Buckling analysis of a cantilever single-walled carbon nanotube embedded in an elastic medium with an attached spring

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Published in Micro & Nano Letters; Received on 4th November 2016; Revised on 10th December 2016; Accepted on 13th December 2016

The buckling analysis of a cantilever single-walled carbon nanotube embedded in an elastic medium with an attached spring is researched. The effects of axial compression load, attached spring, small size and surrounding elastic medium are taken into account at the same time. Theoretical formulation is carried out on the basis of the Bernoulli–Euler beam theory in conjunction with Eringen's non-local elasticity theory. Fourier sine series is selected for the simulation of single-walled carbon nanotube deflections. Winkler elastic foundation type is used to simulate the interaction between single-walled carbon nanotube and elastic medium. 2×2 coefficient matrix is derived with the aid of applying Stokes' transformation to corresponding non-local boundary conditions. The critical buckling loads are calculated by using this coefficient matrix. Different validation studies are performed to endorse and corroborate the usefulness of the presented analytical method.

1. Introduction: Single-walled carbon nanotubes have aroused great interest in the scientific community because of their exceptional electrochemical, electronic, mechanical, and thermal properties [1]. They are thin and long cylinders of molecules composed of atoms in a periodic hexagonal arrangement and they have been widely used in nanomechanical systems. Much effort has been recently devoted to the study of the various aspects of carbon nanotubes such as mechanical, bending, buckling, electrical, and chemical properties. Study of buckling and vibrational behaviour of single-walled carbon nanotubes is of practical interest for better understanding of stability and dynamical responses of single-walled carbon nanotubes [2, 3].

The experimental studies reveal that classical elasticity models do not have the ability to predict the size-dependent mechanical behaviour of nanosized structures. It has been seen in some experimental results that the structures become stiffer in smaller sizes [4–6]. To be able to understand the nanomechanical characteristics of these type structures, different continuum mechanic theories have recently attracted researchers' attention such as couple stress theory [7–9], micropolar theory [10], strain gradient theories [11, 12] and non-local elasticity theory [13, 14]. Several research papers have been published during the last years investigating the higher-order continuum models [15–22].

In non-local continuum theory, the small size effects are captured by assuming that the stress at a point as a function not only of the strains at that point, but also a function of the strains at every points of the domain [14]. This size-dependent elasticity theory is a growing popular technique for modelling nanomechanical behaviour of carbon nanotubes [23, 24]. Pradhan and Murmu [25] have proposed a single non-local beam model to investigate the static and dynamic characteristics of a nanocantilever beam. Reddy and Pang [26] have investigated Timoshenko beam and the Euler–Bernoulli theories using the non-classical constitutive relations of Eringen and Edelen [14]. Dynamical characteristics of carbon nanotubes embedded in an elastic medium have been examined by some of the researchers like [27, 28]. The static behaviours of single-walled and multi-walled carbon nanotubes have been investigated by some of the researchers in [29, 30]. Some researchers have considered free vibration behaviour of single-walled carbon nanotubes like [31–33].

In this work, a stability model is proposed to analyse the critical buckling loads of single-walled carbon nanotube embedded in an elastic medium with an attached spring using Eringen's non-local elasticity theory and Euler–Bernoulli theory. A Winkler foundation model is assumed for simulating the interaction of the single-walled carbon nanotube and the elastic medium. A coefficient matrix is calculated with the aid of applying Stokes' transformation to corresponding non-local boundary conditions. The influence of non-local effects, attached spring and Winkler modulus parameter on the critical buckling load is investigated and discussed.

2. Theoretical formulation of the research problem: This theoretical formulation is carried out on the basis of the Bernoulli–Euler beam theory in conjunction with the non-local elasticity theory of Eringen and Edelen [14]. The constitutive equation is represented by the following relation:

$$(1-\mu\nabla^2)\sigma^{\mathrm{nl}} = \sigma^{\mathrm{l}}, \tag{1}$$

where μ is the non-local parameter which is a factor to consider the effect of small length scale; σ^{l} is the local stress tensor. σ^{nl} denotes the non-local tensor related to strain

$$\sigma^{\mathrm{l}} = \kappa(x):\varepsilon(x), \tag{2}$$

where $\kappa(x)$ is the fourth-order tensor of elasticity. The : symbol denotes the double dot product. $\varepsilon(x)$ denotes the deformation. The following equilibrium equations in terms of the lateral deflections can be written:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = k_{\mathrm{w}}w, \tag{3}$$

$$V - \frac{\mathrm{d}M}{\mathrm{d}x} + P\frac{\mathrm{d}w}{\mathrm{d}x} = 0, \tag{4}$$

where V represents the shear force and M is the bending moment. P is the in-plane axial load. k_{w} represents the constant of the foundation, known as Winkler's constant. w denotes the lateral displacement of the carbon nanotube. The constitutive relation in non-local elasticity is given by

$$M - \mu\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = -EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}. \tag{5}$$

