

Approach to solve multi-criteria group decision-making problems by exponential operational law in generalised spherical fuzzy environment

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Abstract: The construction of generalised spherical fuzzy number (GSFN) is a reliable philosophy to design and understanding of vagueness and impreciseness. In this study, at first, the authors have defined the generalised spherical fuzzy set and discussed its several properties. Then, a new score and accuracy functions have been introduced in the generalised spherical fuzzy environment which leads to a new method of conversion of fuzzy number into a crisp number. New exponential operational law has been defined for GSFNs where the bases are positive real numbers & components are GSFNs and its various algebraic properties have been studied explicitly. Using this exponential operational law, a generalised spherical weighted exponential averaging operator has been proposed, which is used to develop a multi-criteria group decision-making (MCGDM) method in the generalised spherical fuzzy environment. The newly developed MCGDM has been demonstrated through a real-life problem and its effectiveness and rationality have been shown through sensitivity analysis.

1 Introduction

Due to the intricacy and uncertainty of detached concepts and dilemma of human assumptions, Zadeh [1] introduced a remarkable notion of the fuzzy set theory, which has been successfully applied on distinctive domains of applied mathematics and engineering techniques. Chang and Zadeh [2] introduced the conception of fuzzy structure and lots of researchers from different fields have been studying the ideology of one-dimensional or n -dimensional fuzzy numbers, established in the articles [3–5] with the countless improvement and advancement of postulation of fuzzy set theory. The subject eventually became a matter of immense academic concern. Gradually, as research running in this domain, the perception of uncertainty is elongated into interval-valued fuzzy sets [6] and later, Atanassov [7] established the idea of intuitionistic fuzzy set where both membership and non-membership functions are considered simultaneously. Smarandache [8] demonstrated the architecture of neutrosophic set (NS) theory where the idea of the intuitionistic fuzzy set was extended into three distinctive components namely truth, indeterminacy and falsity functions independently lying in the interval $[0^-, 1^+]$. Demonstration of the NS has a great impact on our real-life system and several researchers like [9–19] enriched the NS theory. Day by day, as improvement goes on, researchers fabricated the conception of triangular [20, 21], trapezoidal [22, 23], pentagonal [24, 25] fuzzy numbers to capture many real-life problems in a robust way. Liu and Yuan [26] and Ye [27] established the conception of triangular intuitionistic fuzzy set and trapezoidal intuitionistic fuzzy set, respectively, which are the gracious mixture of the triangular and trapezoidal fuzzy sets with intuitionistic fuzzy set. However, researchers noticed some drawback in the construction of intuitionistic fuzzy set as it disobeyed the definition and its properties in some cases; for example, if we consider fuzzy numbers like $\langle 0.5, 0.7 \rangle$, $\langle 0.4, 0.8 \rangle$, then sum of the components is not less than equal to 1. To overcome this drawback, Yager [28] introduced the conception of Pythagorean fuzzy number (PyFN) where the sum of the square of

its membership and non-membership functions is less than or equal to 1². Garg [29–31], Peng and Yang [32, 33] & many others have done some interesting work on Pythagorean fuzzy set and many more developments are still possible in this area. Cuong [34] inflated the idea of IFS into three-dimensional field and manifested picture fuzzy set (PFS) which is interpreted by positive-membership, neutral-membership and negative-membership degrees with the sum of these degrees is less equal to 1. This concept of PFS is more suitable to human assumption than the previous existing concept of fuzzy theory. PFS is one of the richest fields in the recent times and researchers like Mahmood *et al.* [35] and Jan *et al.* [36] work in this environment to solve the MCDM problem. Furthermore, Mahmood *et al.* [37] unfolded the conception of Pythagorean fuzzy set into three-dimensional space and they initiated the idea of spherical fuzzy set and T-spherical fuzzy set which are characterised by positive-membership degree, neutral-membership degree and negative-membership degree. In spherical fuzzy set, they took the sum of the square of membership degrees is less than equal to 1² and for T-spherical fuzzy set, the n th sum of membership degrees is less than equal to 1². Gündoğdu and Kahramana [38] introduced the TOPSIS method in spherical fuzzy environment. Rafiq *et al.* [39] presented the cosine similarity measures from PyFS & PFS into spherical fuzzy environment and apply it to decision-making problems. Spherical fuzzy sets and T-spherical fuzzy sets are also studied by several other researchers [40–44]. T-spherical fuzzy set can nicely handle few of the drawbacks of spherical fuzzy numbers but it is not easy to implement from practical point of view. Thus, we introduced a new concept namely generalised spherical fuzzy number (GSFN) where the radius of the sphere has been extended appropriately. It is to be noted that value of membership function of each component of a spherical fuzzy number lies between zero and one and the sum of the squares of all the three components of a spherical fuzzy number will be less than or equal to 3. We, therefore, consider a sphere of radius $\sqrt{3}$ instead of radius 1 (as it was considered in spherical fuzzy number) and introduced the perception of new GSFN.

In this research paper, we proposed the idea of the GSFN and its properties. Later, we established the new exponential operational laws on the GSFN which has a great impact on decision-making problem. Apart from this, we also developed score and accuracy functions on the generalised spherical fuzzy domain and lastly, we consider a multi-criteria group decision-making (MCGDM) problem in which there are four different kinds of decision-makers present, different kinds of weight are given to the attributes and according to their opinion, we need to find out the best alternative among all of them.

1.1 Multi-criteria decision-making (MCDM) problem

The MCDM problem is the paramount topic in decision scientific research. In the current scenario, it is more essential in such problems where a group of criteria is appraised. For such problems involving MCDM problems have come into existence. MCDM can be enforced in distinct fields under different imprecise or crisp environment. For example, Karai and Cheikhrouhou [45] have recognised a multi-criteria decision-making approach for collaborative software selection problems. Wanga *et al.* [46] have utilised a multi-criteria decision-making method based on intuitionistic linguistic aggregation operators. Chen *et al.* [47] reviewed a fuzzy MCDM method with new entropy of interval-valued intuitionistic fuzzy sets. Intuitionistic interval fuzzy information in the MCDM has been surveyed by Wang *et al.* [48]. Recently, the researcher finds interest in MCDM. Chao [49] obtained multi-criteria decision-making methodology using type 2 fuzzy linguistic judgments. Wibowo *et al.* [50] examined MCDM for selecting human resources management information systems projects. New web-based framework development for fuzzy multi-criteria decision-making is illuminated by Hanine *et al.* [51]. Büyüközkan and Güleriyüz [52] wielded the decision-making conception in smartphone selection using intuitionistic fuzzy TOPSIS. Efe [53] employed an integrated fuzzy multi-criteria group decision-making approach for the ERP system selection. Seddikia *et al.* [54] served the use of MCDM approach in the thermal innovation of masonry buildings: the case of Algeria. An outranking sorting method for MCGDM using intuitionistic fuzzy sets was given by Shen *et al.* [55]. A better approach namely the Best-Worst-Method is correlated by ELECTRE by [56]. The method WASPAS-SVNS are delivered by Bausys and Juodagalviene [57].

