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HARDY'S INEQUALITIES WITH REMAINDERS AND LAMB-TYPE EQUATIONS

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Abstract: We study Hardy-type integral inequalities with remainder terms for smooth compactly-supported functions in convex domains of finite inner radius. New L_1- and L_p-inequalities are obtained with constants depending on the Lamb constant which is the first positive solution to the special equation for the Bessel function. In some particular cases the constants are sharp. We obtain one-dimensional inequalities and their multidimensional analogs. The weight functions in the spatial inequalities contain powers of the distance to the boundary of the domain. We also prove that some function depending on the Bessel function is monotone decreasing. This property is essentially used in the proof of the one-dimensional inequalities. The new inequalities extend those by Avkhadiev and Wirths for $p = 2$ to the case of every $p \geq 1$.

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Introduction

This article is devoted to the variational Hardy-type inequalities with remainder terms. The classical one-dimensional Hardy inequality for an absolutely continuous function $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ and $f' \in L^2(0, \infty)$ looks as follows:

$$\int_0^\infty \frac{|f(x)|^2}{x^2}\,dx \leq 4\int_0^\infty |f'(x)|^2\,dx.$$

The constant 4 is unimprovable, even though there exists no extremal function $f \not\equiv 0$ at which equality is attained (see, for instance, [1]). The sharp constant in this inequality is the norm of the corresponding linear operator and, as it is well known, the norm of an operator is an upper bound for its spectrum. For instance, if we rewrite the latter inequality in terms of operators; i.e., if we put

$$h(x) = f'(x) \quad \text{and} \quad (Hh)(x) = x^{-1}\int_0^x h(t)\,dt,$$

then we find that H is a bounded linear operator in $L^2(0, \infty)$ and the following equality is valid for its norm:

$$\|H\|^2 = \|H : L^2(0, \infty) \to L^2(0, \infty)\|^2 = 4.$$

Moreover, the spectrum of H lies in the disk of radius $\|H\|$.

The classical Hardy inequality has been generalized and modified in various ways, and the bibliography on this topic is rather extensive (see [1–46]), because Hardy's inequalities have a wide range of applications in mathematics and mathematical physics (see [2–7, 46]). Let us give several examples of applications of the inequalities. For instance, Sobolev [2] used the Hardy-type inequalities in the embedding

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theory of function spaces; thus, the estimates of the form are perceived as the tools of function theory which are used in proofs of the corresponding assertions (see also [3]). In [4] Avkhadiev applied Hardy's inequalities to estimate the torsional rigidity of a beam with cross-section for arbitrary simply connected domains. Dubinskii showed in [6] that the well-posedness of the problem for the Poisson equation with a special normalization condition is equivalent to the validity of the corresponding two-sided scales of Hardy's inequalities (see also [7, 45]).

The spatial Hardy inequalities are extensively studied. Multidimensional Hardy-type inequalities in convex domains [8–11], in domains whose boundaries meet some special conditions [12–14], and even in arbitrary open sets of the Euclidean space [15–18] are well known. In the case of Hardy's inequality in a spatial domain Ω, the function f and its derivative f' are replaced with a smooth function $f : \Omega \to \mathbb{R}$ compactly-supported in Ω and its gradient $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$, and the powers of x are replaced with the powers of the distance $\delta = \delta(x) = \operatorname{dist}(x, \partial\Omega)$ from a point to the boundary of Ω. Moreover, the Hardy-type inequalities with weight functions depending on the hyperbolic radius (see [19–24]) are well studied in the spectral theory of the Laplace–Beltrami operator on Riemannian manifolds of constant negative curvature.

As was already mentioned, in this article we consider Hardy's inequalities with remainder terms. Mazya [3] was the first to prove such inequalities in the case when the integration domain is the upper half-plane. Later some interesting results on Hardy's inequalities with remainder terms followed the article by Brezis and Marcus [10] in which they proved that

$$\int\limits_{\Omega} |\nabla f|^2\,dx \geq \frac{1}{4}\int\limits_{\Omega} \frac{|f|^2}{\delta^2}\,dx + \frac{1}{4(\operatorname{diam}\Omega)^2}\int\limits_{\Omega} |f|^2\,dx \quad \forall f \in H_0^1(\Omega), \tag{1}$$

where Ω is a convex bounded domain, $H_0^1(\Omega)$ is the closure of the space $C_0^1(\Omega)$ of smooth functions $f : \Omega \to \mathbb{R}$ with the finite Dirichlet integral and compact support in Ω, and $\operatorname{diam}\Omega$ is the diameter of Ω.

Note that the constant in (1) depends on the diameter $\operatorname{diam}\Omega$ of Ω. Similar inequalities are obtained in [11, 25] with the constant depending on the volume of Ω. For instance, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and Laptev proved in [11] the inequality

$$\int\limits_{\Omega} |\nabla f(x)|^2\,dx \geq \frac{1}{4}\int\limits_{\Omega} \frac{|f|^2}{\delta(x)^2}\,dx + \frac{1}{4}\frac{K(n)}{|\Omega|^{2/n}}\int\limits_{\Omega} |f(x)|^2\,dx \tag{2}$$

for every $f \in H_0^1(\Omega)$, where Ω is a convex domain in $\mathbb{R}^n$ $(n \geq 2)$, $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, while $|\Omega|$ is the volume of Ω, and

$$K(n) = n\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{2/n}.$$

The constant $1/4$ is sharp. Later, Evans and Lewis in [26] improved the constant in the remainder term of (2), and Tidblom in [25] obtained an L_p-analog of this inequality in the corresponding Sobolev spaces for $p > 1$.

It is clear that if $\operatorname{diam}\Omega$ or $|\Omega|$ tends to infinity, then (1) and (2) do not contain remainder terms.

Observe also that (1) and (2) are different as classes of extremal problems. Inequality (1) is proven in domains of finite diameter; inequality (2), of finite volume. The class of extremal problems in domains of finite inner radius

$$\delta_0 = \delta_0(\Omega) = \sup_{x\in\Omega} \delta(x)$$

is also well known; i.e., the inequalities that are proven in the domains with $\delta_0 < \infty$ (see, e.g., [8, 9, 18, 39]). The class of extremal problems in domains of finite interior radius is wider than the class of problems with finite volume or diameter, since there exist domains of finite interior radius whose volume or diameter are not bounded (for instance, a strip).

In the authors' opinion, among these inequalities in domains of finite inner radius, the sharp inequality of [8] plays a significant role constituting a bridge between Hardy's and Poincaré's inequalities. Namely, it was demonstrated there that if $q > 0$, $0 < \nu \le 1/q$, and Ω is a convex domain such that $\delta_0(\Omega) < \infty$, then

$$\int\limits_{\Omega} |\nabla f(x)|^2\, dx \ge \frac{1-\nu^2 q^2}{4} \int\limits_{\Omega} \frac{|f(x)|^2}{\delta(x)^2}\, dx + \frac{C^2 q^2}{4\delta_0(\Omega)^q} \int\limits_{\Omega} \frac{|f(x)|^2}{\delta(x)^{2-q}}\, dx \quad \text{for all } f \in C_0^1(\Omega),$$

where $C_0^1(\Omega)$ is the space of smooth functions compactly-supported in Ω and the constant $C = C_\nu(q)$ is the first positive root of the following equation for the Bessel function J_ν of order ν:

$$J_\nu(C) + CqJ'_\nu(C) = 0.$$

Later, in [9] this result was generalized to the case of more general weights. Namely, it was proven that

