Nonlinear microstructure-dependent Bernoulli–Euler beam model based on the modified couple stress theory and finite rotation of section

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Based on Hamilton's principle and the modified couple stress theory, a Bernoulli–Euler microbeam model is developed with the finite rotation of the cross-section. The present model includes three couple-stress-induced nonlinear terms, and these nonlinear terms have a significant influence on the mechanic response of the beam.

1. Introduction: In the microelectromechanical systems, the elements of the structure are in the microsize [1]. In this size, the scale effect will occur and modify macroscopic mechanical properties considerably [2–7]. In fact, the classical continuum mechanics may not be applied to the micro and sub-microscale directly because the theory cannot interpret the small scale effect [3–7]. Therefore, the strain gradient theory and the modified couple stress theory are proposed to consider the scale effect in the constitutive of the continuum. The experiments and theories show that the two theories are successful to deal with the topic at the microscale [7, 8]. On the other hand, the researches of the dynamics of the microbeams are insufficient in comparison with the statics. In most studies of the microbeam, it is supposed that the rotation angles of the cross-section are small and its effect can be neglected [9–11]. The recent development, which the modified couple stress is applied to the microbeams and plates, can be found in [12–18]. In fact, a small but finite bending deformation may bring about a significant rotation angle of the section in the microbeam, and the rotation may significantly influence the mechanic response of the microbeam. In this Letter, a new nonlinear dynamic modal of Bernoulli–Euler microbeam is developed using the modified couple stress theory. The model takes account of the finite bending deformation, the finite section rotation and the scale effect. Then the static bending and the frequencies of the free vibrations will be investigated through the new model.

2. Formulation: The potential density of the modified couple stress theory is given by [6]

$$U = \frac{1}{2}\left[(\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}):\boldsymbol{\varepsilon} + \boldsymbol{m}:\boldsymbol{\chi}\right] \tag{1}$$

Here $\boldsymbol{\sigma}_0$ is the initial stress, and

$$\begin{aligned} &\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon}) + 2G\boldsymbol{\varepsilon}, \quad \boldsymbol{m} = 2Gl^2\boldsymbol{\chi}, \\ &\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{u}\nabla + \nabla\boldsymbol{u} + \nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla) \end{aligned} \tag{2}$$

Here ∇ is the Hamiltonian differential operator. $\boldsymbol{u}$ is the position vector of the beam, and $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors in the directions of x, y and z axes, respectively. $\boldsymbol{\sigma}$ is the stress tensor. $\boldsymbol{\varepsilon}$ is the strain tensor. G, λ are Lame parameters, $G = E/[2(1+\nu)]$, and E, ν are Young's modulus and Poisson ratio, respectively. l is a material length scale parameter acquired from experimentations and the value is about $1-20\,\mu\mathrm{m}$ [6–10]. In fact, it is an open question how to determine the length scale. In most cases, the length scale may be determined using a bend test for a clamped-clamped or cantilever microbeam [19]. Here $\boldsymbol{m}$ is the component of the deviatoric part of the couple stress tensor, and $\boldsymbol{\chi}$ is the symmetric part of the curvature tensor, and are defined as

$$\boldsymbol{\chi} = \frac{1}{2}\left[\nabla\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})^T\right], \quad \boldsymbol{\omega} = \frac{1}{2}\nabla\times\boldsymbol{u} \tag{3}$$

For establishing the equation of motion of the beam, as shown in Fig. 1, three hypotheses are employed as follows: (i) the plane sections perpendicular to the undeformed reference line remain plane and perpendicular to the deformed reference line (Bernoulli–Euler hypothesis); (ii) neglect the longitudinal displacement (This means that the present model may apply to the beam hinged or clamped at two ends, but it cannot be employed to the simply supported or cantilevered beam.); (iii) the cross-section of the beam is symmetrical about the y and z axes. So the displacement field is written as

$$u_1 = -y\sin\theta, \quad u_2 = w(x,t) - y(1-\cos\theta), \quad u_3 = 0 \tag{4}$$