1.2 Motivation

The realistic concept and impact of impreciseness plays an important appreciation in mathematical modelling, various complex engineering and medical science problems etc. Spherical fuzzy number has been embedded and demonstrated as an extension of Pythagorean fuzzy set. However, if anyone considers fuzzy numbers like $\langle 0.5, 0.7, 0.9 \rangle$, $\langle 0.7, 0.9, 0.9 \rangle$ then it disobeyed the definition of spherical fuzzy number as the sum of the square of its component is not less than or equal to 1². To tackle this kind of problem, we extended the concept of spherical fuzzy set into the generalised spherical fuzzy set (GSFS). On the other hand, it is observed from the existing literature that the most operational laws are based on algebraic sum and product in the aggregation procedure where weights appear as a power that are taken as crisp number lying in $[0, 1]$. Now, there might be cases where weights are a fuzzy number. To deal with this situation exponential operational laws are defined with real base in case of intuitionistic fuzzy set [58], interval-valued intuitionistic fuzzy set [59] and Pythagorean fuzzy set [60]. Naturally, the question arises whether exponential operational laws can be defined in the case of the spherical fuzzy set or not? Unfortunately, exponential operational laws cannot be embedded directly into the spherical fuzzy set due to the restriction of the sum of squares of three membership functions less or equal to one. Thus, we first extended spherical fuzzy set into the GSFS by increasing the radius of the sphere from 1 to $\sqrt{3}$ and then defined exponential operational law in generalised spherical fuzzy environment.

1.3 Novelties

Several works are already published in this spherical fuzzy set arena. Researchers are already improved lots of formulations and applications in different fields of spherical fuzzy set. However, in the case of GSFS theory, distinctive kinds of works can be developed which are still unknown. Our work is to try to build up the conception on this unknown point.

- (i) Demonstration of GSFSs.
- (ii) Development of score and accuracy functions.
- (iii) Construction of exponential operational law in generalised spherical fuzzy environment.
- (iv) Utilisation of GSFN in multi-criteria group decision-making problem.

2 Mathematical preliminaries

In this section, some elementary definitions and operations related to IFSs, PyFSs, and SFSs have been discussed.

Definition 1: Let W be a universe of discourse. Then

$$\tilde{I} = \{ \langle w, \tau(w), \sigma(w) \rangle; \quad w \in W \}$$

is said to be IFS [7] on W , where $\tau: W \rightarrow [0, 1]$ and $\sigma: W \rightarrow [0, 1]$ with the condition $\tau(w) + \sigma(w) \leq 1$ for all $w \in W$. The numbers $\tau(w)$ and $\sigma(w)$ are called the membership and non-membership degrees, respectively of the element to the set \tilde{I} . For convenience, we represent this pair $\tilde{I} = \{ \langle \tau, \sigma \rangle$, where $\tau, \sigma \in [0, 1]$, $\tau + \sigma \leq 1$ } and called as an intuitionistic fuzzy number.

Definition 2: Let W be a universe of discourse. Then

$$\tilde{P} = \{ \langle w, \tau(w), \sigma(w) \rangle; \quad w \in W \}$$

is said to be PyFS [28] on W , where $\tau: W \rightarrow [0, 1]$ and $\sigma: W \rightarrow [0, 1]$ with the condition $\tau^2(w) + \sigma^2(w) \leq 1$ for all $w \in W$. The numbers $\tau(w)$ and $\sigma(w)$ are called the membership and non-membership degrees, respectively, of the element to the set \tilde{P} . For convenience, we represent this pair $\tilde{P} = \{ \langle \tau, \sigma \rangle$, where $\tau, \sigma \in [0, 1]$, $\tau^2 + \sigma^2 \leq 1$ } and called as a Pythagorean fuzzy number (PyFN).

Definition 3: Let W be a universe of discourse. Then

$$\tilde{S} = \{ \langle w, \tau(w), \eta(w), \sigma(w) \rangle; \quad w \in W \}$$

is said to be SFS [37] on W , where $\tau: W \rightarrow [0, 1]$, $\eta: W \rightarrow [0, 1]$ and $\sigma: W \rightarrow [0, 1]$ with the condition $\tau^2(w) + \eta^2(w) + \sigma^2(w) \leq 1$ for all $w \in W$. The numbers $\tau(w)$, $\eta(w)$ and $\sigma(w)$ are called the positive-membership, neutral-membership and negative-membership degrees, respectively, of the element to the set \tilde{S} . For convenience, we represent this pair $\tilde{S} = \{ \langle \tau, \eta, \sigma \rangle$, where $\tau, \eta, \sigma \in [0, 1]$, $\tau^2 + \eta^2 + \sigma^2 \leq 1$ } and called as a spherical fuzzy number (SFN).

3 Generalised spherical fuzzy sets

Definition 4: Let W be a universe of discourse. Then

$$\tilde{G} = \{ \langle w, \tau(w), \eta(w), \sigma(w) \rangle; \quad w \in W \}$$

is said to be GSFS on W , where $\tau: W \rightarrow [0, 1]$ and $\eta: W \rightarrow [0, 1]$ $\sigma: W \rightarrow [0, 1]$ with the condition $\tau^2(w) + \eta^2(w) + \sigma^2(w) \leq (\sqrt{3})^2$ for all $w \in W$. The numbers $\tau(w)$, $\eta(w)$ and $\sigma(w)$ are called the positive-membership degree, neutral-membership degree and

negative-membership degree, respectively, of the element to the set \tilde{G} . For convenience, we represent this pair $\tilde{G} = \{\langle \tau, \eta, \sigma \rangle\}$, where $\tau, \eta, \sigma \in [0, 1]$, $\tau^2 + \eta^2 + \sigma^2 \leq (\sqrt{3})^2$ and called as a generalised spherical fuzzy number (GSFN).

Remark 1: The space of GSFNs is greater than the space of spherical fuzzy numbers.

Remark 2: In SVNS, the sum of the three membership functions (truth, indeterminacy and falsity) is less than equal to 3, i.e. the sum is taken as linearly and it represents a plane in space. However, in case of GSFN, we consider the non-linear form of membership functions as in Definition 4 which represent a sphere in space (Fig. 1).

