Free longitudinal vibration of a nanorod with elastic spring boundary conditions made of functionally graded material

Mustafa Özgür Yayli ✉

Faculty of Engineering, Department of Civil Engineering, Uludag University, 16059 Görükle Campus, Bursa, Turkey
✉ *E-mail: ozguryayli@uludag.edu.tr*

Published in Micro & Nano Letters; Received on 4th March 2018; Revised on 31st March 2018; Accepted on 9th April 2018

The elastic spring boundary conditions play an important role in dynamical analysis of functionally graded (FG) nanorods. However, these special issues have not been properly paid attention to in the previously developed non-local models. In this work, longitudinal vibration analysis of FG restrained nanorods is presented via non-local elasticity theory. Two axial springs are attached to a FG nanorod at both ends. By considering the non-local differential relations for the FG nanorod, a coefficient matrix is derived and analysed via an exact eigenvalue method. Finally, the results calculated from finite-element method are used to validate the present method. The influence of FG index, non-local parameter and boundary conditions on the axial frequencies of FG nanorods is discussed.

1. Introduction: Over the past decades, free vibration characteristics of functionally graded (FG) nanorods have received a great deal of attention, especially the carbon nanostructures. There are many papers on carbon nanostructures. These types of structures are used in many engineering devices, mainly nano and micro electromechanic systems, due to their specific physical, chemical and mechanical properties [1, 2]. At micro or nanometre length scales, small size effects become important. Both theoretical and experimental test results have shown that the size effects in the analysis of mechanical properties of micro or nanosized structures cannot be neglected. Consequently, classical elasticity theories cannot be applied to these types of structures. In recent years, numerous scientific researches based on higher order elasticity theories have been implemented to analyse these type of nanosized structures and its correlates, utilising different definitions and approaches [3, 4]. Molecular dynamics method is a more convenient technique to simulate the nanomechanical response of small sized structures. However, it takes too much time for computation.

The FG nanorods (carbon nanotubes) have several advantages in different field of application, including corrosion and enhanced thermal resistance. Structures made of FG materials have a great application in engineering and industrial fields [5–8]. Several potential applications of FG nanorods (carbon nanotubes) have been made to various engineering applications on account of its useful properties, such as chemical sensors, graphene transistors, field-gas detection, effect transistor, logic circuits, solar cells, ultracapacitors, diagnosis devices, conductive and ultrastrength composite materials and transparent films. FG nanorods are produced from mixing of two materials. This type of nanorods provides the benefits of both of the materials. Recently, numerous higher order continuum theories have been used by researchers in order to understand the effect of the small size [9–21] and different theoretical methods have been used to solve for the classical and the higher order models [22–24].

In this Letter, axial vibration analysis of the FG nanorod with deformable boundary conditions is studied. The analytical calculation method of vibration frequencies of such nanorod is formed by using Fourier sine series and Stoke transformation. The advantage of this method is that a coefficient matrix including material index and boundary conditions is obtained first, and then the eigenvalue approximation is used to solve the dynamic characteristics of the FG nanorod. Compared with the finite-element method, the proposed approach in this Letter is more efficient, in the process of forming the characteristic equation of the FG restrained nanorod, there is no any assumption introduced, thus the method can be considered as an exact method.

2. Background theory: In this Letter, the free axial vibrations of FG carbon nanotubes have been investigated based on non-local elasticity theory.

2.1. Non-local elasticity theory: In this Letter, the FG nanotube with elastic boundary conditions can be taken as a nanorod of circular cross-section with end restraints which are considered as axial springs (see Fig. 1). The normality assumption for stresses in the classical elasticity model is discarded in the non-local elasticity model. Consequently, the non-local elasticity theory is capable of capturing the size effect in contrast to the classical elasticity theory. The following higher order mathematical expressions have been used in non-local elastic theory [25]:

$$\sigma_{kl,l} + \rho\left(f_l - \frac{\partial^2 u_l}{\partial t^2}\right) = 0, \tag{1}$$

$$\sigma_{kl}(x) = \int_V \alpha(|x - x'|, \chi)\tau_{kl}(x')\mathrm{d}V(x'), \tag{2}$$