Here θ is the rotation angle of the cross-section, as shown in Fig. 2.
Expanding $\cos\theta$, $\sin\theta$ and keeping up with cubic terms, have

$$u_1 = -y\left[\frac{\partial w}{\partial x} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^3\right], \quad u_2 = w - \frac{y}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{5}$$

Neglecting the nonlinear terms of (5), it becomes classical displacements field [20]. Using the von-Karman strain tensor [20] and keeping up to cubic terms, the axial strain is obtained as

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_2}{\partial x}\right)^2 = -y\left[\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}\right] + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{6}$$

The stress may write as $\sigma_{xx} = E\varepsilon_{xx}$. And have

$$\begin{aligned} &\chi_{xz} = \chi_{zx} = 2\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} - y\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{3}{2}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right], \\ &\chi_{yz} = \chi_{zy} = -\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{7}$$

Using the strain tensor and the curvature tensor, the potential energy of the beam, U, can be calculated. Let $U = U_1 + U_2 - W$. U_1 is the

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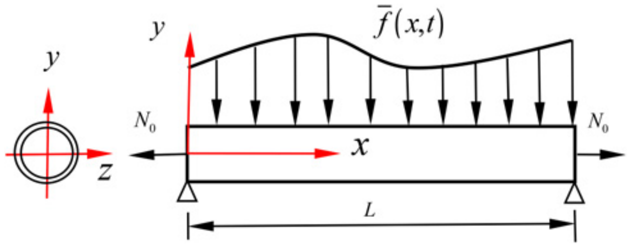

Fig. 1 *Modal of the structure*

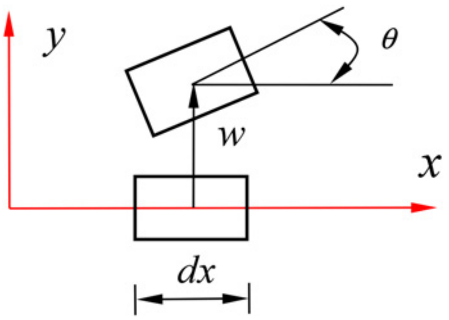

Fig. 2 *Rotation angle of the cross-section*

strain energy, U_2 is the potential energy due to the initial axial load, and the external force potential energy is W. So have

$$U = \int_0^L \int_A \left[\frac{1}{2}E\varepsilon_{xx}^2 + 2Gl^2\left(\chi_{zx}^2 + \chi_{zy}^2\right)\right] \mathrm{d}A\,\mathrm{d}x + \int_0^L \int_A \sigma_{xx}^0 \varepsilon_{xx}\,\mathrm{d}A\,\mathrm{d}x - \int_0^L \bar{f}(x,t) w\,\mathrm{d}x \tag{8}$$

where $\bar{f}(x, t)$ is an external load. The velocity is obtained using the displacement field (5). For a slender microbeam, the radius of gyration of the section is small, so the nonlinear inertia terms are negligible and the kinetic energy is written as

$$\begin{aligned} T &= \frac{m}{2}\int_0^L \left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2\right] \mathrm{d}x \\ &= \frac{m}{2}\int_0^L \left[I\left(\frac{\partial^2 w}{\partial t \partial x}\right)^2 + A\left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}x \end{aligned} \tag{9}$$

Here the equation of motion is established by the Hamiltonian principle. The principle can be described as follows [20]: the actual movement between the initial state at the time t_0 and the final state at a time t takes the least possible value of the Lagrangian action, namely $\delta\int_{t_0}^{t}(T - U)\,\mathrm{d}t = 0$. So the motion equation is obtained as

$$\begin{aligned} & m\frac{\partial^2 w}{\partial t^2} + \left(EI + Gl^2 A\right)\frac{\partial^4 w}{\partial x^4} - N_0\frac{\partial^2 w}{\partial x^2} - \frac{3EA}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} \\ & - \left(2EI + \frac{5Gl^2 A}{4}\right)\left[\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^4 w}{\partial x^4} + \left(\frac{\partial^2 w}{\partial x^2}\right)^3 + 4\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right] = \bar{f} \end{aligned} \tag{10}$$