Definition 5: Let $\tilde{G} = \langle \tau, \eta, \sigma \rangle$, $\tilde{G}_1 = \langle \tau_1, \eta_1, \sigma_1 \rangle$ and $\tilde{G}_2 = \langle \tau_2, \eta_2, \sigma_2 \rangle$ be any three GSFNs. Then, their operational rules are as follows:

- (i) $(\tilde{G})^c = \langle \sigma, \eta, \tau \rangle$ (complement of \tilde{G});
- (ii) $\tilde{G}_1 \leq \tilde{G}_2$ iff $\tau_1 \leq \tau_2, \eta_1 \geq \eta_2, \sigma_1 \geq \sigma_2$;
- (iii) $\tilde{G}_1 = \tilde{G}_2$ iff $\tilde{G}_1 \leq \tilde{G}_2$ and $\tilde{G}_1 \geq \tilde{G}_2$;
- (iv) $\tilde{G}_1 \oplus \tilde{G}_2 = \langle \sqrt{\tau_1^2 + \tau_2^2 - \tau_1 \cdot \tau_2}, \eta_1 \cdot \eta_2, \sigma_1 \cdot \sigma_2 \rangle$;
- (v) $\tilde{G}_1 \otimes \tilde{G}_2 = \langle \tau_1 \cdot \tau_2, \eta_1 \cdot \eta_2, \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \cdot \sigma_2} \rangle$;
- (vi) $l\tilde{G} = \langle \sqrt{1 - (1 - \tau^2)^l}, (\eta)^l, (\sigma)^l \rangle, l \geq 0$;
- (vii) $\tilde{G}^l = \langle \tau^l, \eta^l, \sqrt{1 - (1 - \sigma^2)^l} \rangle, l \geq 0$.

Theorem 1: Let $\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle, (i=1, 2, \dots, n)$ be any collection of GSFNs. Then the generalised spherical weighted averaging (GSWAA) operators are given as

$$\text{GSWAA}(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n) = \left\langle \left[1 - \prod_{i=1}^n (1 - \tau_i^{w_i}) \right], \left[\prod_{i=1}^n \eta_i^{w_i} \right], \left[\prod_{i=1}^n \sigma_i^{w_i} \right] \right\rangle$$

where $w_i (i=1, 2, \dots, n)$ is the weight of $\tilde{G}_i (i=1, 2, \dots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$.

Proof: The proof of this is similar to the proof of Theorem 4.1.2 in [40]. \square

3.1 Newly defined score and accuracy functions on the GSFN

The score and accuracy functions are important index for ranking of GSFNs. Here, we first defined score and accuracy functions in a meaningful way and then defined ranking method for comparing any two GSFNs.

Definition 6: Let $\tilde{G} = \langle \tau, \eta, \sigma \rangle$ be any GSFN. Then, we define a score function S of a GSFN \tilde{G} as follows:

$$S(\tilde{G}) = \frac{3\tau^2 - 2\eta^2 - \sigma^2}{3}; \quad \text{where } S(\tilde{G}) \in [-1, 1]$$

If $\tilde{G} = \langle 0, 1, 1 \rangle$, then $S(\tilde{G}) = 0$

$\tilde{G} = \langle 1, 1, 1 \rangle$, then $S(\tilde{G}) = -1$

$\tilde{G} = \langle 1, 0, 0 \rangle$, then $S(\tilde{G}) = 1$

Definition 7: Let $\tilde{G} = \langle \tau, \eta, \sigma \rangle$ be any GSFN. Then, we define an accuracy function A of a GSFN \tilde{G} as follows:

$$A(\tilde{G}) = \frac{1 + 3\tau^2 - \sigma^2}{4}; \quad \text{where } A(\tilde{G}) \in [0, 1]$$

If $\tilde{G} = \langle 0, 0, 1 \rangle$, then $A(\tilde{G}) = 0$

$\tilde{G} = \langle 1, 0, 0 \rangle$, then $A(\tilde{G}) = 1$

Definition 8: Let $\tilde{G}_1 = \langle \tau_1, \eta_1, \sigma_1 \rangle$ and $\tilde{G}_2 = \langle \tau_2, \eta_2, \sigma_2 \rangle$ be any two GSFNs, then the ranking method is defined as

If $S(\tilde{G}_1) > S(\tilde{G}_2)$, then $\tilde{G}_1 > \tilde{G}_2$;

If $S(\tilde{G}_1) < S(\tilde{G}_2)$, then $\tilde{G}_1 < \tilde{G}_2$;

If $S(\tilde{G}_1) = S(\tilde{G}_2)$, then

If $A(\tilde{G}_1) > A(\tilde{G}_2)$, then $\tilde{G}_1 > \tilde{G}_2$

If $A(\tilde{G}_1) < A(\tilde{G}_2)$, then $\tilde{G}_1 < \tilde{G}_2$

If $A(\tilde{G}_1) = A(\tilde{G}_2)$, then $\tilde{G}_1 = \tilde{G}_2$.

4 Exponential operational law for GSFN

Definition 9: Let the universal set be W and $\tilde{G} = \langle \tau, \eta, \sigma \rangle$ be any GSFN, then the exponential operational law of \tilde{G} is defined as

$$a^{\tilde{G}} = \begin{cases} \left\langle a^{\sqrt{1-\tau^2}}, \sqrt{1-a^2\eta}, \sqrt{1-a^2\sigma} \right\rangle; & a \in (0, 1) \\ \left\langle \left(\frac{1}{a} \right)^{\sqrt{1-\tau^2}}, \sqrt{1-a^2\eta}, \sqrt{1-a^2\sigma} \right\rangle; & a \geq 1 \end{cases}$$

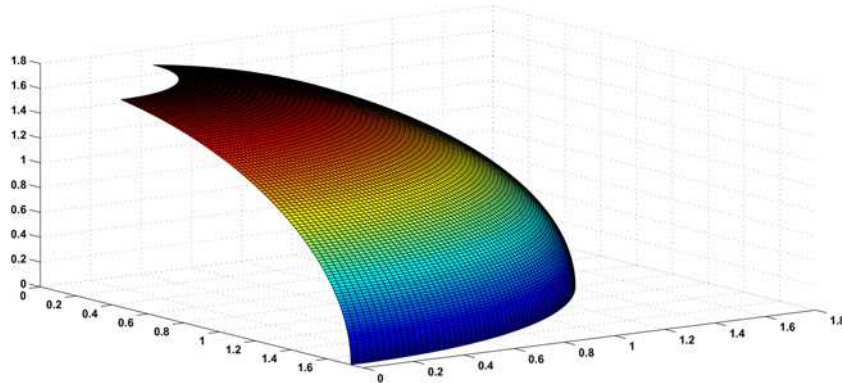


Fig. 1 Generalised spherical fuzzy space

Theorem 2: For any GSFN \tilde{G} , the value of $a^{\tilde{G}}$ is also a GSFN.

Proof: Let $\tilde{G} = \langle \tau, \eta, \sigma \rangle$ be any GSFN, where $0 \leq \tau, \eta, \sigma \leq 1$ with the condition $\tau^2 + \eta^2 + \sigma^2 \leq 3$. \square

Case 1: Let $a \in (0, 1)$, then we can get

$$0 \leq a^{\sqrt{1-\tau^2}}, \sqrt{1-a^2\eta}, \sqrt{1-a^2\sigma} \leq 1$$

$$\Rightarrow 0 \leq (a^{\sqrt{1-\tau^2}})^2 + (\sqrt{1-a^2\eta})^2 + (\sqrt{1-a^2\sigma})^2 \leq 3.$$

Thus, according to Definition 4, $a^{\tilde{G}}$ is also a GSFN.

Case 2: When $a \geq 1$, i.e. $0 \leq 1/a \leq 1$, then in this case proceeding in the exact similar way as in case 1, we can prove that $a^{\tilde{G}}$ is also a GSFN.

