Free vibration analysis of a single-walled carbon nanotube embedded in an elastic matrix under rotational restraints

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Free vibration analysis of a restrained carbon nanotube in an elastic matrix subjected to rotationally restrained boundary conditions is investigated based on Eringen's non-local elasticity. The analytical solution for free vibration frequencies and corresponding mode shapes of single-walled carbon nanotubes are established. Using Fourier series and Stokes' transformation, a useful coefficient matrix is derived. The present analytical formulation permits to have more efficient coefficient matrix for calculating the vibration frequencies of carbon nanotubes with different boundary conditions (rigid or restrained). The eigen values of this matrix give the vibration frequencies. Comparisons between the free vibration frequency results of the present solutions and previous works in the literature are performed. The calculated results show an excellent agreement with other solutions available in the literature.

1. Introduction: Single-walled carbon nanotubes and nanostructures have attracted attention of researchers because of their superior properties, mechanical, exceptional electrochemical, thermal and electronic properties [1]. Making experiments with nanoscale sized structures is found to be expensive and difficult. Consequently, development of analytical models for carbon nanostructures might be an important research points concerning application of this type of structures.

Nanostructures possess extraordinary thermal, mechanical, chemical and electrical properties that are superior to classical ones. The classical continuum mechanics theory does not take into account the small size effect in nanostructures. The classical elasticity-based models over predict the real responses of nanostructures. So, a new non-classical elasticity model that captures size effect is required. It has been shown in some published papers that the nano-sized structures become stiffer [2–4]. Several researchers have investigated the higher order elasticity theories [5–14]. Different non-classical elasticity theories have attracted researchers attention such as micropolar theory [15], strain gradient theories [16, 17], couple stress theory [18–20] and non-local elasticity theory [21, 22].

In non-local elasticity theory, the size effects are included in the constitutive equation by assuming that the stress is a function of the strains at every points in the explored domain [23]. Wang and Liew [24] have investigated static analysis of nanostructures based on non-local elasticity using Timoshenko and Euler Bernoulli beam theories. Pradhan and Murmu [25] have presented a non-classical beam model and have investigated the vibration analysis of rotating nano-sized cantilevers. Differential quadrature method has been used to calculate the numerical analysis of non-dimensional frequencies. Vibration behaviour of single-walled carbon nanotubes has been explored recently by different researchers using wave propagation approach [26, 27]. Free vibration behaviour and mechanical properties of a single-walled carbon nanotubes have been of interest to scientific researchers due to its practical applications. The static analysis of carbon nanotubes has been performed by some of researchers [28, 29]. Thai [30] has presented a non-local higher order beam model for stability, static and vibration of nano-sized beams using the non-local elasticity theory. Free vibration behaviour of single-walled carbon nanotubes has been investigated by some researchers [31–35].

In this work, an attempt is made to investigate the free vibration of the simply supported single-walled carbon nanotubes with rotational restrained boundary conditions using non-local elasticity theory. Both the Fourier sine series and Stoke transformation have been incorporated in the vibration analysis. The proposed method can be extended to the analysis of carbon nanotubes with rigid boundary conditions. Influence of small-scale effects, rotational restraints and Winkler modulus coefficient on the free vibration frequencies are investigated and discussed.

2. Theoretical formulation: According to Eringen's [22] non-local elasticity theory, the constitutive equation is represented by

$$(1 - \lambda\nabla^2)\sigma^{nl} = \sigma^l, \tag{1}$$

where λ is the small-scale parameter (non-local parameter), σ^l is the local stress tensor and σ^{nl} is the non-local stress related to strain matrix

$$\sigma^l = \eta(x):\epsilon(x), \tag{2}$$

where $\eta(x)$ is the elasticity tensor. The ':' symbol is the double dot product. $\epsilon(x)$ is the deformation. The moment equation in non-local elasticity may be derived by

$$M - \lambda^2\frac{\partial^2 M}{\partial x^2} = -EI\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