$$\tau_{kl}(x') = \lambda_{mm}(x')\delta_{kl} + 2\mu_{kl}(x'), \tag{3}$$

$$\epsilon_{kl}(x') = \frac{1}{2}\left(\frac{\partial u_k(x')}{\partial x'_l} + \frac{\partial u_l(x')}{\partial x'_k}\right), \tag{4}$$

where $\alpha|x - x'|$ represents the distance form of Euclidean. f_l denotes the applied force density, ρ is the mass density of the body, $\epsilon_{kl}(x')$ denotes the strain tensor, t is the time, V indicates the volume occupied by the body, μ and λ represent Lame constants, σ_{kl} indicates the non-local stress tensor, $\tau_{kl}(x')$ is the Cauchy stress tensor at any point x' and u_l indicates the displacement vector. According to the non-local elasticity, the differential form of above equations can be displayed as

$$\aleph = \left[1 - (e_0 a)^2\nabla^2\right], \tag{5}$$

in which, a expresses the internal length. e_0 represents a material constant, ∇^2 expresses the Laplacian and $\aleph$ is a differential operator. This higher order partial differential equation can be

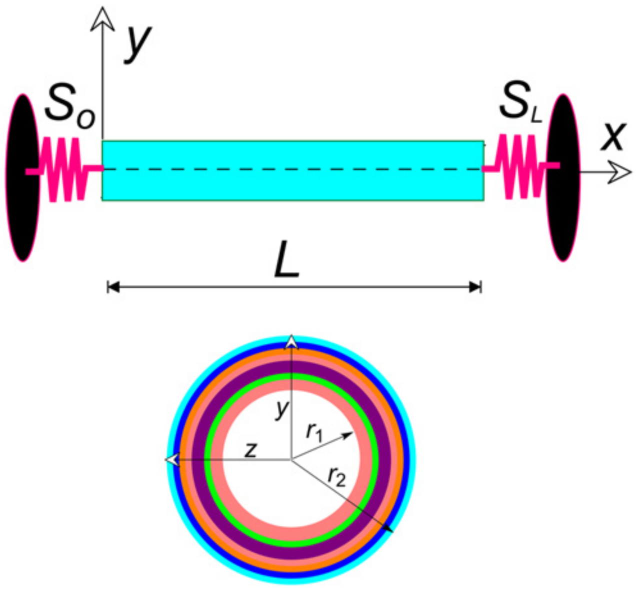

Fig. 1 *Schematic drawing of a FG nanorod with elastically restrained by the means of axial springs*

expressed in terms of small-scale parameter

$$\left[1-(e_0a)^2\nabla^2\right]\sigma_{kl}=\tau_{kl}. \tag{6}$$

2.2. Equation of motion: By using (6), axial force equation may be expressed as

$$N_x-(e_0a)^2\frac{\partial^2 N_x}{\partial x^2}=E(x)H_{rx}\frac{\partial\varphi}{\partial x} \tag{7}$$

in which, N_x denotes the axial force. In contrast to homogeneous nanotubes, the effective elasticity modulus H_{rx} of the FG carbon nanotube should be integrated over the circular cross-section

$$H_{rx}=\int_A E_r(z,y)(z^2+y^2)\mathrm{d}A=2\pi\int_{r_1}^{r_2}E_r(r)r^3\mathrm{d}r \tag{8}$$

where E_r represents the axial rigidity for the FG material. It is taken as follows:

$$E_r=\left(E_i-E_0\right)\left(\frac{r_0-r}{r_0-r_i}\right)^{\beta}+G_0 \tag{9}$$

where β expresses the FG index. The outer radius and the inner radius are assumed to be r_o and r_i, respectively. By the help of equilibrium equation, the following mathematical formulation can be obtained:

$$N_x=H_{rx}\frac{\partial\varphi}{\partial x}+(e_0a)^2m_r\frac{\partial^3\varphi}{\partial x\partial t^2} \tag{10}$$

where

$$m_r=\int_A \rho_r(z,y)(z^2+y^2)\mathrm{d}A=2\pi\int_{r_1}^{r_2}\rho_r(r)r^3\mathrm{d}r \tag{11}$$

Combining the equilibrium equation and (10) yields the axial motion for a FG nanotube

$$H_{rx}\frac{\partial^2\varphi}{\partial x^2}+(e_0a)^2m_r\frac{\partial^4\varphi}{\partial x^2\partial t^2}-m_r\frac{\partial^2\varphi}{\partial t^2}=0 \tag{12}$$

Equation (12) is the higher order partial differential equation for the longitudinal vibration of a FG tube.