Example 1: Let $\tilde{G} = \langle 0.7, 0.9, 0.9 \rangle$ be a GSFN and $a = 0.6$, then

$$a^{\tilde{G}} = 0.6^{(0.7, 0.9, 0.9)}$$

$$= \langle 0.6^{\sqrt{1-0.7^2}}, \sqrt{1-0.6^{2 \times 0.9}}, \sqrt{1-0.6^{2 \times 0.9}} \rangle$$

$$= \langle 0.69, 0.78, 0.78 \rangle$$

If $a=6$, then

$$a^{\tilde{G}} = 6^{(0.7, 0.9, 0.9)}$$

$$= \left\langle \left(\frac{1}{6}\right)^{\sqrt{1-0.7^2}}, \sqrt{1-\left(\frac{1}{6}\right)^{2 \times 0.9}}, \sqrt{1-\left(\frac{1}{6}\right)^{2 \times 0.9}} \right\rangle$$

$$= \langle 0.28, 0.98, 0.98 \rangle$$

Now, we will discuss some basic properties of $a^{\tilde{G}}$ for $a \in (0, 1)$. When $a \geq 1$, it can be proved similarly.

Theorem 3: Let $\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle$ ($i=1, 2$) be any two GSFNs and $a \in (0, 1)$, then

- (i) $a^{\tilde{G}_1} \oplus a^{\tilde{G}_2} = a^{\tilde{G}_1 \oplus \tilde{G}_2}$;
- (ii) $a^{\tilde{G}_1} \otimes a^{\tilde{G}_2} = a^{\tilde{G}_1 \otimes \tilde{G}_2}$.

Proof: The proof of the theorem directly follows from Definitions 5 and 9. \square

Theorem 4: Let $\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle$ ($i=1, 2, 3$) be any three GSFNs and $a \in (0, 1)$, then

- (i) $(a^{\tilde{G}_1} \oplus a^{\tilde{G}_2}) \oplus a^{\tilde{G}_3} = a^{\tilde{G}_1 \oplus (\tilde{G}_2 \oplus \tilde{G}_3)}$;
- (ii) $(a^{\tilde{G}_1} \otimes a^{\tilde{G}_2}) \otimes a^{\tilde{G}_3} = a^{\tilde{G}_1 \otimes (\tilde{G}_2 \otimes \tilde{G}_3)}$.

Proof: The proof of the theorem directly follows from Definitions 5 and 9. \square

Theorem 5: Let $\tilde{G} = \langle \tau, \eta, \sigma \rangle$, $\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle$ ($i=1, 2$) be any three GSFNs, $l, l_1, l_2 \geq 0$ be three real numbers and $a, a_1, a_2 \in (0, 1)$, then

- (i) $l(a^{\tilde{G}_1} \oplus a^{\tilde{G}_2}) = la^{\tilde{G}_1} \oplus la^{\tilde{G}_2}$;
- (ii) $(a^{\tilde{G}_1} \otimes a^{\tilde{G}_2})^l = (a^{\tilde{G}_1})^l \otimes (a^{\tilde{G}_2})^l$;
- (iii) $l_1 a^{\tilde{G}} \oplus l_2 a^{\tilde{G}} = (l_1 + l_2) a^{\tilde{G}}$;
- (iv) $(a^{\tilde{G}})^{l_1} \otimes (a^{\tilde{G}})^{l_2} = (a^{\tilde{G}})^{l_1 + l_2}$;
- (v) $a_1^{\tilde{G}} \otimes a_2^{\tilde{G}} = (a_1 a_2)^{\tilde{G}}$.

Proof: Now, we have

$$a^{\tilde{G}_1} = \langle a^{\sqrt{1-\tau_1^2}}, \sqrt{1-a^2\eta_1}, \sqrt{1-a^2\sigma_1} \rangle$$

$$a^{\tilde{G}_2} = \langle a^{\sqrt{1-\tau_2^2}}, \sqrt{1-a^2\eta_2}, \sqrt{1-a^2\sigma_2} \rangle$$

$$\therefore a^{\tilde{G}_1} \oplus a^{\tilde{G}_2} = \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau_1^2}})(1 - (a)^{2\sqrt{1-\tau_2^2}}))}, \right.$$

$$\left. \sqrt{(1 - a^2\eta_1)(1 - a^2\eta_2)}, \sqrt{(1 - a^2\sigma_1)(1 - a^2\sigma_2)} \right\rangle.$$

and

$$a^{\tilde{G}_1} \otimes a^{\tilde{G}_2} = \left\langle (a)^{\sqrt{1-\tau_1^2} + \sqrt{1-\tau_2^2}}, \sqrt{1 - ((1 - (1 - a^2\eta_1))(1 - (1 - a^2\eta_2)))}, \right.$$

$$\left. \sqrt{1 - ((1 - (1 - a^2\sigma_1))(1 - (1 - a^2\sigma_2)))} \right\rangle.$$

(i) Now for any $l > 0$, we have

$$l(a^{\tilde{G}_1} \oplus a^{\tilde{G}_2}) = \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau_1^2}})(1 - (a)^{2\sqrt{1-\tau_2^2}}))^l}, \right.$$

$$\left. (\sqrt{(1 - a^2\eta_1)(1 - a^2\eta_2)})^l, (\sqrt{(1 - a^2\sigma_1)(1 - a^2\sigma_2)})^l \right\rangle$$

$$= \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau_1^2}})^l, \sqrt{(1 - a^2\eta_1)}^l, \sqrt{(1 - a^2\sigma_1)}^l)}, \right.$$

$$\left. \oplus \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau_2^2}})^l, \sqrt{(1 - a^2\eta_2)}^l, \sqrt{(1 - a^2\sigma_2)}^l) \right\rangle \right\rangle$$

$$= la^{\tilde{G}_1} \oplus la^{\tilde{G}_2}.$$

(ii) Again for any $l > 0$, we have

$$(a^{\tilde{G}_1} \otimes a^{\tilde{G}_2})^l = \left\langle (a^{\sqrt{1-\tau_1^2}})^l (a^{\sqrt{1-\tau_2^2}})^l, \right.$$

$$\left. \sqrt{1 - ((1 - (1 - a^2\eta_1))^l (1 - (1 - a^2\eta_2))^l)}, \right.$$

$$\left. \sqrt{1 - ((1 - (1 - a^2\sigma_1))^l (1 - (1 - a^2\sigma_2))^l)} \right\rangle$$

$$= \left\langle (a^{\sqrt{1-\tau_1^2}})^l, \sqrt{1 - (a^2\eta_1)}^l, \sqrt{1 - (a^2\sigma_1)}^l \right\rangle$$

$$\otimes \left\langle (a^{\sqrt{1-\tau_2^2}})^l, \sqrt{1 - (a^2\eta_2)}^l, \sqrt{1 - (a^2\sigma_2)}^l \right\rangle = (a^{\tilde{G}_1})^l \otimes (a^{\tilde{G}_2})^l.$$