Here A and I are the area and the moment of inertia of the section, respectively. Equation (10) shows that the scale effect induces three nonlinear terms that come from the finite rotation of the section. However, these terms are neglected in [10–18, 21]. In fact, some nonlinear terms may arise from the scale effect if the transverse shear effect is considered [22]. However, the shear stress is generally ignored in the slender beam [20]. Compared with cantilevered microbeams [23, 24], there is a nonlinear term $(3/2)EA(\partial w/\partial x)^2(\partial^2 w/\partial x^2)$ in (10). This term has remarkable consequence in microbeams for $A \ll I$, as shown in Figs. 3 and 4. A linear modified couple stress model may be obtained by omitting all nonlinear terms of (10). Neglecting the scale-related terms in (10), let $l = 0$, one may obtain a classical dynamic model that is consistent with (4.5.32) in [20] with the longitudinal displacement $u = 0$. When the longitudinal displacement of the beam cannot be ignored, the dynamic model will become nonlinear partial differential equations with two unknown displacement functions. We will research this case in another paper. Besides, the effect of damping is important in the microscale, but it is not clear whether the classical damping model can be applied to the microstructure directly. So the effect of the damping is neglected in this Letter.

Equation (10) may be written in the dimensionless through defining the quantities $(\tilde{x}, \tilde{w}) = (x/L,\ w/L)$, and $\tilde{t} = \bar{\omega}_0 t$. Here $\bar{\omega}_0 = \pi^2\sqrt{(EI)/(mL^4)}$ is the nature frequency of the hinged-hinged beam. Substituting these dimensionless quantities into (10), and drop the cap of $\tilde{x}$, $\tilde{w}$ and $\tilde{t}$, have

$$\begin{aligned} & \frac{\partial^2 w}{\partial t^2} + H_0\frac{\partial^4 w}{\partial x^4} - H_1\frac{\partial^2 w}{\partial x^2} - H_2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} \\ & - H_3\left[\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^4 w}{\partial x^4} + \left(\frac{\partial^2 w}{\partial x^2}\right)^3 + 4\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right] = F \end{aligned} \tag{11}$$

Here

$$H_0 = \frac{EI + Gl^2 A}{mL^4\bar{\omega}_0^2}, \quad H_1 = \frac{N_0}{mL^2\bar{\omega}_0^2},$$
$$H_2 = \frac{3EA}{2mL^2\bar{\omega}_0^2}, \quad H_3 = \frac{8EI + 5Gl^2 A}{4mL^4\bar{\omega}_0^2}, \quad F = \frac{\bar{f}(x,t)}{mL\bar{\omega}_0^2}$$

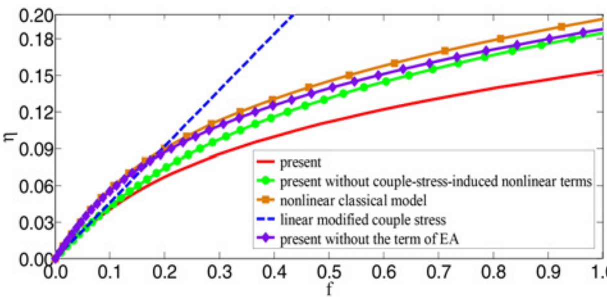

Fig. 3 *Static deformation as a function of the loads*

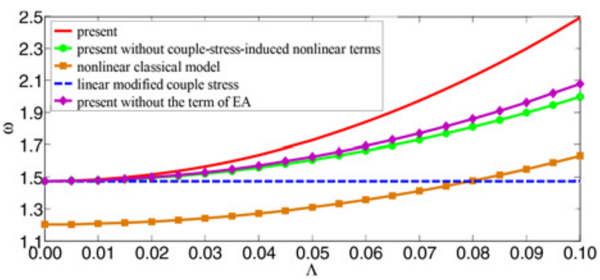

Fig. 4 *Frequencies varying with vibration amplitudes*

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For a hinged-hinged beam, the boundary conditions of (11) are

$$w(0) = w(1) = \frac{\partial^2 w}{\partial x^2}(0) = \frac{\partial^2 w}{\partial x^2}(1) = 0 \tag{12}$$