3. Formulation for deformable boundary conditions: The main idea of this Letter is to present a general analytical method for eigenvalue solution for the FG nanotubes including the axial spring parameters.

3.1. Axial deformation function: It is assumed that the vibration modes are harmonic in time. Consequently, the axial deformation function can be expressed as

$$\varphi(x,t)=\delta(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{13}$$

where $\delta(x)$ denotes the axial deformation function and ω represents the angular frequencies of the vibration modes. Substituting (13) into equation (12), the following second-order ordinary differential equation results in

$$H_{rx}\frac{\mathrm{d}^2\delta(x)}{\mathrm{d}x^2}-(e_0a)^2\omega^2m_r^2\frac{\mathrm{d}^2\delta(x)}{\mathrm{d}x^2}+m_r\omega^2\delta(x)=0, \tag{14}$$

in this Letter, the axial deformation function $\delta(x)$ is defined as follows:

$$\delta(x)=\begin{bmatrix}\delta_0 & x=0\\ \delta_L & x=L\\ \sum_{n=1}^{\infty}A_n\sin(\alpha_n x) & 0<x<L\end{bmatrix}, \tag{15}$$

where

$$\alpha_n=\frac{n\pi}{L}. \tag{16}$$

3.2. Stokes' transformation: In this section, Stokes' transformation is described [26–30]. Fourier coefficient $\left(A_n\right)$ in (15) can be expressed by

$$A_n=\frac{2}{L}\int_0^L\delta(x)\sin(\alpha_n x)\,\mathrm{d}x. \tag{17}$$

First derivative of axial deformation function with respect to x gives

$$\delta'(x)=\sum_{n=1}^{\infty}\alpha_nA_n\cos(\alpha_n x). \tag{18}$$

Fourier cosine series can be related to (18)

$$\delta'(x)=\frac{\xi_0}{L}+\sum_{n=1}^{\infty}\xi_n\cos(\alpha_n x). \tag{19}$$

where ξ_0, ξ_n are the Fourier constants. These constants are described by

$$\xi_0=\frac{2}{L}\int_0^L\delta'(x)\,\mathrm{d}x=\frac{2}{L}\left[\delta(L)-\delta(0)\right], \tag{20}$$

$$\xi_n=\frac{2}{L}\int_0^L\delta'(x)\cos(\alpha_n x)\,\mathrm{d}x\quad(n=1,2\ldots), \tag{21}$$

utilising the integrating parts rule, the following mathematical

Micro & Nano Letters, 2018, Vol. 13, Iss. 7, pp. 1031–1035
doi: 10.1049/mnl.2018.0181

expressions are obtained:

$$f_n = \frac{2}{L}\left[\delta(x)\cos(\alpha_n x)\right]_0^L + \frac{2}{L}\left[\alpha_n\int_0^L \delta(x)\sin(\alpha_n x)\mathrm{d}x\right], \tag{22}$$

$$f_n = \frac{2}{L}\left[(-1)^n\delta(L) - \delta(0)\right] + \alpha_n H_n. \tag{23}$$

Finally, it is worth pointing out that axial deflections at the ends of nanotube are taken into account through the proposed solution method without any simplifications. This algorithm can be repeated for the higher derivatives as follows:

$$\frac{\mathrm{d}\delta(x)}{\mathrm{d}x} = \frac{\delta_L - \delta_0}{L} + \sum_{n=1}^{\infty}\cos(\alpha_n x) \times \left(\frac{2\left((-1)^n\delta_L - \delta_0\right)}{L} + \alpha_n A_n\right), \tag{24}$$

$$\frac{\mathrm{d}^2\delta(x)}{\mathrm{d}x^2} = -\sum_{n=1}^{\infty}\alpha_n\sin(\alpha_n x) \times \left(\frac{2\left((-1)^n\delta_L - \delta_0\right)}{L} + \alpha_n A_n\right). \tag{25}$$