$$\int\limits_{\Omega} \frac{|\nabla f(x)|^2}{\delta(x)^{s-1}}\, dx \ge \frac{s^2-\nu^2 q^2}{4} \int\limits_{\Omega} \frac{|f(x)|^2}{\delta(x)^{s+1}}\, dx + \frac{C^2 q^2}{4\delta_0(\Omega)^q} \int\limits_{\Omega} \frac{|f(x)|^2}{\delta(x)^{s+1-q}}\, dx \tag{3}$$

for every $f \in C_0^1(\Omega)$, where $s > 0$, $q > 0$, $\nu \in [0, s/q]$, and now the constant $C = C_\nu(s,q)$ is the first positive root of the more general equation

$$sJ_\nu(C) + CqJ'_\nu(C) = 0.$$

The aim of this article is to obtain analogs of (3) in L_p for $p \ge 1$. We prove one-dimensional inequalities and their multidimensional analogs in domains of finite inner radius; i.e., the extremal problem of the third kind is considered. It is well known that in L_p for $p \in (0,1)$ Hardy's inequalities do not hold for arbitrary nonnegative measurable functions; although they are valid, for instance, for nonnegative nonincreasing functions; cp. [27, 28] (see also [29, 43]). Emphasize that we managed to obtain L_1-inequalities. Hardy's inequalities in L_1 are of a special interest (see, for instance, [30]), because they are connected with partial differential equations with the so-called 1-Laplacian, an analog of the usual Laplace operator and p-Laplacian for $p > 1$.

In the inequalities considered here, the constants depend on the first positive root of the following equation for the Bessel function:

$$(s - q\mu)J_\nu(\lambda_\nu) + q\lambda_\nu J'_\nu(\lambda_\nu) = 0, \tag{4}$$

where $s > 0$, $q > 0$, and $\mu \in \left[0, \frac{s+\nu q}{q}\right)$. Observe that we could include the case of $\mu = \frac{s+\nu q}{q}$ into the latter interval for μ; but, as we will show later, $\lambda_\nu \to 0$ in that case.

Equations (4) are considered in [31] (see also [32, p. 502]); therefore, in line with [8, 9, 33–36], we call λ_ν the *Lamb constant*; and (4), the *parametric Lamb equation*.

Solving the parametric Lamb equation directly is relatively difficult, because it is necessary to consider particular cases and make preliminary transformations. For instance, since

$$\nu J_\nu(\lambda_\nu) + \lambda_\nu J'_\nu(\lambda_\nu) = \lambda_\nu J_{\nu-1}(\lambda_\nu),$$

equation (4) can be rewritten in simpler form:

$$\lambda_\nu J_{\nu-1}(\lambda_\nu) = 0$$

for $s > \nu q$ and $\mu = (s - \nu q)/q$. Clearly, the Lamb constant λ_ν is equal to $j_{\nu-1}$ in this case, where $j_{\nu-1}$ is the first positive root of the Bessel function $J_{\nu-1}$ of order $\nu - 1$.

If we consider (4) for $\nu = 1/2$, i.e., for $J_{1/2}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{\sqrt{x}}$; then we obtain the following equation for the Lamb constant $\lambda_{1/2}$:

$$2q\lambda_{1/2}\cos\lambda_{1/2} + (2s - 2q\mu - q)\sin\lambda_{1/2} = 0.$$

Some interesting approach to solving Lamb equations in general form was found by Avkhadiev and Wirths in [9]. Namely, they proved that the Lamb constant $z = \lambda_\nu(r)$ defined as the first positive root of the equation

$$rJ_\nu(z) + 2zJ_\nu'(z) = 0$$

for r can be found as a solution to the following Cauchy problem for which numerical methods are well developed:

$$\frac{dz}{dr} = \frac{2z}{r^2 - 4\nu^2 + 4z^2}.$$

Usually, the differential equations connecting the weight functions in the integral inequalities are used to prove the inequalities in L_2 and L_p for $p > 1$ (see, for instance, [8, 9, 37, 38]). We are unaware of any other articles in which differential equations are applied to prove the Hardy-type L_1-inequalities.

The present article is structured as follows: In Section 1 we obtain the one-dimensional L_1- and L_p-inequalities. At that, the monotonicity is essential of the specially introduced function depending on the Bessel function. In particular, the following inequality is proven for every absolutely continuous function f on $[0,1]$ such that $f(0) = 0$ and

$$(s^2 - \nu^2 q^2) \int_0^1 \frac{|f(x)|}{x^{s+1}} dx + q^2 \lambda_\nu^2\left(\frac{2s}{q}\right) \int_0^1 \frac{|f(x)|}{x^{s-q+1}}\, dx$$
$$\leq \int_0^1 \frac{|f'(x)|}{x^s} \left(2(s+\nu q) - \frac{q^2\mu^2}{s} + q\mu\left(\frac{q\mu}{s} - 2\right)x^s\right) dx$$

for $s, q > 0$, where $\mu \in \left[0; \frac{s+\nu q}{q}\right)$ and the constant $z = \lambda_\nu(r)$ is the first positive root of the equation

$$(r - 2\mu)J_\nu(z) + 2zJ_\nu'(z) = 0.$$

As a consequence of this result for $s > 0$, $q > 0$, and $\mu = \frac{s-\nu q}{q} > 0$, for every absolutely continuous $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$, we obtain

$$(s - \nu q) \int_0^1 \frac{|f(x)|}{x^{s+1}}\, dx + \frac{q^2 j_{\nu-1}^2}{s+\nu q} \int_0^1 \frac{|f(x)|}{x^{s-q+1}}\, dx$$
$$\leq \left(2 - \frac{(s-\nu q)^2}{s(s+\nu q)}\right) \int_0^1 \frac{|f'(x)|}{x^s}\, dx - \frac{s-\nu q}{s} \int_0^1 |f'(x)|\, dx,$$

where $j_{\nu-1}$ is the first positive root of the Bessel function $J_{\nu-1}$.

Section 2 is devoted to multidimensional inequalities in convex domains in $\mathbb{R}^n$ of finite inner radius. To prove these assertions, we use Avkhadiev's approach from [16, 17] which bases on the classical approximation of a domain by cubes. This method makes it possible to extend the corresponding one-dimensional inequalities to the case of spatial domains.

Remark. The proof of sharpness of the constants in the one-dimensional inequalities is a very difficult problem, all the more so in the multidimensional case. As a consequence of our estimates, we obtain the available sharp inequalities or inequalities comparable with the well-known sharp inequalities (see, for instance, Examples 1–3 below). Therefore, we only claim that, in particular cases, the constants in the proven inequalities are sharp.

1. One-Dimensional Inequalities

Henceforth we need the function

$$y = F_{\nu,s,q}(x) = x^{\frac{s}{2}} J_\nu(\lambda_\nu(2s/q)x^{\frac{q}{2}}),$$

where J_ν is the Bessel function of order ν defined as follows:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}, \quad \nu > 0.$$

Recall that the Lamb constant $z = \lambda_\nu(r)$ is determined as the first positive root of the equation

$$rJ_\nu(z) + 2zJ'_\nu(z) = 0, \quad \nu \geq 0.$$

It is well known (see [47, p. 440; 8, 9]) that $y = F_{\nu,s,q}(x)$ is a solution to the differential equation

$$x^2 y'' + (1-s)xy' + \left(\frac{s^2 - \nu^2 q^2}{4} + \frac{q^2 \lambda_\nu^2(2s/q)}{4x^{-q}}\right) y = 0, \quad x \in [0,1], \tag{5}$$

and

$$\lim_{t \to 0+} \frac{tF'_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} = \frac{s+\nu q}{2} > 0.$$

The reader can learn more details about the properties and examples of using $y = F_{\nu,s,q}(x)$ in [8, 9].