(iii) We know

$$a^{\tilde{G}} = \langle a^{\sqrt{1-\tau^2}}, \sqrt{1-a^2\eta}, \sqrt{1-a^2\sigma} \rangle.$$

\therefore For any $l_1 > 0$ and $l_2 > 0$, we have

$$l_1(a^{\tilde{G}}) \oplus l_2(a^{\tilde{G}})$$

$$= \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau^2}})^{l_1}, \sqrt{(1 - a^2\eta)}^{l_1}, \sqrt{(1 - a^2\sigma)}^{l_1})} \right.$$

$$\left. \oplus \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau^2}})^{l_2}, \sqrt{(1 - a^2\eta)}^{l_2}, \sqrt{(1 - a^2\sigma)}^{l_2})} \right\rangle \right\rangle$$

$$= \left\langle \sqrt{1 - ((1 - (a)^{2\sqrt{1-\tau^2}})^{l_1+l_2}, \sqrt{(1 - a^2\eta)}^{l_1+l_2}, \sqrt{(1 - a^2\sigma)}^{l_1+l_2})} \right\rangle$$

$$= (l_1 + l_2) a^{\tilde{G}}.$$

(iv) Again for any $l_1 > 0$ and $l_2 > 0$, we know

$$(a^{\tilde{G}})^{l_1} = \left\langle (a^{\sqrt{1-\tau^2}})^{l_1}, \sqrt{1 - (a^2\eta)^{l_1}}, \sqrt{1 - (a^2\sigma)^{l_1}} \right\rangle,$$

and

$$(a^{\tilde{G}})^{l_2} = \left\langle (a^{\sqrt{1-\tau^2}})^{l_2}, \sqrt{1-(a^{2\eta})^{l_2}}, \sqrt{1-(a^{2\sigma})^{l_2}} \right\rangle,$$

$$\begin{aligned} & \therefore (a^{\tilde{G}})^{l_1} \otimes (a^{\tilde{G}})^{l_2} \\ &= \left\langle (a^{\sqrt{1-\tau^2}})^{l_1}, \sqrt{1-(a^{2\eta})^{l_1}}, \sqrt{1-(a^{2\sigma})^{l_1}} \right\rangle \\ & \otimes \left\langle (a^{\sqrt{1-\tau^2}})^{l_2}, \sqrt{1-(a^{2\eta})^{l_2}}, \sqrt{1-(a^{2\sigma})^{l_2}} \right\rangle \\ &= \left\langle (a^{\sqrt{1-\tau^2}})^{l_1+l_2}, \sqrt{1-(a^{2\eta})^{l_1+l_2}}, \sqrt{1-(a^{2\sigma})^{l_1+l_2}} \right\rangle \\ &= (a^{\tilde{G}})^{l_1+l_2}. \end{aligned}$$

(v) Now

$$\begin{aligned} & a_1^{\tilde{G}} \otimes a_2^{\tilde{G}} \\ &= \left\langle a_1^{\sqrt{1-\tau^2}}, \sqrt{1-a_1^{2\eta}}, \sqrt{1-a_1^{2\sigma}} \right\rangle \otimes \left\langle a_2^{\sqrt{1-\tau^2}}, \sqrt{1-a_2^{2\eta}}, \sqrt{1-a_2^{2\sigma}} \right\rangle \\ &= \left\langle a_1 a_2^{\sqrt{1-\tau^2}}, \sqrt{1-(a_1^{2\eta}(a_2^{2\eta}))}, \sqrt{1-(a_1^{2\sigma}(a_2^{2\sigma}))} \right\rangle \\ &= (a_1 a_2)^{\tilde{G}}. \end{aligned}$$

□

5 Aggregation operator

Definition 10: Let $\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle$ ($i = 1, 2, \dots, n$) be any collection of GSFNs and a_i ($i = 1, 2, \dots, n$) be any collection of real numbers and generalised spherical weighted exponential aggregation (GSWEA): $\Lambda^n \rightarrow \Lambda$. If

$$\text{GSWEA}(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n) = a^{\tilde{G}_1} \oplus a^{\tilde{G}_2} \oplus \dots \oplus a^{\tilde{G}_n}$$

where Λ is the collection of GSFNs, then the function GSWEA is called the generalised spherical weighted exponential averaging operator, where \tilde{G}_i are the exponential weights of a_i ($i = 1, 2, \dots, n$).

Theorem 6: Let $\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle$ ($i = 1, 2, \dots, n$) be any collection of GSFNs, then the aggregated value by using the GSWEA operator is also GSFN and is given by

(see (1))

where \tilde{G}_i are the exponential weights of a_i ($i = 1, 2, \dots, n$).

Proof: We can prove the result given in (1) by using mathematical induction on n and let $a \in (0, 1)$. Since for each i ,

$\tilde{G}_i = \langle \tau_i, \eta_i, \sigma_i \rangle$ is a GSFN, then $\tau_i, \eta_i, \sigma_i \in [0, 1]$ and $\tau_i^2 + \eta_i^2 + \sigma_i^2 \leq 1$. Then, we have the following steps as follows:

Step 1: when $n = 2$, we have

$$\text{GSWEA}(\tilde{G}_1, \tilde{G}_2) = a_1^{\tilde{G}_1} \oplus a_2^{\tilde{G}_2}$$

From Theorem 2, we can see that $a_1^{\tilde{G}_1}$ and $a_2^{\tilde{G}_2}$ are GSFNs and the value of $a_1^{\tilde{G}_1} \oplus a_2^{\tilde{G}_2}$ are also a GSFN.

Now, we have

$$a_1^{\tilde{G}_1} = \left\langle a_1^{\sqrt{1-\tau_1^2}}, \sqrt{1-a_1^{2\eta_1}}, \sqrt{1-a_1^{2\sigma_1}} \right\rangle$$

and

$$a_2^{\tilde{G}_2} = \left\langle a_2^{\sqrt{1-\tau_2^2}}, \sqrt{1-a_2^{2\eta_2}}, \sqrt{1-a_2^{2\sigma_2}} \right\rangle$$

$$\begin{aligned} \therefore \text{GSWEA}(\tilde{G}_1, \tilde{G}_2) &= a_1^{\tilde{G}_1} \oplus a_2^{\tilde{G}_2} \\ &= \left\langle \sqrt{1-(1-(a_1)^2\sqrt{1-\tau_1^2})(1-(a_2)^2\sqrt{1-\tau_2^2})}, \right. \\ & \quad \left. \sqrt{(1-a_1^{2\eta_1})(1-a_2^{2\eta_2})}, \sqrt{(1-a_1^{2\sigma_1})(1-a_2^{2\sigma_2})} \right\rangle \\ &= \left\langle \sqrt{1-\prod_{i=1}^2 (1-(a_i)^2\sqrt{1-\tau_i^2})}, \sqrt{\prod_{i=1}^2 (1-a_i^{2\eta_i})}, \sqrt{\prod_{i=1}^2 (1-a_i^{2\sigma_i})} \right\rangle \end{aligned}$$

\therefore (1) is true for $n = 2$.

