Torsional vibration analysis of nanorods with elastic torsional restraints using non-local elasticity theory

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In this work, torsional vibration of nanorods with torsional elastic boundary conditions is presented via non-local elasticity theory. The present model developed based on non-local elasticity theory gives the opportunity to interpret size effect. Two torsional elastic springs are attached to a nanorod at both ends. A mathematical transformation known as 'Stoke transformation' is utilised to work out the Fourier series for the nanorods with torsional restraints. A coefficient matrix including torsional coefficients is determined by using non-local boundary conditions. A comparison is performed to validate numerical simulations with those given in the literature and the results agree with each other exactly. The non-local effects of torsional end restraints on the free torsional vibration response are investigated for both deformable and rigid boundary conditions.

1. Introduction: Experimental investigations related to nano-sized structures and machines have shown that these type of structures have extremely low weight, high aspect ratio and high stiffness. In recent decades, several scientific investigations have been performed to analyse micro/nano-sized structures and its correlates, using different approaches and definitions [1, 2]. It is agreed that the classical continuum theories are not suitable for modelling of nano-sized structures and machines. In order to design nano-sized structures and machines, size-dependent different elasticity models have been used such as strain and stress type gradient elastic models, peridynamics and modified couple stress theory.

There are different approaches developed for the analyses of sized effect on the mechanical behaviours of nano-sized structures and machines. In recent years, a wide range of higher-order elasticity theories have been utilised for several engineering applications, such as screw dislocations, modelling of micro- or nano-scaled structures, analysis of nanoelectromechanical systems and micro-electromechanical systems, modelling of carbon nanotubes, analysis of ultrathin films and atomic force microscope, and dynamical control for micromachines.

Due to the smooth variation of material scale parameter, the nanorods have many advantages in different areas of application, including enhanced thermal and corrosion resistance. Potential applications of nanorod (carbon nanotube) have been made to various engineering fields on account of its specially properties [3, 4], such as graphene transistors, chemical sensors, field-effect transistor, gas detection, solar cells, logic circuits with filed-effect transistor, diagnosis devices, ultracapacitors, transparent and conductive films, ultrastrength composite materials. Since classical continuum theories cannot predict the characteristic behaviours of nanorods, higher order elasticity theories have been proposed managing to predict mechanical properties of nanorods in recent years [5–10]. Application of non-local elasticity (NE) theory to static analysis of micro and nano-structures has been performed by Wang and Liew [10]. Recently, several higher order elasticity theories have been utilised to help the researchers to understand the effect of the small size [11–19].

Literature review reveals that the conducted theoretical and experimental studies on the torsional free vibration of nanorods are based on the assumptions that the boundary conditions are classical and rigid (fixed-free). Very few studies have been conducted to examine the effects of torsional restraints. The present work is concerned with the derivation of a general eigenvalue solution for the torsional vibration analysis of nanorods modelled as NE theory [20, 21]. This model bridges the gap between rigid and the restrained boundary conditions. A mathematical procedure known as 'Stoke transformation' is employed to work out the Fourier series for the nanorods with general elastic torsional boundary conditions. The direct analytical expressions of the torsional vibrational responses with elastic springs are derived by using the NE theory.

2. NE theory: For isotropic and homogenous elastic solids, the NE theory is described by the following equations [20, 21]:

$$\sigma_{kl,l} + \rho\left(f_l - \frac{\partial^2 \boldsymbol{u}_l}{\partial t^2}\right) = 0 \tag{1}$$

$$\sigma_{kl}(x) = \int_V \alpha\left(\left|x - x'\right|, \chi\right)\tau_{kl}(x')\,\mathrm{d}V(x') \tag{2}$$

$$\tau_{kl}(x') = \lambda\epsilon_{mm}(x')\delta_{kl} + 2\mu\epsilon_{kl}(x') \tag{3}$$

$$\epsilon_{kl}(x') = \frac{1}{2}\left(\frac{\partial u_k(x')}{\partial x'_l} + \frac{\partial \boldsymbol{u}_l(x')}{\partial x'_k}\right) \tag{4}$$

in which, ρ is the mass density of the body, σ_{kl} is the non-local stress tensor, $\boldsymbol{u}_l$ is the displacement vector, f_l is the applied force density, $\varepsilon_{kl}(x')$ is the strain tensor, $\tau_{kl}(x')$ is the Cauchy stress tensor at any point x', V is the volume occupied by the body, t denotes the time, μ and λ are Lame constants, $\alpha\left|x - x'\right|$ is the distance form of Euclidean. $\alpha|x|$ can be displayed by a linear differential operator. This could be shown as the following compact form [20, 21]:

$$\aleph\alpha\left(\left|x - x'\right|\right) = \delta\left(\left|x - x'\right|\right), \tag{5}$$

the following relation can be deduced from (2):

$$\aleph\sigma_{kl} = \tau_{kl} \tag{6}$$

Furthermore, following partial differential equation can be obtained from (1):

$$\tau_{kl,l} + \aleph\left(f_l - \rho\ddot{u}_k\right) = 0 \tag{7}$$

the linear differential operator has been proposed by Eringen given as

$$\aleph = (1 - (e_0 a)^2\nabla^2 = 0 \tag{8}$$

where a denotes internal characteristic length and e_0 is a material constant, ∇^2 is the Laplacian. Then constitutive equation in NE can be written in terms of non-local parameter

$$\left[1-(e_0a)^2\nabla^2\right]\sigma_{kl}=\tau_{kl}. \tag{9}$$

3. Governing equation of torsional vibration: By using the relation in (9), the equation of the motion of the NE theory in terms of the angular rotation is as follows [22]:

$$\begin{aligned} GJ_p\frac{\partial^2\phi(x,t)}{\partial x^2}+(e_0a)\rho J_p^2\frac{\partial^4\phi(x,t)}{\partial x^2\partial t^2}\\ -\rho J_p\frac{\partial^2\phi(x,t)}{\partial t^2}=0, \end{aligned} \tag{10}$$

where ϕ denotes the angular rotation about the centre of twist, J_p is the polar moment of inertia, G is the shear modulus of elasticity

$$G=\frac{E}{2(1+\nu)}, \tag{11}$$

in which, ν is the Poisson's ratio and E is the Young's modulus. Equation (10) is the partial differential equation for the free torsional vibration of nanorod.

4. Free torsional vibration with general elastic boundary conditions: In this section, a nanorod with torsional restraints (see Fig. 1) for a torsional free vibration is investigated based on Eringens' NE theory. The main idea of the proposed analytical method is to derive an eigenvalue problem including the torsional spring coefficients.

4.1. Angular rotation function about the centre of twist: By employing the separation of variables technique, $\phi(x,t)$ in (10) could be rewritten as the following form:

$$\phi(x,t)=\theta(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{12}$$

where $\theta(x)$ is the rotation function about the centre of twist and ω is the angular frequency. By substituting the (12) into equation (10) yields

$$GJ_p\frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2}-(e_0a)\omega^2\rho J_p^2\frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2}+\rho J_p\omega^2\theta(x)=0, \tag{13}$$

the angular rotation function $\theta(x)$ in (13) is defined as follows:

$$\theta(x)=\begin{bmatrix} \theta_0 & x=0\\ \theta_L & x=L\\ \sum_{n=1}^{\infty}A_n\sin(\beta_n x) & 0<x<L \end{bmatrix}, \tag{14}$$

where

$$\beta_n=\frac{n\pi}{L}. \tag{15}$$

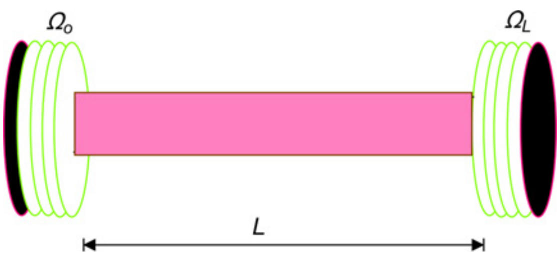

Fig. 1 *Nanorod with torsional springs at both ends*

4.2. Stokes' transformation: In this Letter, in order to see the influences of torsional restraints, a mathematical transformation known as 'Stokes' transformation' is applied to the boundary conditions and the governing equation [6, 23–26]. The coefficients (A_n) in (14) read as

$$A_n=\frac{2}{L}\int_0^L\theta(x)\sin(\beta_n x)\,\mathrm{d}x. \tag{16}$$

Taking the first derivative of (14) with respect to x gives

$$\theta'(x)=\sum_{n=1}^{\infty}\alpha_n A_n\cos(\alpha_n x). \tag{17}$$

By combining Fourier cosine series and (17), the following equation can be obtained:

$$\theta'(x)=\frac{f_0}{L}+\sum_{n=1}^{\infty}f_n\cos(\beta_n x). \tag{18}$$

Fourier constants (f_0, f_n) read as

$$f_0=\frac{2}{L}\int_0^L\theta'(x)\,\mathrm{d}x=\frac{2}{L}\left[\theta(L)-\theta(0)\right], \tag{19}$$

$$f_n=\frac{2}{L}\int_0^L\theta'(x)\cos(\beta_n x)\,\mathrm{d}x(n=1,2\ldots), \tag{20}$$

by integrating parts of (20), the following relations are derived:

$$f_n=\frac{2}{L}\left[\theta(x)\cos(\beta_n x)\right]_0^L+\frac{2}{L}\left[\beta_n\int_0^L\theta(x)\sin(\beta_n x)\mathrm{d}x\right], \tag{21}$$

$$f_n=\frac{2}{L}\left[(-1)^n\theta(L)-\theta(0)\right]+\beta_n A_n. \tag{22}$$

The present method (Fourier series to gather with Stokes' transformation) will be useful when dealing with torsional deformable boundary conditions. Similarly, first and second derivatives of $\theta(x)$ can be calculated as

$$\begin{aligned} \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}&=\frac{\theta_L-\theta_0}{L}\\ &+\sum_{n=1}^{\infty}\cos(\beta_n x)\left(\frac{2\left((-1)^n\theta_L-\theta_0\right)}{L}+\beta_n A_n\right), \end{aligned} \tag{23}$$

$$\frac{\mathrm{d}^2\theta(x)}{\mathrm{d}x^2}=-\sum_{n=1}^{\infty}\beta_n\sin(\beta_n x)\left(\frac{2\left((-1)^n\theta_L-\theta_0\right)}{L}+\beta_n A_n\right). \tag{24}$$

Substituting (14) and (24) into (13), the coefficient A_n and the angular rotation function $\phi(x,t)$ can be written in terms of θ_0 and θ_L as follows:

$$A_n=\frac{2\left((-1)^{n+1}\theta_L+\theta_0\right)\left((e_0a)^2\rho\omega^2-G\beta_n\right)}{L\left(-G\beta_n^2+\gamma^2\rho\omega^2\beta_n+\rho\omega^2\right)} \tag{25}$$

$$\begin{aligned} \phi(x,t)&=\frac{2\left((-1)^{n+1}\theta_L+\theta_0\right)\left((e_0a)^2\rho\omega^2-G\beta_n\right)}{L\left(-G\beta_n^2+\gamma^2\rho\omega^2\beta_n+\rho\omega^2\right)}\\ &\times\sin(\beta_n x)\mathrm{e}^{\mathrm{i}\omega t}, \end{aligned} \tag{26}$$

5. Non-local boundary conditions: By using the relations for rigid boundary conditions in [22], the non-local boundary conditions for

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deformable boundary conditions can be written as

$$GJ_p \frac{d\theta}{dx} - (e_0 a)^2 J_p \rho \omega^2 \frac{d\theta}{dx} = \Omega_0 \theta_0, \quad x = 0 \tag{27}$$

$$GJ_p \frac{d\theta}{dx} - (e_0 a)^2 J_p \rho \omega^2 \frac{d\theta}{dx} = \Omega_L \theta_L, \quad x = L \tag{28}$$

in which, Ω_0 and Ω_L are the torsional spring coefficients. After some mathematical manipulations, the substitution of (23) and (25) into (27) and (28) leads to two equations

$$\begin{aligned}&\left(-\frac{GJ_p}{L} + \frac{(e_0a)^2 J_p \rho\omega^2}{L} - \Omega_0 \right.\\ &\quad \left. + \sum_{n=1}^{\infty} \frac{2J_p L\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2}\right)\theta_0 \\ &\quad + \left(\frac{GJ_p}{L} - \frac{(e_0a)^2 J_p\rho\omega^2}{L} \right.\\ &\quad \left. + \sum_{n=1}^{\infty} \frac{2J_p L(-1)^{n+1}\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2}\right)\theta_L = 0\end{aligned} \tag{29}$$