1.1. Auxiliary assertions. Obtain some new properties of the Bessel function which will be used henceforth.

Lemma 1. *Let $s > 0$, $q > 0$, $\nu \geq 0$, and $\lambda_\nu(2s/q) \in [0, j_\nu)$, where j_ν is the first positive root of the Bessel function J_ν of order ν. If $x \in [0,1]$ and $y = F_{\nu,s,q}(x) = x^{\frac{s}{2}} J_\nu(\lambda_\nu(2s/q)x^{\frac{q}{2}})$, then the function $xy'(x)/y(x)$ decreases and*

$$\sup_{x\in[0,1]} \frac{xy'(x)}{y(x)} = \lim_{x\to 0} \frac{xy'(x)}{y(x)} = \frac{s+\nu q}{2},$$

$$\inf_{x\in[0,1]} \frac{xy'(x)}{y(x)} = \frac{y'(1)}{y(1)} = \frac{s}{2} + q\lambda_\nu(2s/q)\frac{J'_\nu(\lambda_\nu(2s/q))}{2J_\nu(\lambda_\nu(2s/q))}.$$

PROOF. To prove the decrease of $xy'(x)/y(x)$, show that its first derivative is negative. Let $z = \lambda_\nu(2s/q)x^{q/2}$, $x \in [0,1]$, and

$$A := \frac{d}{dx}\left(\frac{xy'(x)}{y(x)}\right) = \frac{y'(x)}{y(x)} + x\left(\frac{y''(x)}{y(x)} - \left(\frac{y'(x)}{y(x)}\right)^2\right).$$

It is obvious that $z \in [0, j_\nu)$ and

$$\frac{y'(x)}{y(x)} = \frac{1}{2x}\left(s + qz\frac{J'_\nu(z)}{J_\nu(z)}\right).$$

Using (1), we obtain

$$\begin{aligned}
A = \frac{1}{2x}\left(s + qz\frac{J'_\nu(z)}{J_\nu(z)}\right) - \Bigg(\frac{s^2-\nu^2q^2}{4x} + \frac{q^2\lambda_\nu^2(2s/q)}{4x^{1-q}}& \\
+\frac{1-s}{2x}\left(s + qz\frac{J'_\nu(z)}{J_\nu(z)}\right)\Bigg) - \frac{1}{4x}\left(s + qz\frac{J'_\nu(z)}{J_\nu(z)}\right)^2& \\
= \frac{1}{2x}\left(s^2 + sqz\frac{J'_\nu(z)}{J_\nu(z)} - \frac{s^2-\nu^2q^2}{2} - \frac{q^2\lambda_\nu^2(2s/q)}{2x^{-q}} - \frac{1}{2}\left(s + qz\frac{J'_\nu(z)}{J_\nu(z)}\right)^2\right)& \\
= \frac{1}{2x}\left(\frac{s^2}{2} - \frac{s^2-\nu^2q^2}{2} - \frac{q^2\lambda_\nu^2(2s/q)}{2x^{-q}} - \frac{q^2z^2}{2}\left(\frac{J'_\nu(z)}{J_\nu(z)}\right)^2\right)& \\
= \frac{q^2}{4x}\left(\nu^2 - \frac{\lambda_\nu^2(2s/q)}{x^{-q}} - z^2\left(\frac{J'_\nu(z)}{J_\nu(z)}\right)^2\right) = \frac{q^2}{4x}\left(\nu^2 - z^2 - z^2\left(\frac{J'_\nu(z)}{J_\nu(z)}\right)^2\right).&
\end{aligned}$$

Consider the two cases:
Case 1: $\nu = 0$. Obviously,

$$A = -\frac{q^2}{4x}\left(z^2 + z^2\left(\frac{J_0'(z)}{J_0(z)}\right)^2\right) \le 0.$$

Case 2: $\nu > 0$. Clearly,

$$A = \frac{q^2}{4x}\left(\nu^2 - z^2 - z^2\left(\frac{J_\nu'(z)}{J_\nu(z)}\right)^2\right) = \frac{q^2\nu^2}{4x}\left(1 - \frac{z^2}{\nu^2} - \frac{z^2}{\nu^2}\left(\frac{J_\nu'(z)}{J_\nu(z)}\right)^2\right).$$

Using the relations (see [32, p. 17])

$$J_\nu'(z) = \frac{1}{2}(J_{\nu-1}(z) - J_{\nu+1}(z)), \quad J_\nu(z) = \frac{z}{2\nu}(J_{\nu-1}(z) + J_{\nu+1}(z)),$$

we derive

$$\begin{aligned} A &= \frac{q^2\nu^2}{4x}\left(1 - \left(\frac{J_{\nu-1}(z) - J_{\nu+1}(z)}{J_{\nu-1}(z) + J_{\nu+1}(z)}\right)^2 - \frac{z^2}{\nu^2}\right) \\ &= \frac{q^2\nu^2}{4x}\left(\frac{4J_{\nu+1}(z)}{J_{\nu-1}(z) + J_{\nu+1}(z)} - \left(\frac{2J_{\nu+1}(z)}{J_{\nu-1}(z) + J_{\nu+1}(z)}\right)^2 - \frac{z^2}{\nu^2}\right) \\ &= \frac{q^2\nu^2}{4x}\left(\frac{z^2}{\nu(\nu+1)}\frac{J_\nu(z) + J_{\nu+2}(z)}{J_\nu(z)} - \frac{z^2}{\nu^2}\frac{J_{\nu+1}^2(z)}{J_\nu^2(z)} - \frac{z^2}{\nu^2}\right) \\ &= \frac{q^2z^2}{4x}\left(\frac{2\nu}{z}\frac{J_{\nu+1}(z)}{J_\nu(z)} - \frac{J_{\nu+1}^2(z)}{J_\nu^2(z)} - 1\right) = \frac{q^2z^2}{4x}\left(-1 + \frac{J_{\nu+1}(z)J_{\nu-1}(z)}{J_\nu^2(z)}\right). \end{aligned}$$

It is well known that the following equality is valid for the Bessel function (see [32, p. 152]):

$$\frac{1}{4}z^2(J_{\nu-1}^2(z) - J_{\nu-2}(z)J_\nu(z)) = \sum_{n=0}^{\infty}(\nu + 2n)J_{\nu+2n}^2(z).$$

Applying it obviously yields

$$A = \frac{q^2z^2}{4xJ_\nu^2(z)}\big(J_{\nu+1}(z)J_{\nu-1}(z) - J_\nu^2(z)\big) = -\frac{q^2}{xJ_\nu^2(z)}\sum_{n=0}^{\infty}(\nu + 2n + 1)J_{\nu+2n+1}^2(z) \le 0.$$

Thus, $xy'(x)/y(x)$ decreases; therefore, its supremum is attained at the left endpoint of $[0, 1]$ and the infimum, at the right endpoint. $\square$

Corollary 1. *Let $s > 0$, $q > 0$, and $\nu \ge 0$. If $\mu \to \frac{s+\nu q}{q}$ then $\lambda_\nu \to 0$.*

1.2. L_1-inequalities on an interval. Let us derive the one-dimensional Hardy-type L_1-inequalities. They are used later to obtain L_p-inequalities. As examples, we also give here several particular cases of general inequalities.