Consequently substituting (1) and (2) in shear force and bending moment is written as

$$V=-EI\frac{d^3w}{dx^3}+\mu\left[P\frac{d^3w}{dx^3}+k_w\frac{dw}{dx}\right]-P\frac{dw}{dx}, \tag{6}$$

$$M=-EI\frac{d^2w}{dx^2}+\mu\left[P\frac{d^2w}{dx^2}+k_w w\right]. \tag{7}$$

Further considering (6) and (7), the fourth-order governing differential equation of a single-walled carbon nanotube embedded in the elastic medium is given by [34, 35]

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}-P\mu\frac{d^4w}{dx^4}-\mu k_w\frac{d^2w}{dx^2}+k_w w=0. \tag{8}$$

3. Deformable boundary conditions: In the case of the deformable boundary conditions, the analytical solution of (8) is difficult to obtain, so Fourier series expansion together with Stokes transformation will be adopted in this Letter for the solution of governing equation. The displacement function is described here in three separate regions, two for boundary points and the other for the intermediate places between the boundary points

$$w(x)=\delta_0 \quad x=0, \tag{9}$$

$$w(x)=\delta_L \quad x=L, \tag{10}$$

$$w(x)=\delta(x)=\sum_{k=1}^{\infty}C_k\sin\left(\frac{k\pi x}{L}\right) \quad 0<x<L, \tag{11}$$

with

$$C_k=\frac{2}{L}\int_0^L\delta(x)\sin\left(\frac{k\pi x}{L}\right)dx. \tag{12}$$

It can be clarified that the two boundary points, $x=L$ and $x=0$, have been excluded in (11) since the Fourier series cannot converge to the true displacement values in the boundary points (cantilever with end spring). Equations (9) and (10) allow freedom in choosing the displacement function. In the contrast to rigid supported non-local beam models, this method can model nanostructures having any arbitrary restrained boundary conditions. Term wise differentiation of (11) yields

$$\delta'(x)=\sum_{k=1}^{\infty}\frac{k\pi}{L}C_k\cos\left(\frac{k\pi x}{L}\right), \tag{13}$$

$\delta'(x)$ can be represented by cosine series

$$\delta'(x)=\frac{b_0}{L}+\sum_{k=1}^{\infty}b_k\cos\left(\frac{k\pi x}{L}\right). \tag{14}$$

The (b_0, b_k) coefficients in are written conveniently as follows:

$$b_0=\frac{2}{L}\int_0^L\delta'(x)dx=\frac{2}{L}[\delta(L)-\delta(0)], \tag{15}$$

$$b_k=\frac{2}{L}\int_0^L\delta'(x)\cos\left(\frac{k\pi x}{L}\right)dx \quad n=1,2,\ldots, \tag{16}$$

by applying integration by parts

$$b_k=\frac{2}{L}\left[\delta(x)\cos\left(\frac{k\pi x}{L}\right)\right]_0^L+\frac{2}{L}\left[\frac{k\pi}{L}\int_0^L\delta(x)\sin\left(\frac{k\pi x}{L}\right)dx\right], \tag{17}$$

$$b_k=\frac{2}{L}\left[(-1)^k\delta(L)-\delta(0)\right]+\frac{k\pi}{L}C_k. \tag{18}$$

The above transformation procedure is known as Stokes' transformation. Then the first derivative of lateral deflection function is calculated as follows:

$$\frac{d\delta(x)}{dx}=\frac{\delta_L-\delta_0}{L}+\sum_{k=1}^{\infty}\cos(\alpha_k x)\left(\frac{2\left((-1)^k\delta_L-\delta_0\right)}{L}+\alpha_k C_k\right). \tag{19}$$

