Gamma(x) \sin\left(\frac{j\pi x}{L}\right) dx, \quad (6)$$

2.2. Stokes' transformation: Termwise differentiation of (5) yields

$$\Gamma'(x) = \sum_{j=1}^{\infty} \frac{j\pi}{L} A_j \cos\left(\frac{j\pi x}{L}\right), \quad (7)$$

Equation (7) can be represented by a Fourier cosine series

$$\Gamma'(x) = \frac{b_0}{L} + \sum_{k=1}^{\infty} b_k \cos\left(\frac{j\pi x}{L}\right), \quad (8)$$

The coefficients in (8) are given by

$$b_0 = \frac{2}{L} \int_0^L \Gamma'(x) dx = \frac{2}{L} [\Gamma(L) - \Gamma(0)], \quad (9)$$

$$b_j = \frac{2}{L} \int_0^L \Gamma'(x) \cos\left(\frac{j\pi x}{L}\right) dx, \quad j = 1, 2, \dots \quad (10)$$

By applying an integration by parts to (10), we obtain

$$b_j = \frac{2}{L} \left[ \Gamma(x) \cos\left(\frac{j\pi x}{L}\right) \right]_0^L + \frac{2}{L} \left[ \frac{j\pi}{L} \int_0^L \Gamma(x) \sin\left(\frac{j\pi x}{L}\right) dx \right] \quad (11)$$

then

$$b_j = \frac{2}{L} [(-1)^j \Gamma(L) - \Gamma(0)] + \frac{j\pi}{L} A_j \quad (12)$$

The higher order derivatives of  $\Gamma(x)$  can be determined by employing Stokes transformation as follows

$$\frac{d\Gamma(x)}{dx} = \frac{\Gamma_L - \Gamma_0}{L} + \sum_{j=1}^{\infty} \cos(\beta_j x) \left( \frac{2((-1)^j \Gamma_L - \Gamma_0)}{L} + \beta_j A_j \right) \quad (13)$$

$$\frac{d^2\Gamma(x)}{dx^2} = - \sum_{j=1}^{\infty} \beta_j \sin(\beta_j x) \left( \frac{2((-1)^j \Gamma_L - \Gamma_0)}{L} + \beta_j A_j \right) \quad (14)$$

where

$$\beta_j = \frac{j\pi}{L} \quad (15)$$

By substituting (2), (5) and (14) in (1), the Fourier coefficient  $A_j$  is written in terms of  $\Gamma_0$  and  $\Gamma_L$  as follows

$$A_j = \frac{2\pi j(\gamma^2(-\lambda^2) + \gamma^2 K + 1)(\Gamma_0 - (-1)^j \Gamma_L)}{-\lambda^2 + K^2 + \pi^2 \gamma^2 K j^2 - \pi^2 \gamma^2 \lambda^2 j^2 + \pi^2 j^2} \quad (16)$$

where

$$\gamma = \frac{e_0 a}{L} \quad (17)$$

$$K = \frac{k_w L^2}{EA} \quad (18)$$

$$\lambda^2 = \frac{m \omega^2 L^2}{EA} \quad (19)$$

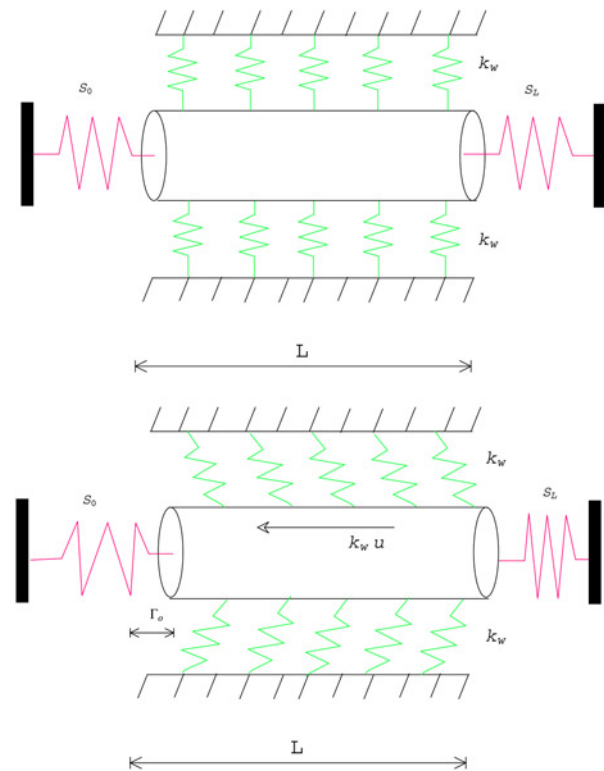
The axial displacement function for the longitudinal vibration of a nanorod embedded in elastic medium becomes