Lemma 2. *Suppose that $s > 0$, $q > 0$, $\nu \ge 0$, and f is an absolutely continuous function on $[0, 1]$ such that $f(0) = 0$ and $f'(x)/x^s \in L^1[0, 1]$. If $\mu \in \big[0; \frac{s+\nu q}{q}\big)$, then*

$$\begin{aligned} &\frac{s^2 - \nu^2q^2}{4}\int_0^1 \frac{|f(x)|}{x^{s+1}}\,dx + \frac{q^2\lambda_\nu^2\big(\frac{2s}{q}\big)}{4}\int_0^1 \frac{|f(x)|}{x^{s-q+1}}\,dx \\ &\le \left(\frac{s+\nu q}{2} - \frac{q^2\mu^2}{4s}\right)\int_0^1 \frac{|f'(x)|}{x^s}\,dx + \left(\frac{q^2\mu^2}{4s} - \frac{q\mu}{2}\right)\int_0^1 |f'(x)|\,dx; \end{aligned}$$

and if $\mu \le 0$, then

$$(s^2-\nu^2q^2)\int\limits_0^1 \frac{|f(x)|}{x^{s+1}}\,dx + q^2\lambda_\nu^2(2s/q)\int\limits_0^1 \frac{|f(x)|}{x^{s-q+1}}\,dx \le 2(s+\nu q)\int\limits_0^1 \frac{|f'(x)|}{x^s}\,dx - 2q\mu\int\limits_0^1 |f'(x)|\,dx,$$

where the constant $z=\lambda_\nu(r)$ is the first positive solution to the equation

$$(r-2\mu)J_\nu(z)+2zJ'_\nu(z)=0, \quad z\in(0,j_\nu).$$

PROOF. Using the inequality $|f(x)| \le \int_0^x |f'(t)|\,dt$ and changing the order of integration in the repeated integral, we obtain

$$\int\limits_0^1 \frac{|f(x)|}{x^{s+1}}\left(\frac{s^2-\nu^2q^2}{4}+\frac{q^2\lambda_\nu^2(2s/q)}{4x^{-q}}\right)dx \le \int\limits_0^1 |f'(t)|T(t)\,dt,$$

where

$$T(t)=\int\limits_t^1\left(\frac{s^2-\nu^2q^2}{4x^{s+1}}+\frac{q^2\lambda_\nu^2(2s/q)}{4x^{s-q+1}}\right)dx.$$

By (5), we get

$$T(t)=-\int\limits_t^1 \frac{1}{x^{s+1}}\left(\frac{x^2y''(x)}{y(x)}+\frac{(1-s)xy'(x)}{y(x)}\right)dx = -\int\limits_t^1 \left(\frac{y'(x)}{x^{s-1}y(x)}\right)' + \frac{1}{x^{s-1}}\left(\frac{y'(x)}{y(x)}\right)^2 dx,$$

where the obvious relation

$$\frac{d}{dx}\left(\frac{y'(x)}{x^{s-1}y(x)}\right)=\frac{y''(x)}{x^{s-1}y(x)}+\frac{(1-s)y'(x)}{x^s y(x)}-\frac{1}{x^{s-1}}\left(\frac{y'(x)}{y(x)}\right)^2$$

was used.

Consider the two cases.

CASE 1: $xy'(x)/y(x) \ge 0$, $x\in[0,1]$. Since $xy'(x)/y(x)$ is decreasing by Lemma 1, we obtain

$$\inf_{x\in[0,1]} x\frac{y'(x)}{y(x)}=\frac{y'(1)}{y(1)} \quad \text{and} \quad \inf_{x\in[0,1]}\left(x\frac{y'(x)}{y(x)}\right)^2=\left(\frac{y'(1)}{y(1)}\right)^2 \ge 0;$$

hence,

$$T(t)=-\int\limits_t^1 \left(\frac{y'(x)}{x^{s-1}y(x)}\right)' + \frac{1}{x^{s-1}}\left(\frac{y'(x)}{y(x)}\right)^2 dx$$

$$\le \frac{y'(t)}{t^{s-1}y(t)}-\frac{y'(1)}{y(1)}-\left(\frac{y'(1)}{y(1)}\right)^2\int\limits_t^1\frac{dx}{x^{s+1}} = \frac{y'(t)}{t^{s-1}y(t)}-\frac{y'(1)}{y(1)}+\frac{1}{s}\left(\frac{y'(1)}{y(1)}\right)^2-\frac{1}{st^s}\left(\frac{y'(1)}{y(1)}\right)^2.$$

By Lemma 1, the equality

$$\sup_{x\in[0,1]}\frac{xy'(x)}{y(x)}=\frac{s+\nu q}{2}$$

is also valid. Consequently,

$$T(t) \le \frac{1}{t^s} \sup_{t\in[0,1]} \frac{ty'(t)}{y(t)} - \frac{y'(1)}{y(1)} + \frac{1}{s}\left(\frac{y'(1)}{y(1)}\right)^2 - \frac{1}{st^s}\left(\frac{y'(1)}{y(1)}\right)^2$$
$$= \left(\frac{s+\nu q}{2} - \frac{1}{s}\left(\frac{y'(1)}{y(1)}\right)^2\right)\frac{1}{t^s} + \frac{1}{s}\left(\frac{y'(1)}{y(1)}\right)^2 - \frac{y'(1)}{y(1)}.$$

Note that by the condition of Lemma 2, we chose $z = \lambda_\nu(2s/q)$ as a constant meeting the conditions

$$\frac{2}{q}\frac{y'(1)}{y(1)} = \frac{s}{q} + z\frac{J'_\nu(z)}{J_\nu(z)} = \mu \quad \text{and} \quad z \in (0, j_\nu).$$

Thus,

$$\int_0^1 \frac{|f(x)|}{x^{s+1}}\left(\frac{s^2-\nu^2q^2}{4} + \frac{q^2\lambda_\nu^2(2s/q)}{4x^{-q}}\right)dx \le \left(\frac{s+\nu q}{2} - \frac{q^2\mu^2}{4s}\right)\int_0^1 \frac{|f'(x)|}{x^s}\,dx + \left(\frac{q^2\mu^2}{4s} - \frac{q\mu}{2}\right)\int_0^1 |f'(x)|\,dx.$$

Case 2: there exists $x_0 \in (0,1)$ such that $x_0y'(x_0)/y(x_0) = 0$. In this case, since $xy'(x)/y(x)$ decreases, we obtain

$$\inf_{x\in[0,1]} x\frac{y'(x)}{y(x)} = \frac{y'(1)}{y(1)} \quad \text{and} \quad \inf_{x\in[0,1]}\left(x\frac{y'(x)}{y(x)}\right)^2 = 0;$$

hence,

$$T(t) = -\int_t^1 \left(\frac{y'(x)}{x^{s-1}y(x)}\right)' + \frac{1}{x^{s-1}}\left(\frac{y'(x)}{y(x)}\right)^2 dx \le \frac{y'(t)}{t^{s-1}y(t)} - \frac{y'(1)}{y(1)}.$$

By computations as in Case 1, we derive

$$\int_0^1 \frac{|f(x)|}{x^{s+1}}\left(\frac{s^2-\nu^2q^2}{4} + \frac{q^2\lambda_\nu^2(2s/q)}{4x^{-q}}\right)dx \le \int_0^1 \frac{|f'(x)|}{x^s}\left(\frac{s+\nu q}{2} - \frac{q\mu}{2}x^s\right)dx,$$

which concludes the proof of Lemma 2. $\square$

Observe that in [9] the authors considered the Lamb equation under the condition $\mu = 0$. In that case, Lemma 2 takes the following form:

Corollary 2. *Suppose that $s, q > 0$, $\nu \ge 0$, and f is an absolutely continuous function on $[0,1]$ such that $f(0) = 0$ and $f'(x)/x^s \in L^1[0,1]$. Then*

$$(s^2-\nu^2q^2)\int_0^1 \frac{|f(x)|}{x^{s+1}}\,dx + q^2\lambda_\nu^2(2s/q)\int_0^1 \frac{|f(x)|}{x^{s-q+1}}\,dx \le (2s+2\nu q)\int_0^1 \frac{|f'(x)|}{x^s}\,dx,$$

where the constant $z = \lambda_\nu(r)$ is the first positive solution to the equation

$$rJ_\nu(z) + 2zJ'_\nu(z) = 0, \quad z \in (0, j_\nu).$$

Corollary 3. *Suppose that $s > 0$, $q > s$, $\nu \geq 0$, and f is an absolutely continuous function on $[0,1]$ such that $f(0) = 0$ and $f'(x)/x^s \in L^1[0,1]$. If $\mu \in \left(0; \frac{2s}{q}\right]$ and $\nu \in \left[0, \frac{s}{q}\right]$, then*

$$(s^2 - \nu^2 q^2) \int_0^1 \frac{|f(x)|}{x^{s+1}}\, dx + \left(q^2 \lambda_\nu^2\left(\frac{2s}{q}\right) + q\mu(q-s)\left(2 - \frac{q\mu}{s}\right)\right) \int_0^1 \frac{|f(x)|}{x^{s-q+1}}\, dx$$
$$\leq \left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right) \int_0^1 \frac{|f'(x)|}{x^s}\, dx,$$

where the constant $z = \lambda_\nu(r)$ is the first positive solution to the equation

$$(r - 2\mu) J_\nu(z) + 2z J_\nu'(z) = 0, \quad z \in (0, j_\nu).$$

Proof. It is well known that if $\sigma < 1$ and f is an absolutely continuous function on $[0,1]$ such that $f(0) = 0$, then the following sharp inequality from [18, Lemma 1] is valid:

$$(1 - \sigma) \int_0^1 \frac{|f(x)|}{x^\sigma}\, dx < \int_0^1 |f'(x)|\, dx,$$

which leads to the sought assertion after inserting $\sigma = s - q + 1$ and using Lemma 2. $\square$

Establish the inequalities on an arbitrary interval $[a,b]$.

Theorem 1. *Suppose that $s, q > 0$, $\nu \geq 0$, and f is an absolutely continuous function on $[a,b]$ such that $f(a) = f(b) = 0$ and $f'(x)/x^s \in L^1[0,1]$. If $\mu \in \left(0; \frac{s+\nu q}{q}\right)$, then*

$$(s^2 - \nu^2 q^2) \int_a^b \frac{|f(x)|}{\delta(x)^{s+1}}\, dx + \frac{q^2 \lambda_\nu^2(2s/q)}{\delta_0^q} \int_a^b \frac{|f(x)|}{\delta(x)^{s-q+1}}\, dx$$
$$\leq \left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right) \int_a^b \frac{|f'(x)|}{\delta(x)^s}\, dx + \left(\frac{q^2\mu^2}{s\delta_0^s} - \frac{2q\mu}{\delta_0^s}\right) \int_a^b |f'(x)|\, dx;$$

and if $\mu \leq 0$, then

$$(s^2 - \nu^2 q^2) \int_a^b \frac{|f(x)|}{\delta(x)^{s+1}}\, dx + \frac{q^2 \lambda_\nu^2(2s/q)}{\delta_0^q} \int_a^b \frac{|f(x)|}{\delta(x)^{s-q+1}}\, dx \leq 2(s + \nu q) \int_a^b \frac{f'(x)}{\delta^s(x)}\, dx - 2\frac{q\mu}{\delta_0^s} \int_a^b |f'(x)|\, dx,$$

where $\delta(x) = \min\{b - x, x - a\}$, $\delta_0 = \frac{b-a}{2}$, and the constant $z = \lambda_\nu(r)$ is the first positive solution to the equation

$$(r - 2\mu) J_\nu(z) + 2z J_\nu'(z) = 0, \quad z \in (0, j_\nu).$$

Proof. Given $\rho > 0$, execute the change of variable $x = \rho\tau$ in the first inequality of Lemma 2 and get

$$(s^2 - \nu^2 q^2) \int_0^\rho \frac{|f(x)|}{x^{s+1}}\, dx + \frac{q^2 \lambda_\nu^2(2s/q)}{\rho^q} \int_0^\rho \frac{|f(x)|}{x^{s-q+1}}\, dx$$
$$\leq \left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right) \int_0^\rho \frac{|f'(x)|}{x^s}\, dx + \left(\frac{q^2\mu^2}{s\rho} - \frac{2q\mu}{\rho}\right) \int_0^\rho |f'(x)|\, dx.$$

Applying the latter to $f(x) = g(x+a)$ and $f(x) = g(b-x)$ с $\rho = \delta_0 = (b-a)/2$ yields

$$(s^2 - \nu^2 q^2) \int\limits_a^{(a+b)/2} \frac{|g(t)|}{(t-a)^{s+1}}\,dt + \frac{q^2\lambda_\nu^2(2s/q)}{\rho^q} \int\limits_a^{(a+b)/2} \frac{|g(t)|}{(t-a)^{s-q+1}}\,dt$$
$$\le \left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right) \int\limits_a^{(a+b)/2} \frac{|g'(t)|}{(t-a)^s}\,dt + \left(\frac{q^2\mu^2}{s\rho} - \frac{2q\mu}{\rho}\right) \int\limits_a^{(a+b)/2} |g'(t)|\,dt,$$

$$(s^2 - \nu^2 q^2) \int\limits_{(a+b)/2}^{b} \frac{|g(t)|}{(b-t)^{s+1}}\,dt + \frac{q^2\lambda_\nu^2(2s/q)}{\rho^q} \int\limits_{(a+b)/2}^{b} \frac{|g(t)|}{(b-t)^{s-q+1}}\,dt$$
$$\le \left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right) \int\limits_{(a+b)/2}^{b} \frac{|g'(t)|}{(b-x)^s}\,dt + \left(\frac{q^2\mu^2}{s\rho} - \frac{2q\mu}{\rho}\right) \int\limits_{(a+b)/2}^{b} |g'(t)|\,dt.$$

By summing up the last two inequalities, we derive the sought assertion. The second inequality of the theorem is obtained similarly. This concludes the proof of Theorem 1. $\square$

Corollary 4. *Suppose that $s, q > 0$, $\nu \ge 0$, and f is an absolutely continuous function on $[a, b]$ such that $f(a) = f(b) = 0$ and $f'(x)/(1-|x|)^s \in L^1[0,1]$. Then*

$$\frac{s^2 - \nu^2 q^2}{s + 2\nu q} \int\limits_{-1}^{1} \frac{|f(x)|}{(1-|x|)^{s+1}}\,dx + \frac{q^2 {j'_\nu}^2}{s + 2\nu q} \int\limits_{-1}^{1} \frac{|f(x)|}{(1-|x|)^{s-q+1}}\,dx \le \int\limits_{-1}^{1} \frac{|f'(x)|}{(1-|x|)^s}\,dx - \frac{s}{s+2\nu q} \int\limits_{-1}^{1} |f'(x)|\,dx,$$

where j'_ν is the first positive root of the derivative J'_ν of the Bessel function of order ν.

Proof. It is obvious that if $\mu = s/q$ and $r = 2s/q$, then the equation $\frac{r}{2} + zJ'_\nu(z)/J_\nu(z) = \mu$ takes the form

$$zJ'_\nu(z)/J_\nu(z) = 0.$$

Consequently, the Lamb constant $\lambda_\nu(2s/q)$ is equal to j'_ν in this case, where j'_ν is the first positive root of the derivative of the Bessel function of order J'_ν. Thus, the sought inequality follows from the assertion of Theorem 1 for $a = -1$, $b = 1$, and $\mu = s/q$. $\square$

Give several particular cases of the inequality from Theorem 1.