It is seen clearly from (3) that small-scale parameter enters into the present problem through the constitutive relation, where w is the lateral deflection. Displacement field in classical elasticity is given by

$$u = -z\frac{\partial w}{\partial x} = 0, \tag{4}$$

$$w = w(x, t) = 0, \tag{5}$$

$$v = 0, \tag{6}$$

where u represents the in-plane axial displacement. The stress–strain equations according to classical beam theory can be written as follows:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \tag{7}$$

$$\sigma_{xx} = -Ez\frac{\partial^2 w}{\partial x^2}, \tag{8}$$

$$M = \int_A z\sigma_{xx}\mathrm{d}x. \tag{9}$$

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By applying Hamilton's principle [36] and taking the first variation of resulting functional, the following equation is derived:

$$\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2} + k_w w(x, t) \tag{10}$$

where k_w is the Winkler coefficient. Bending moment in non-local elasticity can be found as follows:

$$M = \lambda^2 \left(\rho A \frac{\partial^2 w}{\partial t^2} + k_w w(x, t) \right) - EI \frac{\partial^2 w}{\partial x^2} \tag{11}$$

The equation of motion can be obtained by substituting (11) into (3)

$$\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[\lambda^2 \left(\rho A \frac{\partial^2 w}{\partial t^2} + k_w w(x, t) \right) - EI \frac{\partial^2 w}{\partial x^2} \right] - k_w w(x, t) = 0 \tag{12}$$

The above equation is the non-local governing equation of the present carbon nanotube (nanobeam) in terms of the displacements.

3. Fourier series with Stokes transformation: Navier type of solutions can be applicable simply supported boundary conditions. In case of the spring boundary conditions, the solution of (12) is difficult to obtain, so Fourier series expansion together with Stokes transformation will be adopted in this work for the solution of (12). The displacement function is described in three separate regions as follows:

$$W(x, t) = w(x) \cos[\omega t]. \tag{13}$$

$$w(x) = \Delta_0 \quad x = 0, \tag{14}$$

$$w(x) = \Delta_L \quad x = L, \tag{15}$$

$$w(x) = \sum_{n=1}^{\infty} Z_n \sin\left(\frac{n\pi x}{L}\right) \quad 0 < x < L, \tag{16}$$

with

$$Z_n = \frac{2}{L} \int_0^L \Delta(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{17}$$

It should be noted that the points $x = 0$ and $x = L$ are not included in (16) since this function cannot converge to the second derivative of lateral displacement function. Equations (14) and (15) allow freedom in second derivatives of this function. In the contrast to simply supported non-classical beam theories, present analytical method can be used nano-sized structures having different deformable boundary conditions. First derivative of (16) yields

$$\Delta'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} Z_n \cos\left(\frac{n\pi x}{L}\right), \tag{18}$$

($\Delta'(x)$) can be shown by cosine series

$$\Delta'(x) = \frac{b_0}{L} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right). \tag{19}$$

The Fourier coefficients in (19) are written properly as follows:

$$b_0 = \frac{2}{L} \int_0^L \Delta'(x) \mathrm{d}x = \frac{2}{L} [\Delta(L) - \Delta(0)], \tag{20}$$

$$b_n = \frac{2}{L} \int_0^L \Delta'(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad n = 1, 2 \ldots, \tag{21}$$

Applying the partial integration rule

$$b_n = \frac{2}{L} \left[\Delta(x) \cos\left(\frac{n\pi x}{L}\right) \right]_0^L + \frac{2}{L} \left[\frac{n\pi}{L} \int_0^L \Delta(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \right], \tag{22}$$

$$b_n = \frac{2}{L} [(-1)^n \Delta(L) - \delta(0)] + \frac{n\pi}{L} Z_n. \tag{23}$$

This is a mathematical procedure known as Stokes' transformation [37–39]. The following derivatives of lateral displacement function are obtained:

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x} = \frac{\Delta_L - \Delta_0}{L} + \sum_{n=1}^{\infty} \cos(\alpha_n x) \left(\frac{2((-1)^n \Delta_L - \Delta_0)}{L} + \alpha_n Z_n \right), \tag{24}$$

$$\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = -\sum_{n=1}^{\infty} \alpha_n \sin(\alpha_n x) \left(\frac{2((-1)^n \Delta_L - \Delta_0)}{L} + \alpha_n Z_n \right), \tag{25}$$

$$\frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} = \frac{\Delta_L'' - \Delta_0''}{L} + \sum_{n=1}^{\infty} \cos(\alpha_n x) \left(\frac{2((-1)^n \Delta_{LL}'' - \Delta_0'')}{L} \right) - \alpha_n^2 \left(\frac{2((-1)^n \Delta_L - \Delta_0)}{L} + \alpha_n Z_n) \right), \tag{26}$$

$$\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = -\sum_{n=1}^{\infty} \alpha_n \sin(\alpha_n x) \left(\frac{2((-1)^n \Delta_L'' - \Delta_0'')}{L} \right) - \alpha_k^2 \left(\frac{2((-1)^n \Delta_L - \Delta_0)}{L} + \alpha_n Z_n) \right), \tag{27}$$

where

$$\alpha_n = \frac{n\pi}{L}. \tag{28}$$

4. Solution procedure: The Navier's-type solution of the equation of motion for the free vibration of a pinned–pinned nanobeam with rotational restraints is presented. The lateral displacement function is chosen to satisfy the equation of motion and the simply-supported conditions. Substituting (16), (25) and (27) into (12), the unknown Z_n can be found as follows:

$$Z_n = -\frac{2\pi EIL^2 n((-1)^{n+1} \Delta_{L''} + \Delta_{0''})}{L^2 (L^2 + \pi^2 \gamma^2 n^2)(k - A\rho\omega^2) + \pi^4 EIn^4}. \tag{29}$$

The deflection function of a non-local beam having free boundaries becomes

$$w(x) = \sum_{n=1}^{\infty} \left(-\frac{2\pi EIL^2 n((-1)^{n+1} \Delta_{L''} + \Delta_{0''})}{L^2 (L^2 + \pi^2 \gamma^2 n^2)(k - A\rho\omega^2) + \pi^4 EIn^4} \right) \times \sin(\alpha_n x). \tag{30}$$

5. Non-local boundary conditions: Consider a non-local beam with rotational restraints and simply supported boundary conditions (see Fig. 1). Following relations can be written as

$$\Omega_0 \frac{\mathrm{d}w(x)}{\mathrm{d}x} = \lambda^2 \left(\rho A \frac{\partial^2 w}{\partial t^2} + k_w w(x, t) \right) - EI \frac{\partial^2 w}{\partial x^2}, \quad x = 0, \tag{31}$$

$$w = 0, x = 0, \tag{32}$$

$$\Omega_L \frac{\mathrm{d}w(x)}{\mathrm{d}x} = \lambda^2 \left(\rho A \frac{\partial^2 w}{\partial t^2} + k_w w(x, t) \right) - EI \frac{\partial^2 w}{\partial x^2}, \quad x = L, \tag{33}$$

$$w = 0, x = L, \tag{34}$$

where Ω_0 and Ω_L denote the spring coefficient at the ends of carbon nanotube. The substitution of (24) and (29) into (31)–(34) leads two

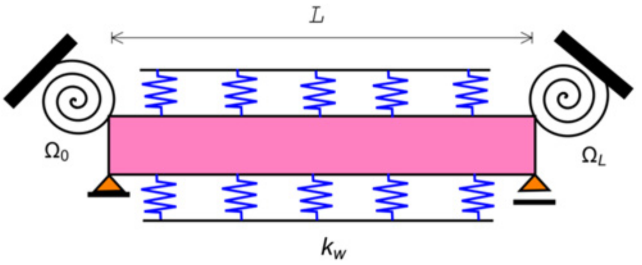

Fig. 1 *Simply supported carbon nanotube embedded in elastic medium with rotational restraints*

simultaneous homogeneous equations as follows:

$$\begin{aligned}&-\left(\sum_{n=1}^{\infty}\frac{2EIL\Omega_0\pi^2n^2}{EI\pi^4n^4+L^2(L^2+\pi^2n^2\lambda^2)(k_{\rm w}-\rho A\omega^2)}\right)\Delta_0''\\&\quad-\left(\sum_{n=1}^{\infty}\frac{2EI(-1)^{1+n}L\Omega_0\pi^2n^2}{EI\pi^4n^4+L^2(L^2+\pi^2n^2\lambda^2)(k_{\rm w}-\rho A\omega^2)}\right)\Delta_L''\\&\quad-EI\Delta_0''=0,\end{aligned} \tag{35}$$