3.3. Fourier constant and axial deformation function: Combining (15) and (25) with (14), the Fourier constant in axial deformation function is calculated to be

$$A_n = \frac{2\left((-1)^{n+1}\delta_L + \delta_0\right)\left((e_0a)^2\omega^2 m_r - H_{rx}\alpha_n\right)}{L\left(\omega^2 m_r\left((e_0a)^2\alpha_n + 1\right) - H_{rx}\alpha_n^2\right)}, \tag{26}$$

By using the Fourier constant in (26), the function $\varphi(x, t)$ can be denoted in terms of δ_0 and δ_L

$$\varphi(x, t) = \sum_{n=1}^{\infty} A_n \times \sin(\alpha_n x)\mathrm{e}^{\mathrm{i}\omega t}. \tag{27}$$

4. Boundary conditions

4.1. General case: The following two equations for non-local continuum elasticity in the x-direction can be written as

$$H_{rx}\frac{\mathrm{d}\delta}{\mathrm{d}x} - (e_0a)^2 m_r\omega^2\frac{\mathrm{d}\delta}{\mathrm{d}x} = S_0\delta_0, \quad x = 0, \tag{28}$$

$$H_{rx}\frac{\mathrm{d}\delta}{\mathrm{d}x} - (e_0a)^2 m_r\omega^2\frac{\mathrm{d}\delta}{\mathrm{d}x} = S_L\delta_L, \quad x = L, \tag{29}$$

in which, S_0 and S_L denote the elastic spring coefficients in the x-direction. By combining (26), (28), (29) with (24), boundary conditions with axial springs are governed by the following systems of equations:

$$\left(S_0 + \Delta_r + \sum_{n=1}^{\infty}\frac{2\Omega_n}{\psi_n}\right)\delta_0 + \left(\Delta_r + \sum_{n=1}^{\infty}\frac{2(-1)^n\Omega_n}{\psi_n}\right)\delta_L = 0, \tag{30}$$

$$\left(\Delta_r + \sum_{n=1}^{\infty}\frac{2(-1)^n\Omega_n}{\psi_n}\right)\delta_0 + \left(S_L + \Delta_r + \sum_{n=1}^{\infty}\frac{2\Omega_n}{\psi_n}\right)\delta_L = 0, \tag{31}$$

where

$$\Delta_r = \frac{(e_0a)^2 m_r\omega^2}{L} - \frac{H_{rx}}{L}, \tag{32}$$

$$\Omega_n = Lm_r\omega^2\left((e_0a)^2 m_r\omega^2 - H_{rx}\right), \tag{33}$$

$$\psi_n = Lm_r\omega^2\left(L + \pi(e_0a)^2 n\right) - \pi^2 n^2 H_{rx}, \tag{34}$$

and the following eigenvalue problem is obtained in terms of axial spring parameters:

$$\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}\begin{bmatrix}\delta_0\\ \delta_L\end{bmatrix} = 0, \tag{35}$$

where

$$\phi_{11} = S_0 + \Delta_r + \sum_{n=1}^{\infty}\frac{2\Omega_n}{\psi_n}, \tag{36}$$

$$\phi_{12} = \Delta_r + \sum_{n=1}^{\infty}\frac{2(-1)^n\Omega_n}{\psi_n}, \tag{37}$$

$$\phi_{21} = \Delta_r + \sum_{n=1}^{\infty}\frac{2(-1)^n\Omega_n}{\psi_n}, \tag{38}$$

$$\phi_{22} = S_L + \Delta_r + \sum_{n=1}^{\infty}\frac{2\Omega_n}{\psi_n}. \tag{39}$$

To obtain a non-trivial solution of (35), the coefficient determinant of the matrix should be zero, or

$$\left|\phi_{ij}\right| = 0(i, j = 1, 2). \tag{40}$$

Letting $S_0 = S_L = \infty$. In the above equation, the axial natural frequency of a fixed-fixed FG nanorod modelled on the basis of the non-local elasticity theory can be obtained. Furthermore, the axial frequency of a non-local fixed-free FG nanorod can be derived from (40) by letting $S_0 = \infty$ and $S_L = 0$.