$$\begin{aligned}&\left(\frac{GJ_p}{L} - \frac{(e_0a)^2 J_p\rho\omega^2}{L} \right.\\ &\quad \left. + \sum_{n=1}^{\infty} \frac{2J_p L(-1)^{n+1}\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2}\right)\theta_0 \\ &\quad + \left(-\frac{GJ_p}{L} + \frac{(e_0a)^2 J_p\rho\omega^2}{L} - \Omega_L \right.\\ &\quad \left. + \sum_{n=1}^{\infty} \frac{2J_p L\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2}\right)\theta_L = 0\end{aligned} \tag{30}$$

and above systems of equations can be written in a matrix form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_L \end{bmatrix} = 0 \tag{31}$$

where

$$a_{11} = -\frac{GJ_p}{L} + \frac{(e_0a)^2 J_p\rho\omega^2}{L} - \Omega_0 + \sum_{n=1}^{\infty} \frac{2J_p L\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2} \tag{32}$$

$$a_{12} = \frac{GJ_p}{L} - \frac{(e_0a)^2 J_p\rho\omega^2}{L} + \sum_{n=1}^{\infty} \frac{2J_p L(-1)^{n+1}\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2} \tag{33}$$

$$a_{21} = \frac{GJ_p}{L} - \frac{(e_0a)^2 J_p\rho\omega^2}{L} + \sum_{n=1}^{\infty} \frac{2J_p L(-1)^{n+1}\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2} \tag{34}$$

$$a_{22} = -\frac{GJ_p}{L} + \frac{(e_0a)^2 J_p\rho\omega^2}{L} - \Omega_L + \sum_{n=1}^{\infty} \frac{2J_p L\rho\omega^2\left((e_0a)^2\rho\omega^2 - G\right)}{L\rho\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 G n^2} \tag{35}$$

The torsional frequencies in NE can be computed by requiring the determinant of the coefficient matrix to zero

$$\left|a_{ij}\right| = 0(i, j = 1, 2) \tag{36}$$

6. Results and discussions: In this section, analytical solutions for free torsional vibration analysis of nanorods are presented, considering the non-local effects and torsional restraints. However, before venturing into mathematical calculations, it is desired to evaluate the accuracy of the present model when implement to some special cases of the torsional restraints. To verify the mathematical results calculated in this Letter, free torsional frequencies are compared predicted by the current method for NE theory with those predicted by following expression available in the literature. The free torsional frequencies according to NE theory and the classical elasticity (CE) theory can be calculated from following two formulations (37) and (38) for fixed-fixed boundary conditions [22, 27]:

$$\varpi_k^{\mathrm{NE}} = \frac{\pi\sqrt{G}k}{L\sqrt{\rho\left(\left((\pi^2(e_0a)^2k^2)/L^2\right) + 1\right)}}, \tag{37}$$

$$\varpi_k^{\mathrm{CE}} = \frac{\pi k}{L}\sqrt{\frac{G}{\rho}}, \tag{38}$$

For numerical illustration, the following properties of nanorod are used in this Letter: Poisson ratio $\nu = 0.25$, Young's modulus $E = 0.72$ TPa, density = 2.3 g/cm^3 [28], inner radius $R_1 = 2.16$ nm and outer radius is $R_2 = 2.50$ nm., the length $L = 10$ nm and the thickness of the nanorod $t = 0.34$ nm [28]. It should be pointed out that non-local parameter e_0a should be smaller than 2 nm for carbon nanotubes [29]. Consequently, the non-local parameter e_0a is selected in the range 0–2 nm [30]. Polar moment of inertia of the cross-section and area could be written as

$$J_p = \frac{\pi}{2}\left(R_2^4 - R_1^4\right), \quad A = \pi\left(R_2^2 - R_1^2\right), \tag{39}$$

Fixed-fixed supports are special case of a nanorod with torsional springs of infinite stiffness. In this Letter, to obtain the solution of fixed-fixed supports, torsional spring coefficients are taken as $\Omega_0 = 90 \times 10^9$ N/mm and $\Omega_L = 90 \times 10^9$ N/mm. A comparison study is performed to validate numerical simulations with those given in the literature (37) and (38) and the results agree with each other exactly.

Next, the influence of non-local parameter, mode numbers on torsional vibration behaviours of the nanorods is studied. Herein, the non-dimensional vibration frequencies (normalised frequencies) are defined as the form of $\Delta_k = \varpi_k^{\mathrm{NE}}/\varpi_k^{\mathrm{CE}}$, $(k = 1, 2, 3, 4\ldots)$. The index k indicates the mode number. The parameter Δ_k is used to give a better illustration of the non-local effects in torsional vibration response of nanorods.