$$\begin{aligned}&-\left(\sum_{n=1}^{\infty}\frac{2EI(-1)^{1+n}L\Omega_L\pi^2n^2}{EI\pi^4n^4+L^2(L^2+\pi^2n^2\lambda^2)(k_{\rm w}-\rho A\omega^2)}\right)\Delta_0''\\&\quad-\left(\sum_{n=1}^{\infty}\frac{2EIL\Omega_L\pi^2n^2}{EI\pi^4n^4+L^2(L^2+\pi^2n^2\lambda^2)(k_{\rm w}-\rho A\omega^2)}\right)\Delta_L''\\&\quad-EI\Delta_L''=0.\end{aligned} \tag{36}$$

Following dimensionless quantities are defined as

$$K=\frac{k_{\rm w}L^4}{EI}, \tag{37}$$

$$R_0=\frac{\Omega_0L}{EI}, \tag{38}$$

$$R_L=\frac{\Omega_LL}{EI}, \tag{39}$$

$$\varpi^4=\frac{\omega^2\rho AL^4}{EI}, \tag{40}$$

$$\gamma^2=\frac{\lambda^2}{L^2}, \tag{41}$$

then following two equations are obtained in dimensionless form:

$$\begin{aligned}&\left(-1-\sum_{n=1}^{\infty}\frac{2R_0\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}\right)\Delta_0''\\&\quad-\left(\sum_{n=1}^{\infty}\frac{2(-1)^{1+n}R_0\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}\right)\Delta_L''=0,\end{aligned} \tag{42}$$

$$\begin{aligned}&\left(\sum_{n=1}^{\infty}\frac{2(-1)^{1+n}R_L\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}\right)\Delta_0''\\&\quad-\left(-1-\sum_{n=1}^{\infty}\frac{2R_L\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}\right)\Delta_L''=0,\end{aligned} \tag{43}$$

where

$$K^n=K+K\pi^2n^2\delta^2. \tag{44}$$

Equations (42) and (43) may be written in a matrix form:

$$\begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix}\begin{bmatrix}\Delta_0''\\ \Delta_L''\end{bmatrix}=0, \tag{45}$$

where

$$\Phi_{11}=-1-\sum_{n=1}^{\infty}\frac{2R_0\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}, \tag{46}$$

$$\Phi_{12}=-\sum_{n=1}^{\infty}\frac{2(-1)^{1+n}R_0\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}, \tag{47}$$

$$\Phi_{21}=-\sum_{n=1}^{\infty}\frac{2(-1)^{1+n}R_L\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}, \tag{48}$$

$$\Phi_{22}=-1-\sum_{n=1}^{\infty}\frac{2R_L\pi^2n^2}{K^n+\pi^4n^4-\varpi^4-\varpi^4\pi^2n^2\gamma^2}. \tag{49}$$

The vibration frequencies can be obtained by following determinant:

$$\left|\Phi_{ij}\right|=0\quad(i,j=1,2). \tag{50}$$

Setting above polynomial to zero gives the dimensionless frequencies.

6. Numerical results and discussions: In this section, free vibration analysis of carbon nanotubes is examined employing present method considering non-local effects. The variations of the first and second frequencies of the carbon nanotubes versus uniform rotational spring parameter rises are, respectively, shown in Figs. 2 and 3 for different non-local parameters ($\lambda/L=0.05$, 0.10, 0.15). It should be noted that rotational spring parameter may lead to nanobeam vibrating at high circular frequencies. It is revealed that for rotational restrained nanobeams, increasing non-local parameter with constant values of elastic medium parameter leads to increase in first two frequencies.

Fig. 4 shows the variation of the first frequency parameter with the mode number for different cases of rotational spring parameter ($R_0=R_L=R=0, 1, 2, 3$). The results for the frequency parameter are in the dimensionless form. From Fig. 5, it can be observed that as the scale coefficient increases frequencies. This implies that for increasing mode number the value of frequency parameter increases.