The higher-order derivatives of lateral deflection function $\delta(x)$ can be separately obtained by employing Stokes' transformation as follows [24]:

$$\frac{d^2\delta(x)}{dx^2}=-\sum_{k=1}^{\infty}\alpha_k\sin(\alpha_k x)\left(\frac{2\left((-1)^k\delta_L-\delta_0\right)}{L}+\alpha_k C_k\right), \tag{20}$$

$$\frac{d^3\delta(x)}{dx^3}=\frac{\delta_{L''}-\delta_{0''}}{L}+\sum_{k=1}^{\infty}\cos(\alpha_k x)\left(\frac{2\left((-1)^k\delta_{L''}-\delta_{0''}\right)}{L}-\alpha_k^2\left(\frac{2\left((-1)^k\psi_L-\psi_0\right)}{L}+\alpha_k C_k\right)\right), \tag{21}$$

$$\frac{d^4\delta(x)}{dx^4}=-\sum_{k=1}^{\infty}\alpha_k\sin(\alpha_k x)\left(\frac{2\left((-1)^k\delta_{L''}-\delta_{0''}\right)}{L}-\alpha_k^2\left(\frac{2\left((-1)^k\delta_L-\delta_0\right)}{L}+\alpha_k C_k\right)\right), \tag{22}$$

where

$$\alpha_k=\frac{k\pi}{L}. \tag{23}$$

The coefficients which simultaneously satisfy the governing equation need to be determined. Therefore, substituting (11), (20) and (22) into (8), the Fourier coefficients C_k can be written in terms of end displacements as follows:

$$C_k=\frac{2\alpha_k\left(P_{k_w,\mu}(\delta_0+(-1)^k\delta_L\right)}{L\left(k_w+P_{k_w,\mu}\alpha_k^2+P_{EI,\mu}\alpha_k^2\right)}+\frac{2\alpha_k(P_{EI,\mu}(\alpha_k^2(\delta_0+(-1)^{1+k}\delta_L))}{L\left(k_w+P_{k_w,\mu}\alpha_k^2+P_{EI,\mu}\alpha_k^2\right)}+\frac{2\alpha_k(P_{EI,\mu}(\delta_{0''}+(-1)^k\delta_{L''})}{L\left(k_w+P_{k_w,\mu}\alpha_k^2+P_{EI,\mu}\alpha_k^2\right)}, \tag{24}$$

where

$$P_{k_w,\mu}=k_w\mu-P, \tag{25}$$

$$P_{EI,\mu}=EI-P\mu. \tag{26}$$

The lateral deflection function for the buckling of a single-walled

doi: 10.1049/mnl.2016.0662

carbon nanotube having no restraints and supports becomes

$$w(x) = \sum_{n=1}^{\infty}\left(\frac{2\alpha_k(P_{k_w,\mu}(\delta_0 + (-1)^k\delta_L)}{L(k_w + P_{k_w,\mu}\alpha_k^2 + P_{EI,\mu}\alpha_k^2)} + \frac{2\alpha_k(P_{EI,\mu}(\alpha_k^2(\delta_0 + (-1)^{1+k}\delta_L))}{L\left(k_w + P_{k_w,\mu}\alpha_k^2 + P_{EI,\mu}\alpha_k^2\right)} + \frac{2\alpha_k(P_{EI,\mu}(\delta_{0''} + (-1)^k\delta_{L''})}{L\left(k_w + P_{k_w,\mu}\alpha_k^2 + P_{EI,\mu}\alpha_k^2\right)}\right) \times \sin(\alpha_n x). \tag{27}$$

The above equation is the more general, fundamental equation describing the lateral deflection.

4. Eigenvalue-based formulation: Consider an embedded cantilever single-walled carbon nanotube with an attached spring at free the end (see Fig. 1). Based on the non-local elasticity, it is then seen that the following force boundary conditions at the ends can be written as:

$$\delta_0 = 0, \ \frac{\mathrm{d}W}{\mathrm{d}x} = 0, \quad x = 0, \tag{28}$$

$$\psi_L\delta_L = EI\frac{\mathrm{d}^3W}{\mathrm{d}x^3}, \ \delta_L = 0, \ x = L, \tag{29}$$

where ψ_L denotes the stiffness of the spring at the free end of the carbon nanotube. After some mathematical manipulations, the substitution of (19), (21) and (24) into (28) and (29) leads to the two homogeneous simultaneous equations