3. Example and discussion: In this section, the static bending and the frequencies of free vibrations of the hinged–hinged beam is considered. It is difficult to solve the equation analytically, so the Galerkin method is employed to discrete (11) as an ordinary differential equation. Assuming the first-order mode is $\Psi(x) = \sin \pi x$ for a hinged–hinged beam, so have $w = \Psi(x)\eta(t)$. Substituting w into (11) and then multiplying $\Psi(x)$ at both sides of the equation and integrating in [0, 1] (the truncation of the Galerkin method), have

$$\ddot{\eta} + \omega_0^2 \eta + a\eta^3 = f \tag{13}$$

The dot represents the differentiation to dimensionless time t. Equation (13) is a forced Duffing equation and its coefficients are

$$\omega_0^2 = 1 + \frac{\pi^4 GAl^2}{mL^4 \bar{\omega}_0^2} + \pi^2 H_1, \quad a = \frac{\pi^4 H_2 + \pi^6 H_3}{4}, \quad f = \frac{2F}{\pi} \tag{14}$$

Here the physical parameters of the polymeric nanofibres are used as an example. They are $E = 1.44$ GPa, $G = 0.523$ GPa, and density $\rho = 2 \times 10^3\ \text{kg/m}^3$ [19, 25]. It is necessary to note that the length scale is about $l = 0.69\ \mu\text{m}$ in [19], but $l = 17.6\ \mu\text{m}$ in [21]. In this Letter, $l = 10\ \mu\text{m}$ is used. Assuming the cross-section is a tube, its diameter and thickness are $R = 20\ \mu\text{m}$ and $h = 1\ \mu\text{m}$. Other parameters are $L = 100\ \mu\text{m}$, $N_0 = 0.002$ N. Then, there are $A = \pi Rh = 62.8(\mu\text{m})^2$, $I = \pi hR^3/8 \simeq 3.14 \times 10^3(\mu\text{m})^4$, $m = A\rho = 1.256 \times 10^{-7}$ kg/m and $\bar{\omega}_0^2 = 3.50 \times 10^{13}$.

For the static bending, the bending amplitudes as a function of the loads can be obtained by neglecting the inertia term in (13), as shown in Fig. 3. The finite rotation accounts for the nonlinear terms in the displacement field (5), therefore the couple stress induce the three nonlinear terms in (10) and (11). From Fig. 3, it is observed that the couple stress increases the rigidity of the beam. And the nonlinear terms, which are induced by the couple stress, have an increasing influence on the enlargement of bending amplitudes. Comparing the red line and the green line in Fig. 3, the effect of the finite rotation of the section can be found because the finite rotation of the section makes small scale effect appear in the nonlinear terms of (10). Here the model, present without couple-stress-induced nonlinear terms, neglect the nonlinear terms that are induced by both the finite section rotation and couple stress.

Generally, the vibration frequency with cubic nonlinear terms is a function of the vibration amplitudes. Letting $f = 0$ and using the perturbation method in (13), the function can be obtained as [26]

$$\omega = \omega_0 \left(1 + 3a\Lambda^2/8\right) \tag{15}$$

Here Λ is the dimensionless vibration amplitude. Using (15), it is found that the finite rotations make the nonlinear frequencies increase, and this effect will more apparent accompanying with the vibration amplitudes enlargement, as shown in Fig. 4. When the beam appears a big deformation, Figs. 3 and 4 indicate too that the nonlinear terms in the classical model have a more significant influence than the couple-stress-induced linear term. This can be found by comparing the blue line and the yellow line in Figs. 3 and 4. The recent research of the microbeam, which the couple-stress-induced nonlinear terms are neglected, can be found in [27–30].

4. Conclusions: In this Letter, a new nonlinear model of the Bernoulli–Euler microbeam is developed based on the modified couple stress theory. The results show that the nonlinear terms, which are induced by the scale effect and the finite rotation of the section, have significantly influence on the static bending and the vibration frequency of the microbeam.

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