To illustrate the influence of non-local parameter with different stiffness of the rotational springs on the vibration frequencies,

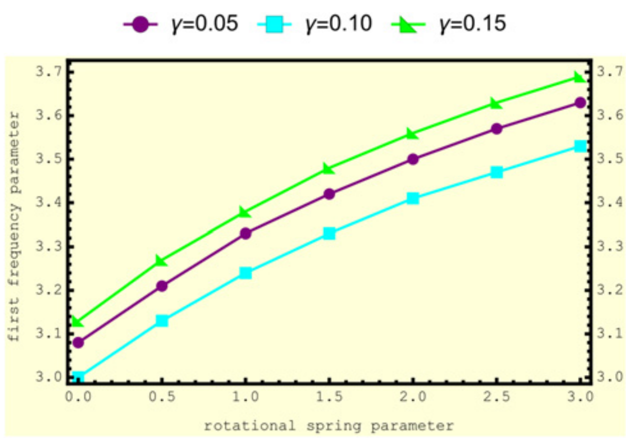

Fig. 2 *Effects of rotational spring parameters on the first frequency parameter for different non-local parameters K = 1.0*

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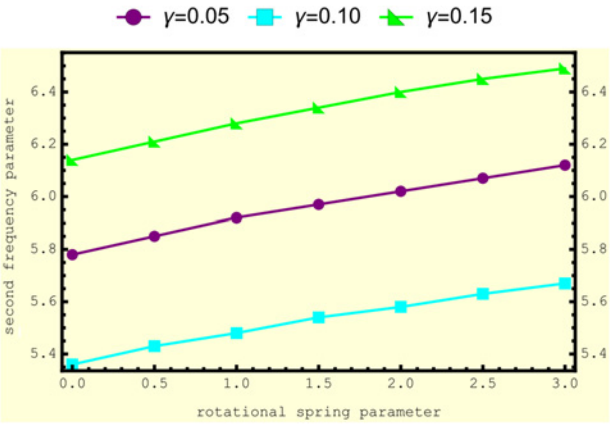

Fig. 3 *Effects of rotational spring parameters on the second frequency parameter for different non-local parameters K = 1.0*