Step 2: Let us assume that (1) is true for $n = k$. Then

$$\begin{aligned} & \text{GSWEA}(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n) \\ &= \left\langle \sqrt{1-\prod_{i=1}^k (1-(a_i)^2\sqrt{1-\tau_i^2})}, \right. \\ & \quad \left. \sqrt{\prod_{i=1}^k (1-a_i^{2\eta_i})}, \sqrt{\prod_{i=1}^k (1-a_i^{2\sigma_i})} \right\rangle; \end{aligned}$$

and the aggregated value is GSFN.

Now for $n = k + 1$, we have

$$\begin{aligned} & \text{GSWEA}(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_k, \tilde{G}_{k+1}) \\ &= a^{\tilde{G}_1} \oplus a^{\tilde{G}_2} \oplus \dots \oplus a^{\tilde{G}_k} \oplus a^{\tilde{G}_{k+1}} \\ &= \left\langle \sqrt{1-\prod_{i=1}^k (1-(a_i)^2\sqrt{1-\tau_i^2})}, \sqrt{\prod_{i=1}^k (1-a_i^{2\eta_i})}, \sqrt{\prod_{i=1}^k (1-a_i^{2\sigma_i})} \right\rangle \\ &= \oplus \left\langle a_{k+1}^{\sqrt{1-\tau_{k+1}^2}}, \sqrt{1-a_{k+1}^{2\eta_{k+1}}}, \sqrt{1-a_{k+1}^{2\sigma_{k+1}}} \right\rangle \\ &= \left\langle \sqrt{1-\prod_{i=1}^{k+1} (1-(a_i)^2\sqrt{1-\tau_i^2})}, \sqrt{\prod_{i=1}^{k+1} (1-a_i^{2\eta_i})}, \sqrt{\prod_{i=1}^{k+1} (1-a_i^{2\sigma_i})} \right\rangle; \end{aligned}$$

and the aggregated value is GSFN.

Therefore, (1) is true for $n = k + 1$.

Hence by mathematical induction, we can say (1) is true for all positive values of n .

$$\begin{aligned} & \text{GSWEA}(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n) \\ &= \begin{cases} \left\langle \sqrt{1-\prod_{i=1}^n (1-(a_i)^2\sqrt{1-\tau_i^2})}, \sqrt{\prod_{i=1}^n (1-a_i^{2\eta_i})}, \sqrt{\prod_{i=1}^n (1-a_i^{2\sigma_i})} \right\rangle; & a \in (0, 1) \\ \left\langle \sqrt{1-\prod_{i=1}^n \left(1-\left(\frac{1}{a_i}\right)^{2\sqrt{1-\tau_i^2}}\right)}, \sqrt{\prod_{i=1}^n \left(1-\left(\frac{1}{a_i}\right)^{2\eta_i}\right)}, \sqrt{\prod_{i=1}^n \left(1-\left(\frac{1}{a_i}\right)^{2\sigma_i}\right)} \right\rangle; & a \geq 1 \end{cases} \end{aligned} \quad (1)$$

Again, if $a_i \geq 1$ and $0 \leq 1/a_i \leq 1$, we can also obtain that

GSWEA($\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n$)

$$\left\langle \sqrt{1 - \prod_{i=1}^n \left(1 - \left(\frac{1}{a_i}\right)^{2\sqrt{1-\tau_i^2}}\right)}, \sqrt{\prod_{i=1}^n \left(1 - \left(\frac{1}{a_i}\right)^{2\eta_i}\right)}, \sqrt{\prod_{i=1}^n \left(1 - \left(\frac{1}{a_i}\right)^{2\sigma_i}\right)} \right\rangle.$$

□

6 Multi-criteria group decision-making method in generalised spherical fuzzy environment

In this paper, we shall crack to find out the best alternatives based on the attribute values which are well-defined by the decision-makers. It is not a very relaxed job to assess the attribute value in terms of crisp number due to the occurrence of impreciseness. The information's of the attribute values are of GSFN in nature. In our regular real lifetime, we often confronted some multi-criteria decision-making problem. Suppose anyone wants to know the most affected disease, according to different doctor's opinion in view of symptoms in the human body. We know that people of our country regularly faced lots of problems in the human body and different kinds of diseases can appear in the human body due to many reasons. Doctors always try to find out the disease after examining the symptoms of the patient's body. They will give different weights which are uncertain parameters for different symptoms and try to give their medicine based on this symptom. Now, someone chooses a different doctor's opinion and wants to know which disease is the most affected one, so this trick becomes a problem of multi-criteria group decision-making problem. In this section, we study a multi-criteria group decision-making problem where there will be a finite number of distinct decision-makers accessible and according to their viewpoints, we want to discover the best alternative. To do so, we create an algorithm such that we can solve the uncertainty problem very easily and using the result of exponential laws and score and accuracy functions.

6.1 Illustration of the MCGDM problem

We consider the MCGDM problem as follows:

Let $D = \{D_1, D_2, \dots, D_t\}$ be the set of t distinct alternatives and $S = \{S_1, S_2, \dots, S_r\}$ be the set of r different attributes. We also consider the set of decision-makers $E = \{E_1, E_2, \dots, E_m\}$ related with alternatives whose weight value is given as $\rho = \{\rho_1, \rho_2, \dots, \rho_t\}$ where each $\rho_m \geq 0$ and also meets the condition $\sum_{i=1}^m \rho_i = 1$, this weight value will be taken according to the decision-maker's ability of judgement, knowledge power, thinking ability etc. According to the decision-makers preferences, we first construct the decision matrices related with alternatives versus attribute functions. We take the element of the matrices in real number lying in the interval $[0, 1]$. Let $\tilde{G}^k = (\tilde{G}_1^k, \tilde{G}_2^k, \dots, \tilde{G}_r^k)^T$ be the weight vector corresponding to the attribute values μ_{ij}^k ($i=1, 2, \dots, t$) ($j=1, 2, \dots, r$) where each of \tilde{G}_i^k is the GSFN and is given by $\tilde{G}_i^k = \langle \tau_i^k, \eta_i^k, \sigma_i^k \rangle$ ($i=1, 2, \dots, r$) and $\mu_{ij}^k \in [0, 1]$, then the associated decision matrix is given as follows:

$$M^k = \begin{matrix} & S_1 & S_2 & \cdots & S_r \\ \begin{matrix} D_1 \\ D_2 \\ \vdots \\ D_t \end{matrix} & \begin{pmatrix} \mu_{11}^k & \mu_{12}^k & \cdots & \mu_{1r}^k \\ \mu_{21}^k & \mu_{22}^k & \cdots & \mu_{2r}^k \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{t1}^k & \mu_{t2}^k & \cdots & \mu_{tr}^k \end{pmatrix} \end{matrix}$$

where $k=1, 2, \dots, m$.