Example 1. Let j'_1 be the first positive root of the derivative J'_1 of the Bessel function. It is well known that $j'_1 \approx 1.8412$. If $a = -1$, $b = 1$, $s = 2$, $q = 2$, $\mu = 1$, and $\nu = 1$, then

$$\frac{2}{3}{j'_1}^2 \int\limits_{-1}^{1} \frac{|f(x)|}{1-|x|}\,dx \le \int\limits_{-1}^{1} \frac{|f'(x)|}{(1-|x|)^2}\,dx - \frac{1}{3} \int\limits_{-1}^{1} |f'(x)|\,dx.$$

The latter inequality is comparable with the sharp inequality from [18], which we can rewrite as

$$2e \int\limits_{-1}^{1} \frac{|f(x)|}{1-|x|}\,dx \le \int\limits_{-1}^{1} \frac{|f'(x)|}{(1-|x|)^2}\,dx.$$

Example 2. Let $a=-1$, $b=1$, $s=1$, $q=1$, $\mu=1$, and $\nu>0$. Then

$$(1-\nu^2)\int\limits_{-1}^{1}\frac{|f(x)|}{(1-|x|)^2}\,dx + {j'_\nu}^2\int\limits_{-1}^{1}\frac{|f(x)|}{(1-|x|)}\,dx \le (1+2\nu)\int\limits_{-1}^{1}\frac{|f'(x)|}{(1-|x|)}\,dx - \int\limits_{-1}^{1}|f'(x)|\,dx,$$

where j'_ν is the first positive root of the derivative J'_ν of the Bessel function of order ν.

The latter inequality for $\nu=1$ is comparable with the following sharp inequality from [18]:

$$e\int\limits_{-1}^{1}\frac{|f(x)|}{1-|x|}\,dx \le \int\limits_{-1}^{1}\frac{|f'(x)|}{1-|x|}\,dx.$$

Example 3. If $a=-1$, $b=1$, $\nu=0$, $s>0$ and $\mu\to\frac{s+\nu q}{q}$, then it follows from Theorem 1 that

$$s\int\limits_{-1}^{1}\frac{|f(x)|}{(1-|x|)^{s+1}}\,dx \le \int\limits_{-1}^{1}\frac{|f'(x)|}{(1-|x|)^{s}}\,dx - \int\limits_{-1}^{1}|f'(x)|\,dx,$$

which is comparable with the sharp inequality from [17, Formula 3], that can be rewritten in the one-dimensional case as

$$s\int\limits_{-1}^{1}\frac{|f(x)|}{(1-|x|)^{s+1}}\,dx \le \int\limits_{-1}^{1}\frac{|f'(x)|}{(1-|x|)^{s}}\,dx - \int\limits_{-1}^{1}|f'(x)|(1-|x|)\,dx.$$

1.3. L_p-inequalities on a segment. Here we obtain the one-dimensional L_p-inequalities. For their proof we use the above-proved inequalities in L_1.

Theorem 2. *Suppose that $p\ge 1$, $r\in[1,p]$, and f is an absolutely continuous function on $[a,b]$ such that $f(a)=f(b)=0$. Let $s>0$, $q>s$, $\mu\in\left(0;\frac{2s}{q}\right]$, and $\nu\in\left[0,\frac{s}{q}\right]$. Then*

$$(s^2-r\nu^2q^2)\int\limits_{a}^{b}\frac{|f(x)|^p}{\delta(x)^{s+1}}\,dx + \frac{qr}{\delta_0^q}\left(q\lambda_\nu^2(2s/q)+\mu(q-s)\left(2-\frac{q\mu}{s}\right)\right)\int\limits_{a}^{b}\frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,dx$$

$$\le p^r s^{2(1-r)}\left(2s+2\nu q-\frac{q^2\mu^2}{s}\right)^r\int\limits_{a}^{b}\frac{|f(x)|^{p-r}\cdot|f'(x)|^r}{\delta(x)^{s-r+1}}\,dx,$$

where $\delta(x)=\min\{b-x,x-a\}$, $\delta_0=\frac{b-a}{2}$, and the constant $z=\lambda_\nu(r)$ is the first positive solution to the equation

$$(r-2\mu)J_\nu(z)+2zJ'_\nu(z)=0,\quad z\in(0,j_\nu).$$

Proof. Let g belong to $C^1[0,1]$. Then $f(x)=|g(x)|^p$ belongs to $C^1[0,1]$, since

$$\frac{d}{dx}|g(x)|^p = p|g(x)|^{p-1}g'(x)\operatorname{sign} g(x)$$

and $|g(x)|^{p-1}\operatorname{sign} g(x)$ is continuous for $p>1$.

Applying Corollary 3 to $f(x)=|g(x)|^p\in C^1[0,1]$, we obtain

$$(s^2-\nu^2q^2)\int\limits_{0}^{1}\frac{|g(x)|^p}{x^{s+1}}\,dx + \left(q^2\lambda_\nu^2\left(\frac{2s}{q}\right)+q\mu(q-s)\left(2-\frac{q\mu}{s}\right)\right)\int\limits_{0}^{1}\frac{|g(x)|^p}{x^{s-q+1}}\,dx$$

$$\le p\left(2s+2\nu q-\frac{q^2\mu^2}{s}\right)\int\limits_{0}^{1}\frac{|g'(x)|\cdot|g(x)|^{p-1}}{x^s}\,dx.$$

Using the elementary inequality (see [1, p. 37])

$$a^{p_1} b^{p_2} \le \left(\frac{p_1 a + p_2 b}{p_1 + p_2}\right)^{p_1+p_2} \tag{6}$$

with

$$a = s^2 \frac{|g(x)|^p}{x^{s+1}}, \quad b = p^r s^{2-2r}\left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right)^r \frac{|g(x)|^{p-r}|g'(x)|^r}{x^{s+1-r}},$$

$p_1 = 1 - \frac{1}{r}$, and $p_2 = \frac{1}{r}$, we derive

$$(s^2 - r\nu^2 q^2)\int_0^1 \frac{|g(x)|^p}{x^{s+1}}\,dx + qr\left(q\lambda_\nu^2(2s/q) + \mu(q-s)\left(2 - \frac{q\mu}{s}\right)\right)\int_0^1 \frac{|g(x)|^p}{x^{s-q+1}}\,dx$$

$$\le p^r s^{2(1-r)}\left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right)^r \int_0^1 \frac{|g(x)|^{p-r}\cdot|g'(x)|^r}{x^{s-r+1}}\,dx. \tag{7}$$

We obtained the inequalities on $[0,1]$. The passage to the interval $[a,b]$ is similar to that of the proof of Theorem 1. Thus, we arrive at the sought assertion. □

Theorem 3. *Suppose that $p \ge 1$, $r \in [1,p]$, and f is an absolutely continuous function on $[a,b]$ such that $f(a) = f(b) = 0$. Let $s > 0$, $q > s$, $\mu \in \left(0; \frac{2s}{q}\right]$, and $\nu \in \left[0, \frac{s}{q}\right]$. Then*

$$\int_a^b \frac{|f(x)|^p}{\delta(x)^{s+1}}\,dx + \frac{qr}{\delta_0^q}\left(q\lambda_\nu^2(2s/q) + \mu(q-s)\left(2 - \frac{q\mu}{s}\right)\right)\int_a^b \frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,dx$$

$$\le \frac{p^r}{s^r}\left(\frac{2s}{s-\nu q} - \frac{q^2\mu^2}{s^2 - \nu^2 q^2}\right)^r \int_a^b \frac{|f(x)|^{p-1}\cdot|f'(x)|^r}{\delta(x)^{s-r+1}}\,dx,$$

where $\delta(x) = \min\{b-x, x-a\}$, $\delta_0 = \frac{b-a}{2}$, and the constant $z = \lambda_\nu(r)$ is the first positive solution to the equation

$$(r - 2\mu)J_\nu(z) + 2zJ'_\nu(z) = 0, \quad z \in (0, j_\nu).$$

Proof. We argue as in the proof of Theorem 2. Inserting

$$a = \frac{|g(x)|^p}{x^{s+1}}, \quad b = p^r\left(\frac{2}{s-\nu q} - \frac{q^2\mu^2}{s^3 - s\nu^2 q^2}\right)^r \frac{|g(x)|^{p-r}|g'(x)|^r}{x^{s+1-r}},$$