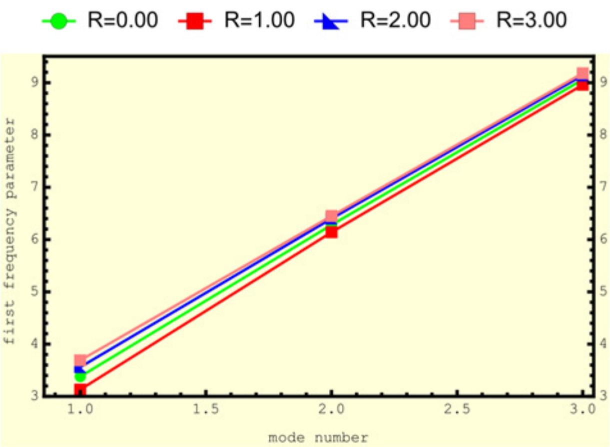

Fig. 4 *Effects of vibration modes on the first frequency parameter K = 1.0*

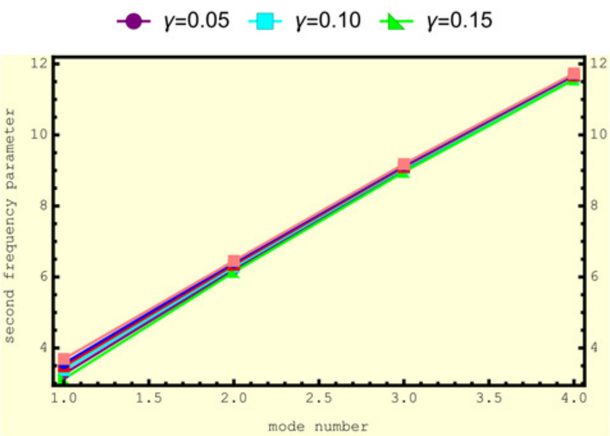

Fig. 5 *Effects of vibration modes on the second frequency parameter K = 1.0*

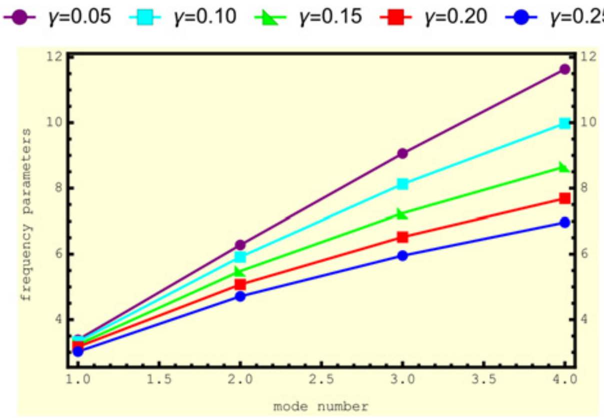

Fig. 6 *Effects of vibration modes on the frequencies for equal rotational spring parameter ($R_0 = R_L = 0.8$), constant elastic medium parameter (K = 1.0)*

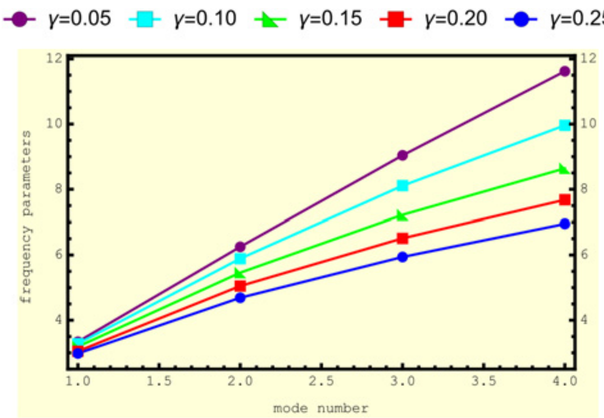

Fig. 7 *Effects of vibration modes on the frequencies for equal rotational spring parameter ($R_0 = R_L = 1.0$), constant elastic medium parameter (K = 1.0)*

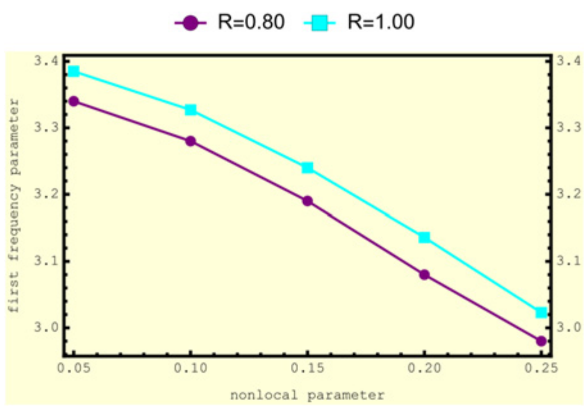

Fig. 8 *Comparison of first frequency parameters with different non-local parameter and constant elastic medium parameter (K = 1.0)*

curves in Figs. 6 and 7 have been plotted for the constant value of elastic spring parameter against the mode number. As the rotational stiffness parameter of the springs increases the frequencies increases.

Fig. 8 shows the variation of first vibration frequency versus the non-local parameter a carbon nanotube with fixed value of $K = 1$. As concluded earlier, it is again seen that smaller natural frequency parameter can be obtained at lower rotational restraint parameters.

7. Conclusion: In this work, application of non-local elastic theory is extended to free vibration behaviour of simply supported carbon nanotubes with rotational restraints. Fourier sine series is selected as a displacement function. An efficient coefficient matrix is derived by using Stoke transformation to non-local boundary conditions. Eigen values of this matrix give the free vibration frequencies of simply supported carbon nanotubes with rotational restraints.

Effect of (i) rotational restraints, (ii) stiffness of Winkler foundation of the single-walled carbon nanotube and (iii) non-local parameter on non-dimensional vibration frequencies is investigated in detail. Size-dependency on free vibration behaviour of the carbon nanotubes is more considerable for higher values of rotational restraint parameters and lower values of length to non-local parameter. It is revealed that for rotational restrained nanobeams, increasing non-local parameter with constant value of elastic medium parameter leads to increase.

8. References

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