Then, our proposed decision-making method under GSFN environment has been given as follows:

Step 1: Now we utilise the GSWEA operator on each individual matrix M^k according to (1) and we get new column matrix $A_{t \times 1}^k$ as follows:

$$A_{t \times 1}^k = \text{GSWEA}(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n) \\ = \begin{matrix} D_1 \\ D_2 \\ \vdots \\ D_t \end{matrix} \begin{pmatrix} \tilde{G}_{11}^k \\ \tilde{G}_{21}^k \\ \vdots \\ \tilde{G}_{t1}^k \end{pmatrix}$$

where each entity of column matrix $A_{t \times 1}^k$ is GSFNs ($k=1, 2, \dots, m$).

Step 2: Here, we obtain final decision matrix after using decision-maker's weight according to operation $\sum_{k=1}^m \rho_k A_{t \times 1}^k$ (scalar multiplication and addition of GSFNs) and we get overall attribute values \tilde{G}_{j1} for the alternatives D_j ($j=1, 2, \dots, t$) as follows:

$$M = \begin{matrix} D_1 \\ D_2 \\ \vdots \\ D_t \end{matrix} \begin{pmatrix} \tilde{G}_{11} \\ \tilde{G}_{21} \\ \vdots \\ \tilde{G}_{t1} \end{pmatrix}$$

Step 3: Utilising the score function, we calculate $S(\tilde{G}_{j1})$ of the alternatives D_j ($j=1, 2, \dots, t$). If two scores $S(\tilde{G}_{i1})$ and $S(\tilde{G}_{j1})$ are equal, then we calculate the corresponding accuracy values $A(\tilde{G}_{i1})$ and $A(\tilde{G}_{j1})$.

Step 4: After getting all the score values of alternatives, the alternatives are ranked and choose the best one.

Step 5: End.

6.2 Illustrative examples

Let us consider a problem related with three different diseases and their distinctive symptoms. We know that, people of our country are affected by many diseases throughout the year and they face disjunctive kind of symptoms in different times. Whenever, they come to meet the doctors they will give their opinion according to their symptoms and other pathological test reports. Now, we want to find out the most affected disease after studying different kinds of doctor's opinion according to their view of the symptoms of patients. So, it is the MCGDM problem with distinct types of decision-makers. We choose the problem as follows: D_1 =Disease 1, D_2 =Disease 2, D_3 =Disease 3 are the alternatives. S_1 =Symptom 1, S_2 =Symptom 2, S_3 =Symptom 3 are the attributes. Let us consider there are four decision-makers E_1 =Junior Doctor (MBBS), E_2 =Senior Doctor (MBBS), E_3 =Junior Doctor (M.D.), E_4 =Senior Doctor (M.D.) having weight function $\rho = (0.15, 0.20, 0.30, 0.35)$ and we also consider different weight vectors associated with different attribute functions for distinct decision-maker. Let $\tilde{G}_1^1 = ((0.8, 0.6, 0.5); \langle 0.55, 0.65, 0.75 \rangle; \langle 0.75, 0.70, 0.65 \rangle)$, $\tilde{G}_2^1 = ((0.45, 0.60, 0.8); \langle 0.8, 0.7, 0.5 \rangle; \langle 0.5, 0.7, 0.9 \rangle)$, $\tilde{G}_3^1 = ((0.45, 0.55, 0.75); \langle 0.7, 0.8, 0.4 \rangle; \langle 0.8, 0.7, 0.7 \rangle)$, $\tilde{G}_4^1 = ((0.55, 0.65, 0.75); \langle 0.6, 0.7, 0.8 \rangle; \langle 0.7, 0.6, 0.7 \rangle)$ be the weights of the attributes in generalised fuzzy spherical environment according to the preferences of the decision-makers E_1, E_2, E_3, E_4 , respectively. The decision matrices M^1, M^2, M^3 and M^4 according to the decision-makers E_1, E_2, E_3 , and

E_4 are given by

$$\begin{aligned} M^1 &= \begin{matrix} & S_1 & S_2 & S_3 \\ \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} & \begin{pmatrix} 0.8 & 0.5 & 0.6 \\ 0.7 & 0.5 & 0.7 \\ 0.3 & 0.6 & 0.4 \end{pmatrix} \end{matrix} \\ M^2 &= \begin{matrix} & S_1 & S_2 & S_3 \\ \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} & \begin{pmatrix} 0.7 & 0.6 & 0.5 \\ 0.4 & 0.7 & 0.6 \\ 0.5 & 0.8 & 0.3 \end{pmatrix} \end{matrix} \\ M^3 &= \begin{matrix} & S_1 & S_2 & S_3 \\ \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} & \begin{pmatrix} 0.8 & 0.6 & 0.7 \\ 0.6 & 0.8 & 0.6 \\ 0.7 & 0.5 & 0.8 \end{pmatrix} \end{matrix} \\ M^4 &= \begin{matrix} & S_1 & S_2 & S_3 \\ \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} & \begin{pmatrix} 0.6 & 0.7 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.5 & 0.5 \end{pmatrix} \end{matrix} \end{aligned}$$

respectively. Now we apply our proposed MCGDM method to select the best one.

Step 1: Now we utilise the GSWEA operator on each matrix M^k according to (1) and we get new column matrix $A_{3 \times 1}^k$ ($k=1, 2, 3, 4$) as follows:

$$\begin{aligned} A_{3 \times 1}^1 &= \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} \begin{pmatrix} \langle 0.9596, 0.267, 0.2505 \rangle \\ \langle 0.954, 0.2851, 0.2682 \rangle \\ \langle 0.8318, 0.5177, 0.5107 \rangle \end{pmatrix} \\ A_{3 \times 1}^2 &= \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} \begin{pmatrix} \langle 0.9215, 0.3324, 0.3521 \rangle \\ \langle 0.914, 0.366, 0.3725 \rangle \\ \langle 0.9241, 0.3513, 0.3445 \rangle \end{pmatrix} \\ A_{3 \times 1}^3 &= \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} \begin{pmatrix} \langle 0.9699, 0.2186, 0.1937 \rangle \\ \langle 0.9618, 0.2568, 0.2114 \rangle \\ \langle 0.9646, 0.2416, 0.2175 \rangle \end{pmatrix} \\ A_{3 \times 1}^4 &= \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} \begin{pmatrix} \langle 0.9653, 0.2117, 0.2499 \rangle \\ \langle 0.8533, 0.4451, 0.5057 \rangle \\ \langle 0.9179, 0.3272, 0.3791 \rangle \end{pmatrix} \end{aligned}$$

Step 2: Now we use decision-makers weight according to operation $\sum_{k=1}^m \rho_k A_{3 \times 1}^k$ (scalar multiplication and addition of GSFNs) and we get overall attribute values \tilde{G}_{j1} for the alternatives D_j ($j=1, 2, 3, 4$) as follows:

$$M = \begin{matrix} D_1 \\ D_2 \\ D_3 \end{matrix} \begin{pmatrix} \langle 0.9541, 0.2617, 0.2658 \rangle \\ \langle 0.9214, 0.3535, 0.3445 \rangle \\ \langle 0.9234, 0.3432, 0.3455 \rangle \end{pmatrix}$$

Step 3: Here, we calculate the score values of \tilde{G}_{j1} ($j=1, 2, 3$) according to Definition 6 of score value and we get

$$S(\tilde{G}_{11}) = 0.8411$$

$$S(\tilde{G}_{21}) = 0.7262$$

$$S(\tilde{G}_{31}) = 0.7344$$

Step 4: Therefore, the ranking order of score value is $S(\tilde{G}_{11}) > S(\tilde{G}_{31}) > S(\tilde{G}_{21})$.