$$\left(\frac{1}{L} + \sum_{k=1}^{\infty}\frac{2(-1)^k k_w L^3}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}\right)\delta_{0''} + \left(-\sum_{k=1}^{\infty}\frac{2L(-1)^k k^2\pi^2(P\mu - EI)}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}\right)\delta_L = 0, \tag{30}$$

$$\left(\frac{\psi_L}{EI} - \sum_{k=1}^{\infty}\frac{2(-1)^k k_w L k^2\pi^2}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}\right)\delta_{0''} + \left(\frac{1}{L} - \sum_{k=1}^{\infty}\frac{2L(-1)^k(k^2P\pi^2 + k_w(L^2 + k^2P\pi^2\mu))}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}\right)\delta_L = 0, \tag{31}$$

where

$$\beta_k = L^4 + L^2k^2\pi^2\mu. \tag{32}$$

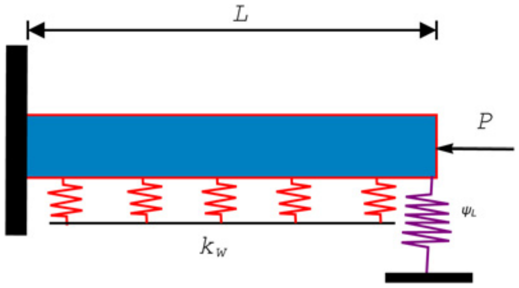

Fig. 1 *Cantilever single-walled carbon nanotube embedded in an elastic medium with an attached spring*

One can obtain the following system of linear equations in matrix form to be solved for the constants (δ_0, δ_L):

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \delta_0 \\ \delta_L \end{bmatrix} = 0, \tag{33}$$

where

$$a_{11} = \frac{1}{L} + \sum_{k=1}^{\infty}\frac{2(-1)^k k_w L^3}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}, \tag{34}$$

$$a_{12} = -\sum_{k=1}^{\infty}\frac{2L(-1)^k k^2\pi^2(P\mu - EI)}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}, \tag{35}$$

$$a_{21} = \frac{\psi_L}{EI} - \sum_{k=1}^{\infty}\frac{2(-1)^k k_w L k^2\pi^2}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}, \tag{36}$$

$$a_{22} = \frac{1}{L} - \sum_{k=1}^{\infty}\frac{2L(-1)^k(k^2P\pi^2 + k_w(L^2 + k^2P\pi^2\mu))}{-L^2k^2P\pi^2 + k^4\pi^4(EI - P\mu) + k_w\beta_k}. \tag{37}$$

Equation (33) defines an eigenvalue problem. The critical buckling loads can be computed by setting the following determinant to zero:

$$\left|a_{ij}\right| = 0 \quad (i, j = 1, 2). \tag{38}$$

Then the characteristic equation in (38) can be solved by assigning the proper values of ψ_L. Present eigenvalue approximations can be degenerate to the rigid supported carbon nanotube (clamped-pinned, clamped-free) in the cases of assigning the infinity and zero values to the attached spring. This is a new eigenvalue approximation for the determination of the critical buckling loads of a cantilever single-walled carbon nanotube embedded in an elastic medium with an attached spring subjected to a comprehensive load. The buckling responses of cantilever non-local beams, whether they have restrained (supported) or free ends, can be predicted using this formula.

5. Results and discussions

5.1. Verification of the present results: The present critical buckling loads of single-walled carbon nanotubes are compared with those of Senthilkumar *et al.* [34] and Pradhan and Reddy [35]. For the comparison purposes, the following quantities are used in computing the numerical results: $E = 1.06$ TPa, $d = 1$ nm, $\psi_L = 10.000$ nN/nm and $k_w = 0$. It should be noted that, by giving larger values to spring coefficient at the end ($\psi_L = 10.000$ nN/nm), (38) will automatically degenerate into the clamped-pinned case. The numerical study has been done and the problem has been solved by using different number of truncated terms. 150 terms of infinite series are used to obtain the accurate results. The critical buckling loads predicted by the present

Table 1 Verification of the proposed method for a clamped-hinged carbon nanotube without elastic medium ($\mu = 0\,\mathrm{nm}^2$)