5. Results and discussion: In the following study, a FG nanotube with axial springs at the ends, as shown in Fig. 1, will be investigated in detail. A plenty of numerical examples are solved and discussed to verify the accuracy of present model in predicting the axial vibration responses of FG restrained nanorods whose inner and outer sides are assumed to be two different materials. In the numerical analysis, a nanorod composed of two different materials is considered. The material properties of inside material are $E_i = 48\,\mathrm{GPa}$, $\rho_i = 2700\,\mathrm{kg/m^3}$, and those of outer material are $E_o = 129\,\mathrm{GPa}$, $\rho_o = 3210\,\mathrm{kg/m^3}$. The outer radius and the inner radius are assumed to be $r_o = 200\,\mathrm{nm}$ and $r_i = 100\,\mathrm{nm}$, respectively. Length of the nanorod is taken as 300 nm.

5.1. Model validation: In this section, the axial frequencies of restrained FG nanorod are compared with the results of finite-element method for different FG index and also for different ten modes.

5.1.1. Clamped-clamped FG nanotube: In order to obtain the solution of fixed-fixed boundary conditions, axial spring coefficients are taken as $S_0 = 10 \times 10^9$ and $S_L = 10 \times 10^9$. In Table 1, the comparative analysis of the first ten modes computed by finite-element and present method is carried out. The numerical results have been presented for constant value of the non-local parameter $(e_0a = 0)$. As it is seen from Table 1, the predicted results of the present

Table 1 Comparisons of the first ten frequencies for a clamped-clamped FG nanorod with different index ($\beta = 1.00$ and $\beta = 2.00$)

	$\beta = 1$		$\beta = 2$	
Mode	FEM	Present	FEM	Present
1	9.6097	9.6097	10.0736	10.0735
2	19.2195	19.2195	20.1471	20.1471
3	28.8292	28.8292	30.2206	30.2204
4	38.4391	38.4386	40.2942	40.2939
5	48.0488	48.0482	50.3677	50.3671
6	57.6585	57.6574	60.4413	60.4401
7	67.2683	67.2665	70.5148	70.5129
8	76.8780	76.8753	80.5885	80.5856
9	86.4877	86.4839	90.6619	90.6083
10	96.0974	96.0922	100.7356	100.7300

FEM, finite-element method.

analytical method (Fourier sine series and Stokes' transformation) are in excellent agreement with those calculated by finite-element method.

5.1.2. Cantilever FG nanotube: In the second validation, the cantilever FG nanorod is considered. In order to compute the frequencies of a clamped-free nanorod, the spring coefficients are taken as $S_0 = 10 \times 10^6$ and $S_L = 0$. It can be seen from Table 2 that the predicted results of the present method are in excellent agreement with those calculated by finite-element method for the cantilever boundary condition.

5.2. Effect of FG index: In this section, in order to delineate the FG material, some numerical case examples are solved and assessed for vibration analysis, using the derived formulations in the previous sections. It is seen from Figs. 2–4 that axial vibration frequencies are affected by the hardness of springs at the ends. This observation is rational, because the axial deflection is neglected in the clamped-clamped boundary conditions and it makes the nanorod behaviour invalidly stiffer than the reality.

Figs. 2, 3 and 4 show that the non-local FG restrained nanorod, with the formulation presented in this work, is stiffer than the homogenous nanorods. This is due to the fact that an extra stiffness is observed in the FG material. It can be said that when the spring parameters at the ends increases, the response of the restrained FG nanorod approaches to the FG nanorod with clamped ends.

From the calculated results in this study it can be deduced that when the FG index decreases, the response of the composite nanorod approaches to the homogeneous one. Figs. 2, 3 and 4 show higher axial vibration frequencies for nanorods with greater index (β). This is because of the fact that the stiffness of the inner material is much less than that of the selected outer material.