Hence, according to Definition 7, the ranking order of the alternatives is

$$D_1 > D_3 > D_2.$$

Therefore, D_1 is the best option (Fig. 2).

6.3 Sensitivity analysis

The main idea of sensitivity analysis is to exchange weights of the decision-makers keeping the rest of the terms are fixed, namely the logical approach. Here, a sensitivity analysis is done to understand how the decision-makers weight affecting the relative matrix and their ranking. The sensitivity analysis result is given in Table 1 and Figs. 3 and 4 represent the associated weights of different decision-makers and the preferences of ranking order respectively.

Remark 3: Since GSFN and exponential operational law are newly defined in this paper, so there is no scope to present any comparative analysis here.

Flowchart:

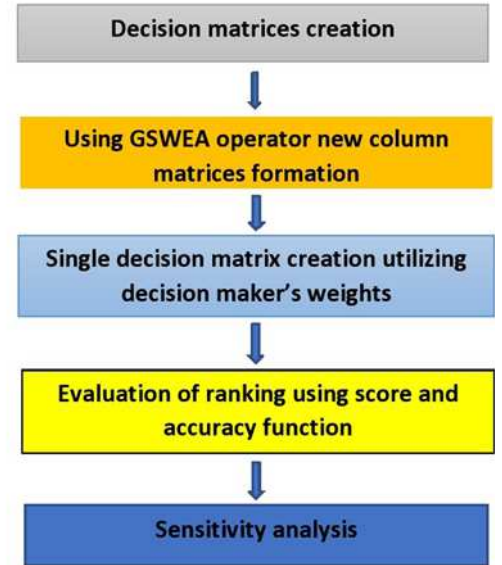


Fig. 2 Flowchart of the proposed MCGDM technique

Table 1 Sensitivity analysis

Decision-makers weight	Final decision matrix	Ranking order
$\langle 0.15, 0.2, 0.3, 0.35 \rangle$	$\begin{pmatrix} \langle 0.9541, 0.2617, 0.2658 \rangle \\ \langle 0.9214, 0.3535, 0.3445 \rangle \\ \langle 0.9234, 0.3432, 0.3455 \rangle \end{pmatrix}$	$D_1 > D_3 > D_2$
$\langle 0.2, 0.25, 0.25, 0.3 \rangle$	$\begin{pmatrix} \langle 0.9577, 0.2503, 0.2556 \rangle \\ \langle 0.9368, 0.3157, 0.307 \rangle \\ \langle 0.9234, 0.3432, 0.3455 \rangle \end{pmatrix}$	$D_1 > D_2 > D_3$
$\langle 0.25, 0.25, 0.25, 0.25 \rangle$	$\begin{pmatrix} \langle 0.9574, 0.2531, 0.2556 \rangle \\ \langle 0.9404, 0.3088, 0.2975 \rangle \\ \langle 0.9325, 0.3225, 0.3216 \rangle \end{pmatrix}$	$D_1 > D_2 > D_3$
$\langle 0.22, 0.27, 0.28, 0.33 \rangle$	$\begin{pmatrix} \langle 0.9674, 0.2174, 0.2223 \rangle \\ \langle 0.9431, 0.3026, 0.2963 \rangle \\ \langle 0.939, 0.3031, 0.3065 \rangle \end{pmatrix}$	$D_1 > D_2 > D_3$
$\langle 0.15, 0.25, 0.28, 0.32 \rangle$	$\begin{pmatrix} \langle 0.9582, 0.2476, 0.2536 \rangle \\ \langle 0.9265, 0.3398, 0.3337 \rangle \\ \langle 0.9353, 0.3278, 0.3313 \rangle \end{pmatrix}$	$D_1 > D_3 > D_2$
$\langle 0.18, 0.22, 0.28, 0.32 \rangle$	$\begin{pmatrix} \langle 0.9646, 0.2344, 0.2329 \rangle \\ \langle 0.9372, 0.3193, 0.3073 \rangle \\ \langle 0.9144, 0.3611, 0.3665 \rangle \end{pmatrix}$	$D_1 > D_2 > D_3$

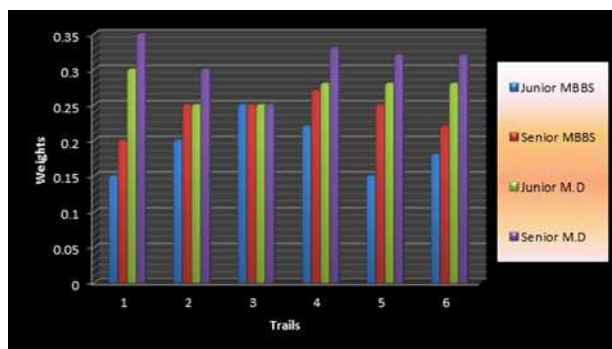


Fig. 3 Input table associative with decision-makers weights

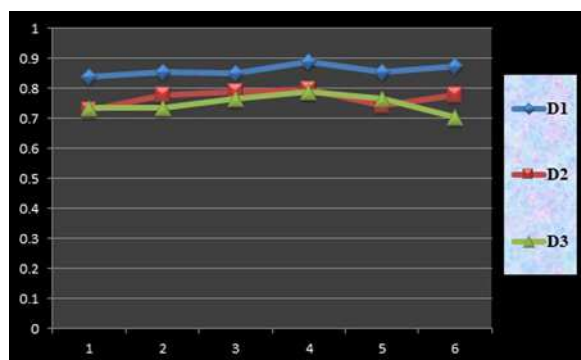


Fig. 4 Output table related with the preferences of the alternatives

7 Conclusion

In this paper, we initiated the concept of GSFNs and their associated properties are also discussed. Then, we have also defined score and accuracy functions on GSFNs. Based on these GSFNs, we invented exponential operational law of GSFNs in which the bases are positive real numbers and its exponents are GSFNs. Then, we have interpreted their associated properties. Next, we proposed the GSWEA operator based on our defined operational law. Then, by using this aggregation operator, we designed a multi-criteria group decision-making method for solving a numerical problem under generalised spherical fuzzy environment. Finally, a numerical example and sensitivity analysis were given to demonstrate the proposed method. From this paper, we conclude that the generalised spherical fuzzy environment is a much more suitable environment to tackle the uncertainty and vagueness theory. Furthermore, exponential operational law is a fruitful supplement of extant operational law and MCGDM method is a new way to solve decision-making problem. In future work, we proposed other operational laws and developed a various method in generalised spherical fuzzy environment.

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