Figs. 2 and 3 show the variation of the first five normalised frequencies (Δ_k) versus the non-local parameter (e_0a) for different two values of the torsional spring coefficients.

The mathematical results in each figure are calculated for a given symmetrical torsional spring coefficients ($\Omega_0 = \Omega_L = 5$ nN/nm, 20 nN/nm). It can be observed, with increasing non-local parameter (e_0a) the normalised frequencies (Δ_k) decrease for all modes. In addition, for a given torsional spring coefficients, the effect of the non-local parameter in decreasing the normalised frequencies for the higher modes is larger than those of the lower ones.

Figs. 4 and 5 display the variation of frequencies with dimensionless length (L/e_0a) change for frequencies computed from classical continuum theory (38) and the Eringens' NE theory: as the increase

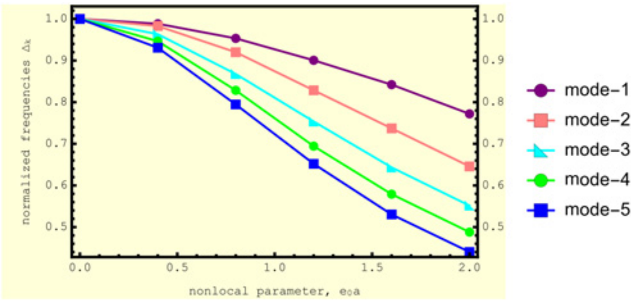

Fig. 2 *First five torsional frequency ratios* (Δ_k) *for different non-local parameter with* $\Omega_0 = 5$ *nN/nm,* $\Omega_L = 5$ *nN/nm*

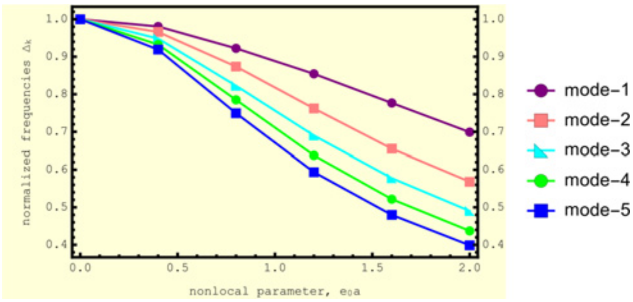

Fig. 3 *First five torsional frequency ratios* (Δ_k) *for different non-local parameter with* $\Omega_0 = 20$ *nN/nm,* $\Omega_L = 20$ *nN/nm*

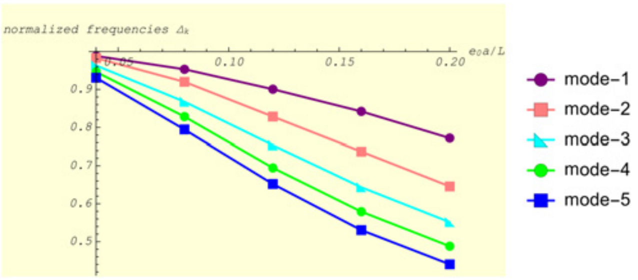

Fig. 4 *Influence of non-local effects on the first five frequencies with* $\Omega_0 = 5$ *nN/nm,* $\Omega_L = 5$ *nN/nm*

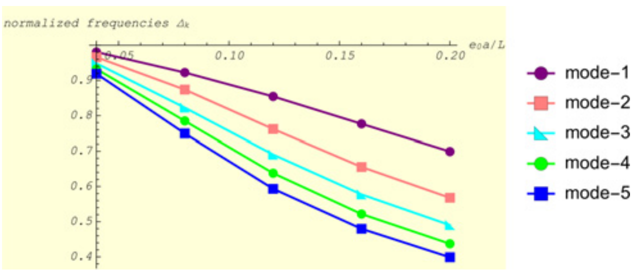

Fig. 5 *Influence of non-local effects on the first five frequencies with* $\Omega_0 = 20$ *nN/nm,* $\Omega_L = 20$ *nN/nm*

in length of nanorod decreases the non-local effects. As found earlier, the NE theory frequencies are always smaller than the CE results.

In Figs. 6 and 7, it can be concluded that the non-local effects increase with increasing mode number, or larger non-local effects result in higher order torsional vibration modes. It is also seen from Figs. 4 and 7, the torsional frequencies of the NE theory are smaller than those of classical continuum theory.