$p_1 = 1 - \frac{1}{r}$, and $p_2 = \frac{1}{r}$ in (6), we derive the sought inequality. □

Theorem 4. *Suppose that $s, q > 0$, $\nu \ge 0$, and f is an absolutely continuous function on $[a,b]$ such that $f(a) = f(b) = 0$. If $\mu \in \left[0; \frac{s+\nu q}{q}\right)$, then*

$$(s^2 - \nu^2 q^2)\int_a^b \frac{|f(x)|^{s+1}}{\delta(x)^{s+1}}\,dx + \frac{q^2\lambda_\nu^2(2s/q)}{\delta_0^q}\int_a^b \frac{|f(x)|^{s+1}}{\delta(x)^{s-q+1}}\,dx$$

$$\le \frac{(s+1)^{s+1}}{s^s}\left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right)\int_a^b |f'(x)|^{s+1}\,dx + \frac{(s+1)}{\delta_0^s}\left(\frac{q^2\mu^2}{s} - 2q\mu\right)\int_a^b |f'(x)|\cdot|f(x)|^s\,dx;$$

if $\mu \le 0$, then

$$(s^2-\nu^2q^2)\int\limits_a^b \frac{|f(x)|^{s+1}}{\delta(x)^{s+1}}\,dx+\frac{q^2\lambda_\nu^2(2s/q)}{\delta_0^q}\int\limits_a^b \frac{|f(x)|^{s+1}}{\delta(x)^{s-q+1}}\,dx$$

$$\le 2\frac{(s+1)^{s+1}}{s^s}(s+\nu q)\int\limits_a^b |f'(x)|^{s+1}\,dx-2\frac{(s+1)q\mu}{\delta_0^s}\int\limits_a^b |f'(x)|\cdot|f(x)|^s\,dx,$$

where $\delta(x)=\min\{b-x,x-a\}$, $\delta_0=\frac{b-a}{2}$, and the constant $z=\lambda_\nu(r)$ is the first positive solution to the equation

$$(r-2\mu)J_\nu(z)+2zJ_\nu'(z)=0,\quad z\in(0,j_\nu).$$

Proof. By Lemma 2,

$$(s^2-\nu^2q^2)\int\limits_0^1 \frac{|g(x)|^{s+1}}{x^{s+1}}\,dx+q^2\lambda_\nu^2(2s/q)\int\limits_0^1 \frac{|g(x)|^{s+1}}{x^{s-q+1}}\,dx$$

$$\le (s+1)\left(2s+2\nu q-\frac{q^2\mu^2}{s}\right)\int\limits_0^1 \frac{|g'(x)|\cdot|g(x)|^s}{x^s}\,dx+(s+1)\left(\frac{q^2\mu^2}{s}-2q\mu\right)\int\limits_0^1 |g'(x)|\cdot|g(x)|^s\,dx$$

for $f(x)=g^{s+1}(x)\in C^1[0,1]$. Using the Opial inequality (see [38, p. 313])

$$\int\limits_a^b \frac{|u(x)|^p\cdot|u'(x)|}{(x-a)^p}\,dx\le\left(\frac{p}{p+1}\right)^{-p}\int\limits_a^b |u'(x)|^{p+1}\,dx,\quad p>0,$$

we derive

$$(s^2-\nu^2q^2)\int\limits_0^1 \frac{|g(x)|^{s+1}}{x^{s+1}}\,dx+q^2\lambda_\nu^2(2s/q)\int\limits_0^1 \frac{|g(x)|^{s+1}}{x^{s-q+1}}\,dx$$

$$\le\frac{(s+1)^{s+1}}{s^s}\left(2s+2\nu q-\frac{q^2\mu^2}{s}\right)\int\limits_0^1 |g'(x)|^{s+1}\,dx$$

$$+(s+1)\left(\frac{q^2\mu^2}{s}-2q\mu\right)\int\limits_0^1 |g'(x)|\cdot|g(x)|^s\,dx. \tag{8}$$

Acting similarly in the case $\mu<0$ yields

$$(s^2-\nu^2q^2)\int\limits_0^1 \frac{|g(x)|^{s+1}}{x^{s+1}}\,dx+q^2\lambda_\nu^2(2s/q)\int\limits_0^1 \frac{|g(x)|^{s+1}}{x^{s-q+1}}\,dx$$

$$\le\frac{(s+1)^{s+1}}{s^s}\left(2s+2\nu q\right)\int\limits_0^1 |g'(x)|^{s+1}\,dx-2(s+1)q\mu\int\limits_0^1 |g'(x)|\cdot|g(x)|^s\,dx. \tag{9}$$

To obtain the sought assertion, we need to pass from $[0,1]$ to the arbitrary interval $[a,b]$ in (8) and (9). $\square$

Now prove the inequalities for L_2.

Theorem 5. *Suppose that f is an absolutely continuous function on $[a,b]$ such that $f(a)=f(b)=0$. If $q>0$, $\nu\ge 0$, and $\mu\in\big[0;\frac{s+\nu q}{q}\big)$, then*

$$(1-\nu^2q^2)\int\limits_0^1 \frac{|f(x)|^2}{\delta(x)^2}\,dx+\frac{q^2\lambda_\nu^2(2/q)}{\delta_0^q}\int\limits_0^1\frac{|f(x)|^2}{\delta(x)^{2-q}}\,dx\le(8+8\nu q-3q^2\mu^2-2q\mu)\int\limits_0^1|f'(x)|^2\,dx;$$

if $q>0$, $\nu\ge 0$, and $\mu\le 0$, then

$$(1-\nu^2q^2)\int\limits_0^1 \frac{|f(x)|^2}{\delta(x)^2}\,dx+\frac{q^2\lambda_\nu^2(2/q)}{\delta_0^q}\int\limits_0^1\frac{|f(x)|^2}{\delta(x)^{2-q}}\,dx\le(8+8\nu q-2q\mu)\int\limits_0^1|f'(x)|^2\,dx;$$

if $q>1$, $\nu\in\big[0,\frac{s}{q}\big]$ and $\mu\in\big(0;\frac{2s}{q}\big]$, then

$$\frac{1-\nu^2q^2}{q}\int\limits_a^b \frac{|f(x)|^2}{\delta(x)^2}\,dx+\frac{q\lambda_\nu^2(2/q)+\mu(q-1)(2-q\mu)}{\delta_0^q}\int\limits_a^b\frac{|f(x)|^2}{\delta(x)^{2-q}}\,dx\le\frac{8+8\nu q-4q^2\mu^2}{q}\int\limits_a^b|f'(x)|^2\,dx.$$

Proof. To obtain the result for $\mu\in\big[0;\frac{s+\nu q}{q}\big)$, put $s=1$ in (8), and get

$$\begin{aligned}(1-\nu^2q^2)\int\limits_0^1 \frac{|f(x)|^2}{x^2}\,dx+q^2\lambda_\nu^2(2/q)\int\limits_0^1\frac{|f(x)|^2}{x^{2-q}}\,dx\\ \le(8+8\nu q-4q^2\mu^2)\int\limits_0^1|f'(x)|^2\,dx+(2q^2\mu^2-4q\mu)\int\limits_0^1|f'(x)|\cdot|f(x)|\,dx.\end{aligned}$$