$$u(x, t) = \sum_{j=1}^{\infty} \frac{2\pi j(\gamma^2(-\lambda^2) + \gamma^2 K + 1)(\Gamma_0 - (-1)^j \Gamma_L)}{-\lambda^2 + K^2 + \pi^2 \gamma^2 K j^2 - \pi^2 \gamma^2 \lambda^2 j^2 + \pi^2 j^2} \times \cos(\omega t) \sin(\beta_j x) \quad (20)$$

This is the more general axial displacement function including the non-local and elastic medium effects.

### 3. General frequency determinant in non-local elasticity

3.1. Non-local boundary conditions: To increase the strength of material, nanorods are usually embedded in an elastic medium, and this elastic medium has strong effect on mechanical behaviour of nanorods (see Fig. 1). To compute the frequency parameter in (19), non-local boundary conditions including elastic restraints at both ends should be given.



**Fig. 1** Schematic of the nanorod embedded in an elastic medium with elastic restraints at both ends

Thus, the non-local boundary conditions are defined as follows [14]

$$EA \frac{\partial u}{\partial x} + (e_0 a)^2 m \frac{\partial^3 u}{\partial x \partial t^2} = s_0 \Gamma_0, \quad x = 0 \quad (21)$$

$$EA \frac{\partial u}{\partial x} + (e_0 a)^2 m \frac{\partial^3 u}{\partial x \partial t^2} = -s_L \Gamma_L, \quad x = L \quad (22)$$

where  $s_0$  and  $s_L$  are the stiffnesses of the springs at the ends.

3.2. Coefficient matrix: The substitution of (13) and (16) into (21) and (22) leads to the two homogeneous simultaneous equations (see (23) and (24))

where

$$S_0 = \frac{s_0 L}{EA}, \quad (25)$$

$$S_L = \frac{s_L L}{EA} \quad (26)$$

$$\left( \beta^2 \lambda^2 - S_0 - 1 + \sum_{j=1}^{\infty} \frac{2(-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \right) \Gamma_0 + \left( 1 - \beta^2 \lambda^2 - \sum_{j=1}^{\infty} \frac{2(-1)^j (-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \right) \Gamma_L = 0 \quad (23)$$

$$\left( 1 - \beta^2 \lambda^2 - \sum_{j=1}^{\infty} \frac{2(-1)^j (-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \right) \Gamma_0 + \left( \beta^2 \lambda^2 - S_L - 1 + \sum_{j=1}^{\infty} \frac{2(-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \right) \Gamma_L = 0 \quad (24)$$

and the following eigenvalue problem is obtained to be solved for the constants ( $\Gamma_0, \Gamma_L$ )

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \Gamma_0 \\ \Gamma_L \end{bmatrix} = 0 \quad (27)$$

where

$$\Phi_{11} = \beta^2 \lambda^2 - S_0 - 1 + \sum_{j=1}^{\infty} \frac{2(-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \quad (28)$$

$$\Phi_{12} = 1 - \beta^2 \lambda^2 - \sum_{j=1}^{\infty} \frac{2(-1)^j (-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \quad (29)$$

$$\Phi_{21} = 1 - \beta^2 \lambda^2 - \sum_{j=1}^{\infty} \frac{2(-1)^j (-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \quad (30)$$

$$\Phi_{22} = \beta^2 \lambda^2 - S_L - 1 + \sum_{j=1}^{\infty} \frac{2(-\beta^2 \lambda^4 + \lambda^2 + \beta^2 \lambda^2 K - K)}{-\lambda^2 + \pi^2 \beta^2 K j^2 + K - \pi^2 \beta^2 \lambda^2 j^2 + \pi^2 j^2} \quad (31)$$