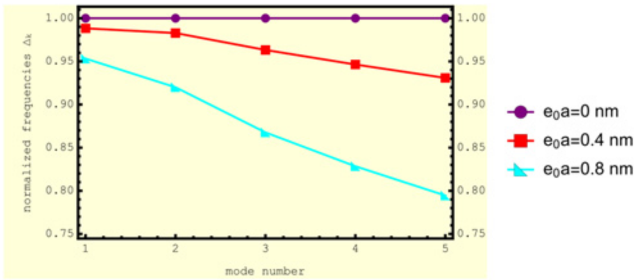

Fig. 6 *Effect of mode number on the first five normalised frequencies with* $\Omega_0 = 5$ *nN/nm,* $\Omega_L = 5$ *nN/nm*

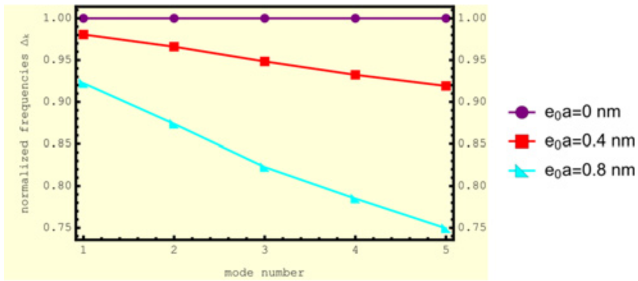

Fig. 7 *Effect of mode number on the first five normalised frequencies with* $\Omega_0 = 20$ *nN/nm,* $\Omega_L = 20$ *nN/nm*

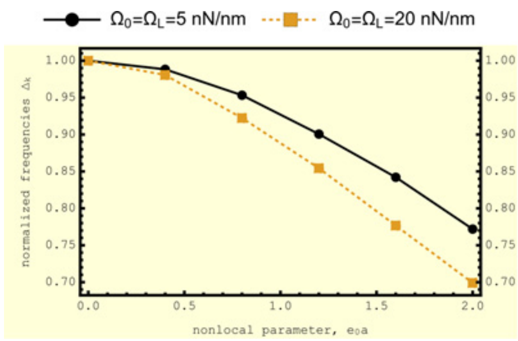

Fig. 8 *Influence of torsional end restraints on the first mode*

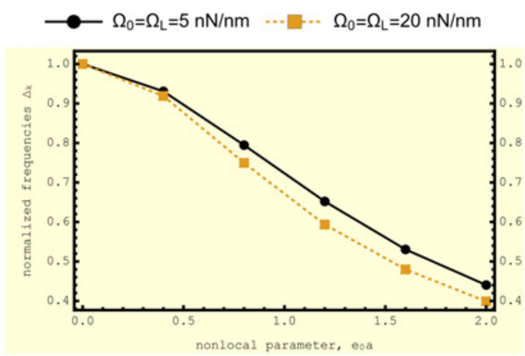

Fig. 9 *Influence of torsional end restraints on the fifth mode*

The variations of the first and fifth non-dimensional frequencies of the torsionally restrained nanorod versus uniform non-local parameter rises are, respectively, shown in Figs. 8 and 9 for different torsional spring coefficients ($\Omega_0 = \Omega_L = 20$ nN/nm, 5 nN/nm). It is found from these figures that increasing non-local parameter

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leads to reduction in the normalised frequencies at a fixed torsional spring coefficients.

One of the chief contributions of this Letter is the derivation of a eigen value problem including both non-local and spring parameters for calculating the frequencies.

7. Conclusion: On the basis of the NE theory, torsional vibrations of nanorods under elastic torsional boundary conditions are investigated. A simplified method is proposed which can be used for a nanorod with any types of elastic torsional boundary conditions. This new method is virtually different from other methods where, instead of classical rigid boundary conditions (free-fixed), deformable boundary conditions (torsional restraints) are used by considering the torsional spring coefficients. Angular rotation is sought as the superposition of Stokes' transformation and Fourier sine series that is used to take care of the torsional supports. A coefficient matrix is obtained with the aid of non-local boundary conditions. The eigenvalues of this matrix give the torsional vibration frequencies. To show the advantages of the proposed analytical method, some numerical examples are solved and investigate the effects of several parameters, such as the non-local parameter and spring coefficients on the torsional responses of the nanorod. It is revealed that for torsional restrained nanorods, increasing non-local parameter with constant value of length leads to decrease.

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