Table 2 Comparisons of the first ten frequencies for a cantilever FG nanorod with $\beta = 1.00$ and $\beta = 2.00$

	$\beta = 1$		$\beta = 2$	
Mode	FEM	Present	FEM	Present
1	4.8096	4.8049	5.0419	5.0368
2	14.4292	14.4146	15.1256	15.1103
3	24.0487	24.0243	25.2094	25.1838
4	33.6681	33.6339	35.2930	35.2573
5	43.2877	43.2435	45.3769	45.3304
6	52.9071	52.8527	55.4605	55.4036
7	62.5265	62.4620	65.5443	65.4766
8	72.1461	72.0709	75.6282	75.5493
9	81.7656	81.6797	85.7118	85.6218
10	91.3850	91.2881	95.7956	95.6940

FEM, finite-element method.

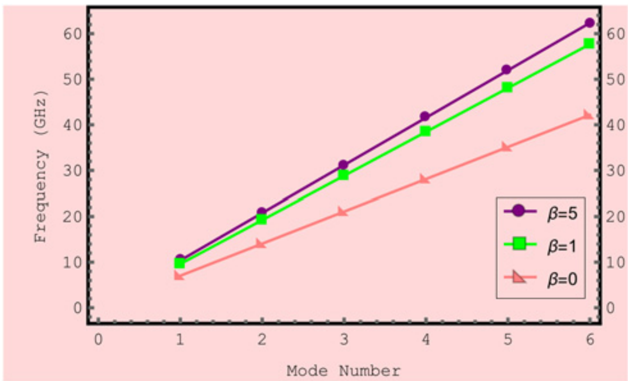

Fig. 2 *Effect of FG index* (β) *for different modes* $(S_0 = S_L = 28 \times 10^{-9} nN/nm)$

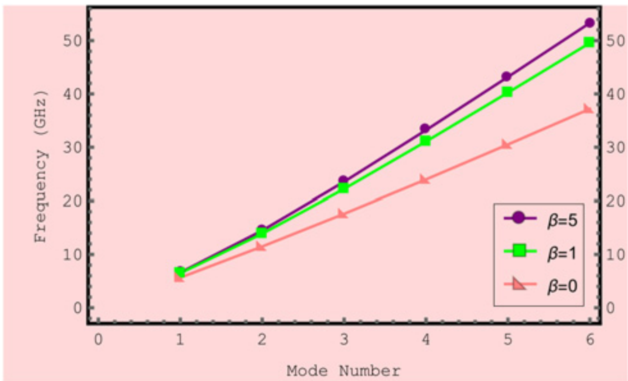

Fig. 3 *Effect of FG index* (β) *for different modes* $(S_0 = S_L = 28 \times 10^{-10} nN/nm)$

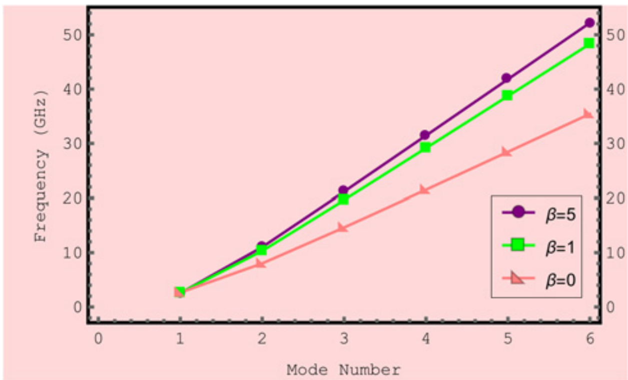

Fig. 4 *Effect of boundary conditions on the vibration frequencies for different deformable boundary conditions*

5.3. Effect of deformable boundary conditions: It can be observed in (40) that the free axial frequency of the FG nanorod is a function of five parameters, the deformable boundary conditions (S_0, S_L), FG index and the non-local parameter $(e_0 a)$. The simultaneous effects of these two major parameters, S_0 and S_L, on the first six axial frequency of the FG nanorod is depicted in Fig. 5. It can be seen that axial frequencies increase with