Free vibration frequencies depend on the eigenvalues of the coefficient matrix in (27)

$$\det[\Phi_{\nu}] = 0 \quad (\tau, \nu = 1, 2) \quad (32)$$

This determinant may not provide a closed-form solution but it is very useful for calculating the vibration response of a nanorod with rigid or restrained boundary conditions. It should be noted that coefficient matrix of without elastic medium case can be obtained by setting the  $K$  parameter equal to zero [16].

#### 4. Results and discussions

**4.1. Validation of proposed method:** To validate the proposed method and in order to demonstrate its capability in predicting vibration response of nanorods embedded in an elastic medium with elastic restraints at both ends, the results obtained from the present method are compared with those available in the literature. Two different boundary conditions are considered for comparison.

$S_0 = 10,000$  and  $S_L = 10,000$  are chosen for clamped-clamped case. A classical nanorod model with  $\beta = 0$  is considered in an elastic medium with the reaction modulus parameter of  $K = 0$ . The results of analysis are tabulated in Table 1. As is obvious from Table 1, there is excellent agreement between the results of the classical model and the presented analytical method with  $\beta = 0$ , which confirms the efficiency and accuracy of the proposed analytical method.

**Table 1** Comparison of the frequency parameters of the clamped-clamped nanorod without an elastic medium obtained using classical elasticity theory

Mode number	Clamped-clamped		$S_0 = S_L = 10,000$
	Aydogdu [14] $\lambda_i$	Kumar and Sujith [17] $\lambda_i$	Present $\lambda_i$
1	3.141	3.141	3.1409
2	6.284	6.283	6.2819
3	9.425	9.424	9.4229

**Table 2** Comparison of the frequency parameters of the cantilever nanorod without an elastic medium obtained using classical elasticity theory

Mode number	Clamped-free		$S_0 = 10,000$ , $S_L = 0$ Present $\lambda_i$
	Aydogdu [14] $\lambda_i$	Kumar and Sujith [17] $\lambda_i$	
1	1.571	1.570	1.5769
2	4.712	4.712	4.7309
3	7.854	7.850	7.8848

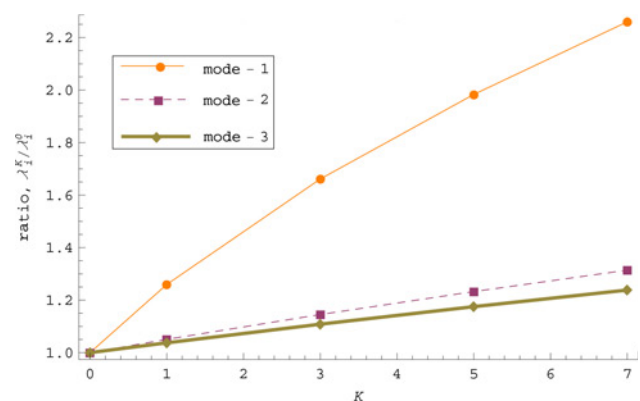
In the second verification, a cantilever nanorod with  $\beta = 0$  non-local parameter and  $K = 0$  elastic medium parameter are assumed. The elastic restraints  $S_0$  and  $S_L$  are taken as 10,000 and 0, respectively. The problem is solved by the proposed analytical method. As obvious from values shown in Table 2, results of the proposed method are closer to the results of the separation of variables method. It should be noted that by increasing the number of terms in series, its accuracy can be improved. In this work, first 50 terms of infinite series are used.