Length, nm	$P_{(\mathrm{exact})}$, nN Senthilkumar *et al.* [34]	$P_{(\mathrm{DTM})}$, nN Pradhan and Reddy [35]	$P_{(\mathrm{FSE})}$, nN Present
10	9.887	9.887	9.937
12	6.886	6.886	6.901
14	5.044	5.044	5.070
16	3.862	3.862	3.882
18	3.052	3.052	3.067
20	2.472	2.472	2.484

Table 2 Verification of the proposed method for a clamped-hinged carbon nanotube without elastic medium ($\mu = 1\,\text{nm}^2$)

Length, nm	$P_{(exact)}$, nN Senthilkumar *et al.* [34]	$P_{(DTM)}$, nN Pradhan and Reddy [35]	$P_{(FSE)}$, nN Present
10	8.229	8.229	8.292
12	6.023	6.023	6.072
14	4.574	4.574	4.612
16	3.580	3.580	3.611
18	2.873	2.873	2.897
20	2.353	2.353	2.373

Table 3 Verification of the proposed method for a clamped-hinged carbon nanotube without elastic medium ($\mu = 2\,\text{nm}^2$)

Length, nm	$P_{(exact)}$, nN Senthilkumar *et al.* [34]	$P_{(DTM)}$, nN Pradhan and Reddy [35]	$P_{(FSE)}$, nN Present
10	7.048	7.048	7.102
12	5.365	5.365	5.403
14	4.184	4.184	4.216
16	3.337	3.337	3.363
18	2.714	2.714	2.736
20	2.245	2.245	2.264

solutions based on non-local elasticity theory are also provided in Tables 1–3. These tables summarise the achieved results for the buckling of clamped-pinned single-walled carbon nanotubes. Good agreement is observed between the present solutions and the results given in [34, 35]. This clearly shows the reliability of the present solution method for the buckling analysis of carbon nanotubes.

5.2. Numerical results and discussions: Figs. 2–5 demonstrate critical buckling load versus length of single-walled carbon nanotube for various values of elastic medium parameter. For the single-walled carbon nanotube, critical buckling load decrease nonlinearly by increasing of L. As expected, the softening effect of non-local parameter is to decrease the critical buckling load. The increasing value of non-local parameter (μ) leads to an increase the buckling load. It is also seen from these figures that the stiffening effect of Winkler coefficient is to increase the critical buckling load.