doi: 10.1049/mnl.2018.0181

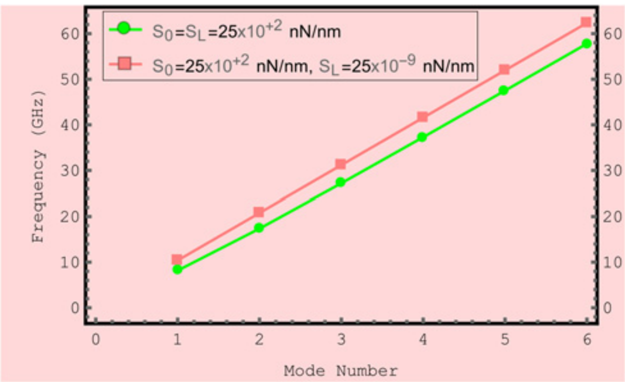

Fig. 5 *Effect of FG index* (β) *for different modes* ($S_0 = S_L = 28 \times 10^2 nN/nm$)

increasing mode number. It is also concluded that the influences of deformable boundary conditions on higher order modes reveal greater than lower order modes.

It may be useful to present a physical interpretation on the non-local deformable boundary conditions (28) and (29) for FG nanorod which are totally new with respect to the nanorod.

6. Conclusion: Idealised boundary conditions (fixed-free, fixed-fixed) on real nanosized structures rarely meet mathematical assumptions exactly. Boundary conditions resist axial deformation in one or two direction. Due to this fact elastic spring boundary conditions are used in this study. This mathematical model bridges the gap between rigid and deformable boundary conditions for FG materials. In this Letter, a general coefficient matrix for eigenvalue analysis with two undetermined coefficients of FG restrained nanorod is obtained first. By using Fourier sine series and Stoke transformation, this Letter presents an analytical approach for free vibration analysis of FG restrained nanorod with general elastic boundary conditions. Then, the results by existing studies are used to verify the reliability and correctness of the approach presented in this study. By using solved examples, the influence of elastic boundary conditions, non-local parameter and material index on vibration frequencies is demonstrated.

7 References

[1] Bachtold A., Hadley P., Nakanihi T., *ET AL.*: 'Logic circuits with carbon nanotube transistors', *Science*, 2001, **294**, pp. 1317–1320

[2] Kim P., Lieber C.M.: 'Nanotube nanotweezers', *Science*, 1999, **286**, pp. 2148–2150

[3] Falvo M.R., Clary G.J., Taylor R.M., *ET AL.*: 'Bending and buckling of carbon nanotubes under large strain', *Nature*, 1997, **389**, pp. 582–584

[4] Dai H., Hafner J.H., Rinzler A.G., *ET AL.*: 'Nanotubes as nanoprobes in scanning probe microscopy', *Nature*, 1996, **384**, pp. 147–150

[5] Tornabene F.: Free vibration analysis of functionally graded conical, cylindrical shell and annular plate structures with a four-parameter power-law distribution. *Comput. Methods Appl. Mech. Eng.*, 2009, **198**, pp. 2911–2935

[6] Tornabene F. Viola E.: 'Free vibration analysis of functionally graded panels and shells of revolution', *Meccanica*, 2009, **44**, pp. 255–281

[7] Tornabene F., Fantuzzi N., Bacciocchi M., *ET AL.*: 'An equivalent layer-wise approach for the free vibration analysis of thick and thin laminated sandwich shells', *Appl. Sci.*, 2017, **7**, (17), pp. 1–34

[8] Tornabene F., Fantuzzi N., Bacciocchi M., *ET AL.*: 'A numerical investigation on the natural frequencies of FGM sandwich shells with variable thickness by the local generalized differential quadrature method', *Appl. Sci.*, 2017, **7**, (131), pp. 1–39

[9] Ece M.C., Aydogdu M.: 'Nonlocal elasticity effect on vibration of in-plane loaded double-walled carbon nano-tubes', *Acta Mech.*., 2007, **190**, pp. 185–195

[10] Akgoz B., Civalek O.: 'Bending analysis of FG microbeams resting on Winkler elastic foundation via strain gradient elasticity', *Compos. Struct.*, 2015, **134**, pp. 294–301

[11] Simsek M.: 'Vibration analysis of a single-walled carbon nanotube under action of a moving harmonic load based on nonlocal elasticity theory', *Phys.-E Low-dimens. Syst. Nanostruct.*, 2010, **43**, pp. 182–191