**4.2. Example 1: effect of elastic medium:** Making use of the parametric analysis presented in the previous section and in order to assess the effects of elastic medium parameter, a numerical study is carried out as following. Spring parameters are taken as  $S_0 = S_L = 1$ . Ratio between with and without elastic foundation is defined as follows

$$\text{Ratio} = \frac{\lambda_i^K}{\lambda_i^0} \quad (33)$$

The index  $K$  denotes the nanorod embedded in an elastic medium,  $i$  denotes the mode number and 0 denotes the without elastic medium case. Fig. 2 shows the comparison of first three frequencies for a non-local parameter of  $\beta = 0.05$ . The presence of elastic medium parameter influences the first three modes. It can be seen that by increasing the elastic medium parameter, the ratio defined in (33) increases. It is more significant for the first mode.

**4.3. Example 2: effect of non-local parameter:** The material properties used in this case are identical to the previous example. In this case, Fig. 2 show that  $K = 0.5$ , elastic spring parameters of  $S_0 = S_L = 1, 3, 5$  and 7. The first frequency ratio ( $\lambda_1^{NL}/\lambda_1^L$ ) is shown in Fig. 3. The index NL denotes the non-local elasticity and L denotes the case local elasticity. Non-local first frequencies are lower than the classical first frequencies. The results show



**Fig. 2** Effect of elastic medium

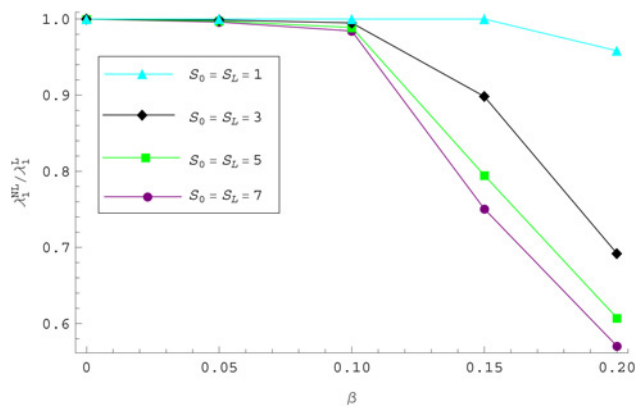


Fig. 3 Effect of non-local parameter

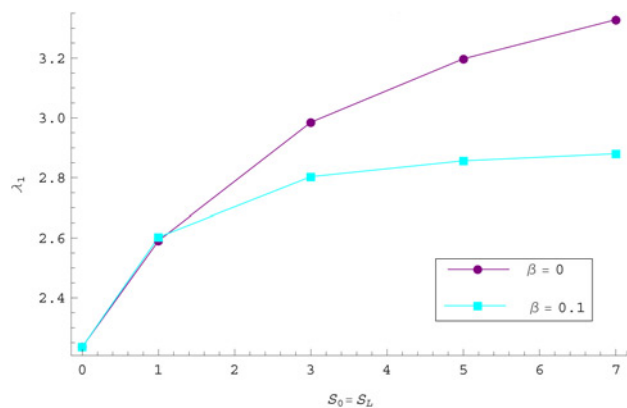


Fig. 4 Effect of elastic restraints at both ends

that small-scale effects should be considered for nanorods with elastically restrained ends. It is clearly shown that hard springs are more sensitive to non-local effects.

4.4. Example 3: effect of elastic restraints at both ends: In the last example, we investigate the effect of elastic restraints at both ends. Fig. 4 presents the effect of the elastic restraints on the dynamic behaviour of nanorod with  $K=5$  for various values of the elastic spring parameters ( $S_0=S_L$ ). There is an abrupt change in the vibrational responses when the spring parameters vary from 1 to 3, but before passing ( $S_0=S_L=1$ ) both of the curves (non-local and local) are monotonically increasing. On the basis of the results in this figure, non-local effects are more prominent for the nanorods with large spring constants.

**5. Conclusion:** The dynamic response of a nanorod embedded in an elastic medium with elastic restraints at both ends has been investigated. A Fourier sine series and Stokes' transform are used to obtain the eigenvalue matrix. This matrix gives more flexibility in boundary conditions and it cannot be found anywhere else. Analyses are performed to investigate the effects of various parameters on the dynamic response, and to examine how the elastic medium and spring parameters affect the vibration frequencies of the system.

## 6 References

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