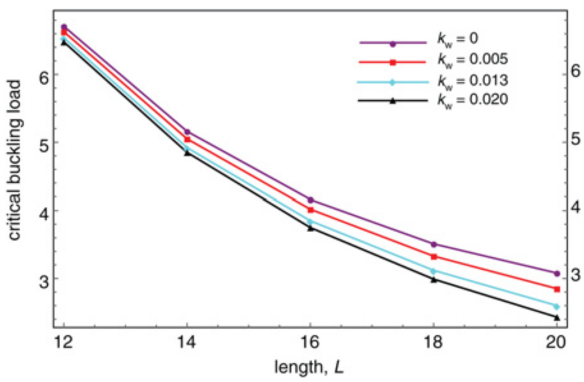

Fig. 2 *Effects of elastic medium parameter for $\mu = 0.5$*

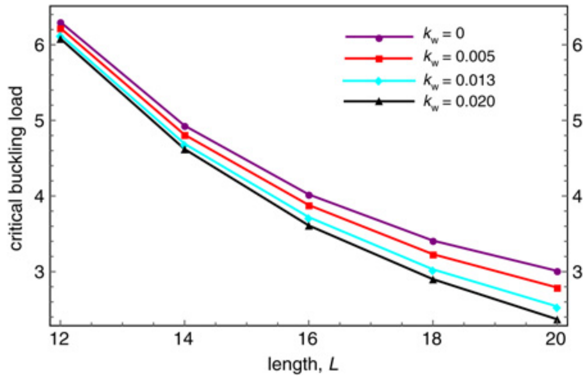

Fig. 3 *Effects of elastic medium parameter for $\mu = 1.0$*

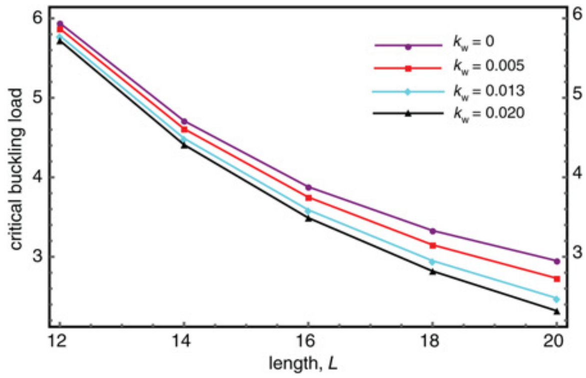

Fig. 4 *Effects of elastic medium parameter for $\mu = 1.5$*

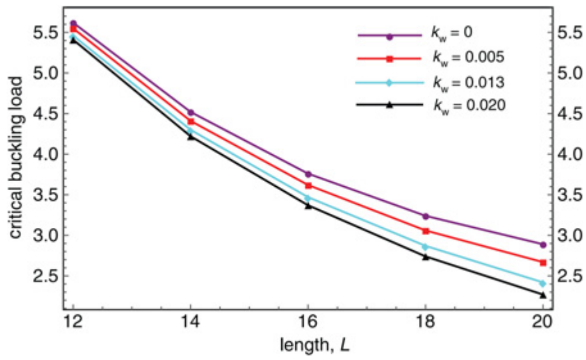

Fig. 5 *Effects of elastic medium parameter for $\mu = 2.0$*

Comparisons between classical and non-local elasticity results are presented in Figs. 6 and 7. The increasing value of non-local parameter leads to a decrease in the magnitude of critical buckling load. It can be concluded from Fig. 7 that if a cantilever carbon nanotube is rested on Winkler's elastic medium with non-local parameter $\mu = 0$, the critical buckling load ratio is constant with respect to the variation of length. The increasing value of the non-local parameter leads to a decrease in the magnitude of critical buckling load.

In fact, this Letter is focused in presenting an efficient analytical method that can model restrained boundary conditions in nanostructures. Also, this method is capable to capture the stability behaviour of cantilever beams by using a zero-stiffness spring at the end. Although the results are presented only for single-walled carbon nanotube embedded in an elastic medium with an attached spring the coefficient matrix in (33) is applicable for carbon nanotubes with different kinds of boundary conditions.

doi: 10.1049/mnl.2016.0662

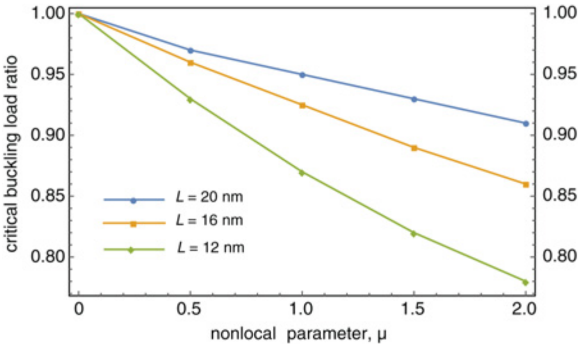

Fig. 6 *Critical buckling load ratio versus non-local parameter*

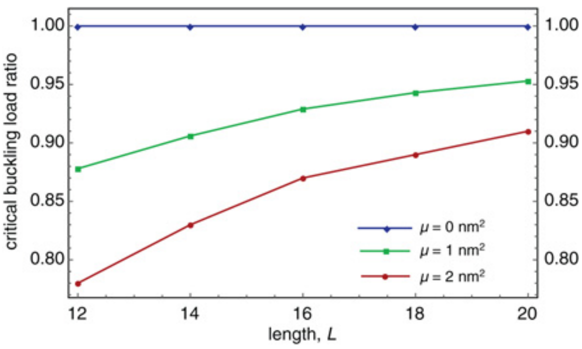

Fig. 7 *Critical buckling load ratio versus length for various non-local parameters*

6. Conclusion: The buckling response of a cantilever single-walled carbon nanotube embedded in an elastic medium with an attached spring is investigated on the basis of non-local elasticity theory in conjunction with Euler–Bernoulli beam theory. A coefficient matrix is derived by implementing Stokes' transformation to non-local boundary conditions. The buckling characteristics of single-walled carbon nanotubes embedded in an elastic medium with an attached spring are investigated. The calculated results are compared with the results of non-local beam theory with no elastic foundation and attached spring effects. A detailed numerical study is performed to determine effects of material length scale parameter, attached spring and Winkler parameter on critical buckling loads of embedded single-walled carbon nanotubes. Size-dependency on buckling behaviour of single-walled carbon nanotubes is more considerable for lower values of length-to-non-local parameter.

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