[12] Ansari R., Gholami R., Sahmani S.: 'Size-dependent vibration of functionally graded curved microbeams based on the modified strain gradient elasticity theory', *Arch. Appl. Mech.*, 2013, **83**, pp. 1439–1449

[13] Akgoz B., Civalek O.: 'Buckling analysis of linearly tapered micro-columns based on strain gradient elasticity', *Struct. Eng. Mech.*, 2013, **48**, pp. 195–205

[14] Kounadis A.N., Mallis J., Sbarounis A.: 'Postbuckling analysis of columns resting on an elastic foundation', *Arch. Appl. Mech.*, 2006, **75**, pp. 395–404

[15] Aydogdu M.: 'Axial vibration of the nanorods with the nonlocal continuum rod model', *Phys.-E Low-dimens. Syst. Nanostruct.*, 2009, **41**, pp. 861–864

[16] Civalek O., Demir C.: 'A simple mathematical model of microtubules surrounded by an elastic matrix by nonlocal finite element method', *Appl. Math. Comput.*, 2016, **289**, pp. 335–352

[17] Liu J.J., Li C., Fan X.L., *ET AL.*: 'Transverse free vibration and stability of axially moving nanoplates based on nonlocal elasticity theory', *Appl. Math. Comput.*, 2017, **45**, pp. 65–84

[18] Li C., Liu J.J., Cheng M., *ET AL.*: 'Nonlocal vibrations and stabilities in parametric resonance of axially moving viscoelastic piezoelectric nanoplate subjected to thermo-electro-mechanical forces', *Compos. Part B-Eng.*, 2017, **116**, pp. 153–169

[19] Wang Q., Liew K.M.: 'Application of nonlocal continuum mechanics to static analysis of micro and nano-structures', *Phys. Lett. A*, 2007, **363**, pp. 236–242

[20] Li C.: 'A nonlocal analytical approach for torsion of cylindrical nanostructures and the existence of higher-order stress and geometric boundaries', *Compos. Struct.*, 2014, **118**, pp. 607–621

[21] Li C, Yao L.Q., Chen W.Q., *ET AL.*: 'Comments on nonlocal effects in nano-cantilever beams', *Int. J. Eng. Sci.*, 2015, **87**, pp. 47–57

[22] Yayli M.O., Aras M., Aksoy S.: 'An efficient analytical method for vibration analysis of a beam on elastic foundation with elastically restrained ends', *Shock Vib.*, 2014, pp. 1–7, Article ID: 159213, doi: 10.1155/2014/159213

[23] Yayli M.O.: Torsion of nonlocal bars with equilateral triangle cross sections. *J. Comput. Theor. Nanosci.*, 2013, **10**, pp. 376–379

[24] Yayli M.O.: Weak formulation of finite element method for nonlocal beams using additional boundary conditions. *J. Comput. Theor. Nanosci.*, 2011, **8**, pp. 2173–2180

[25] Eringen A.C., Edelen D.G.B.: ' on nonlocal elasticity', *Int. J. Eng. Sci.*, 1972, **10**, pp. 233–248

[26] Yayli M.O.: 'On the axial vibration of carbon nanotubes with different boundary conditions', *IET Micro Nano Lett.*, 2014, **9**, pp. 807–811

[27] Yayli M.O., Yanik F., Kandemir S.Y.: 'Longitudinal vibration of nanorods embedded in an elastic medium with elastic restraints at both ends.', *IET Micro Nano Lett.*, 2015, **10**, pp. 641–644

[28] Yayli M.O.: 'Buckling analysis of a microbeam embedded in an elastic medium with deformable boundary conditions', *IET Micro Nano Lett.*, 2016, **11**, pp. 741–745

[29] Yayli M.O.: 'A compact analytical method for vibration analysis of single-walled carbon nanotubes with restrained boundary conditions', *J. Vib. Control*, 2016, **22**, pp. 2542–2555

[30] Yayli M.O.: 'Buckling analysis of a cantilever single-walled carbon nanotube embedded in an elastic medium with an attached spring', *IET Micro Nano Lett.*, 2017, **12**, pp. 255–259