Then use the Opial inequality (see [38])

$$\int\limits_0^\rho |f(x)|\cdot|f'(x)|\,dx\le\frac{\rho}{2}\int\limits_0^\rho|f'(x)|^2\,dx; \tag{10}$$

hence,

$$(1-\nu^2q^2)\int\limits_0^1 \frac{|f(x)|^2}{x^2}\,dx+q^2\lambda_\nu^2(2/q)\int\limits_0^1\frac{|f(x)|^2}{x^{2-q}}\,dx\le(8+8\nu q-3q^2\mu^2-2q\mu)\int\limits_0^1|f'(x)|^2\,dx.$$

In the case $\mu<0$, inserting $s=1$ in (9) and applying (10) yields

$$(1-\nu^2q^2)\int\limits_0^1 \frac{|f(x)|^2}{x^2}\,dx+q^2\lambda_\nu^2(2/q)\int\limits_0^1\frac{|f(x)|^2}{x^{2-q}}\,dx\le(8+8\nu q-2q\mu)\int\limits_0^1|f'(x)|^2\,dx.$$

Put $s=1$, $r=1$, and $p=2$ in (7). Then

$$\begin{aligned}\frac{1-\nu^2q^2}{q}\int\limits_0^1 \frac{|f(x)|^2}{x^2}dx+\big(q\lambda_\nu^2(2/q)+\mu(q-1)(2-q\mu)\big)\int\limits_0^1\frac{|f(x)|^2}{x^{2-q}}\,dx\\ \le\frac{4+4\nu q-2q^2\mu^2}{q}\int\limits_0^1\frac{|f(x)||f'(x)|}{x}\,dx.\end{aligned}$$

Using the Opial inequality (see [38])

$$\int_0^\rho \frac{|f(x)| \cdot |f'(x)|}{x}\, dx \le 2 \int_0^\rho |f'(x)|^2\, dx$$

leads to

$$\frac{1-\nu^2 q^2}{q} \int_0^1 \frac{|f(x)|^2}{x^2}\, dx + \big(q\lambda_\nu^2(2/q) + \mu(q-1)(2-q\mu)\big) \int_0^1 \frac{|f(x)|^2}{x^{2-q}}\, dx$$
$$\le \frac{8+8\nu q - 4q^2\mu^2}{q} \int_0^1 |f'(x)|^2\, dx.$$

It only remains to pass to $[a,b]$ in these three cases. $\square$

2. Inequalities in the Convex Domains of the Euclidean Space $\mathbb{R}^n$

Here we present the multidimensional analogs of the inequalities obtained above. Assume that Ω is a convex domain of finite inner radius. We use Avkhadiev's method from [16] (see also [17, 18]) to prove the inequalities which is based on a special approximation of Ω by cubes. In the case of convex sets, the proof of the multidimensional inequalities reduces to application of the one-dimensional inequalities.

Let Ω be an open convex set in $\mathbb{R}^n$ of finite inner radius

$$\delta_0 = \delta_0(\Omega) = \sup_{z\in\Omega} \delta(x),$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Denote by $C_0^1(\Omega)$ the well-known space of smooth functions $f : \Omega \to \mathbb{R}$ compactly-supported in Ω.

As we mentioned above, the situation is rather simple for convex domains, and the one-dimensional inequalities extend straightforwardly to the spatial case. Namely, Avkhadiev's method is reduced to the following assertion:

Theorem A. *Let Ω be an open compact set in $\mathbb{R}^n$ of finite inner radius $\delta_0 = \delta_0(\Omega)$. If*

$$\int_0^\alpha \frac{|f'(t)|^p}{t^{s-p}}\, dt \ge b^2 \int_0^\alpha \frac{|f(t)|^p}{t^s}\, dt + \frac{c^2}{\delta_0^m} \int_0^\alpha \frac{|f(t)|^p}{t^{s-m}}\, dt, \quad f \in C_0^1(0, 2\alpha),$$

for every $\alpha \in (0, \delta_0]$ and nonnegative b and c, then

$$\int_\Omega \frac{\nabla|f(x)|^p}{\delta(x)^{s-p}}\, dx \ge b^2 \int_0^\alpha \frac{|f(x)|^p}{\delta(x)^s}\, dx + \frac{c^2}{\delta_0^m} \int_0^\alpha \frac{|f(x)|^p}{\delta(x)^{s-m}}\, dx, \quad f \in C_0^1(\Omega).$$

2.1. L_1-inequalities. Combining Theorems A and 1 yields

Theorem 6. *Let Ω be an open convex set in $\mathbb{R}^n$ of finite inner radius $\delta_0 = \delta_0(\Omega)$. Suppose that $s, q > 0$, $\nu \ge 0$, $f \in C_0^1(\Omega)$, and $f'(x)/\delta^s(\cdot) \in L^1(\Omega)$. If $\mu \in \big(0; \frac{s+\nu q}{q}\big)$, then*

$$(s^2 - \nu^2 q^2) \int_\Omega \frac{|f(x)|}{\delta(x)^{s+1}}\, dx + \frac{q^2\lambda_\nu^2(2s/q)}{\delta_0^q} \int_\Omega \frac{|f(x)|}{\delta(x)^{s-q+1}}\, dx$$
$$\le \left(2s + 2\nu q - \frac{q^2\mu^2}{s}\right) \int_\Omega \frac{|\nabla f(x)|}{\delta(x)^s}\, dx + \left(\frac{q^2\mu^2}{s\delta_0^s} - \frac{2q\mu}{\delta_0^s}\right) \int_\Omega |\nabla f(x)|\, dx;$$

if $\mu \le 0$, then

$$(s^2-\nu^2q^2)\int\limits_\Omega \frac{|f(x)|}{x^{s+1}}\,dx+\frac{q^2\lambda_\nu^2(2s/q)}{\delta_0^q}\int\limits_\Omega \frac{|f(x)|}{\delta(x)^{s-q+1}}\,dx$$
$$\le 2(s+\nu q)\int\limits_\Omega \frac{|\nabla f(x)|}{\delta^s(x)}\,dx-\frac{q\mu}{\delta_0^s}\int\limits_\Omega |\nabla f(x)|\,dx,$$

where the constant $z=\lambda_\nu(r)$ is the first positive solution to the equation

$$(r-2\mu)J_\nu(z)+2zJ_\nu'(z)=0,\quad z\in(0,j_\nu).$$

Combining Theorem A and Corollary 3 leads to

Theorem 7. *Let Ω be an open convex set in $\mathbb{R}^n$ of finite inner radius $\delta_0=\delta_0(\Omega)$. Suppose that $s>0$, $q>s$, and $f\in C_0^1(\Omega)$ is such that $|\nabla f(x)|/\delta^s(x)\in L^1(\Omega)$. If $\mu\in\big(0;\frac{2s}{q}\big]$ and $\nu\in\big[0,\frac{s}{q}\big]$, then*

$$(s^2-\nu^2q^2)\int\limits_\Omega \frac{|f(x)|}{\delta(x)^{s+1}}\,dx+\frac{q}{\delta_0^q}\left(q\lambda_\nu^2\left(\frac{2s}{q}\right)+\mu(q-s)\left(2-\frac{q\mu}{s}\right)\right)\int\limits_\Omega \frac{|f(x)|}{\delta(x)^{s-q+1}}\,dx$$
$$\le\left(2s+2\nu q-\frac{q^2\mu^2}{s}\right)\int\limits_\Omega \frac{|\nabla f(x)|}{\delta(x)^s}\,dx,$$

where $z=\lambda_\nu(r)$ is a constant satisfying the condition

$$(r-2\mu)J_\nu(z)+2zJ_\nu'(z)=0,\quad z\in(0,j_\nu).$$

2.2. L_p-inequalities. Combining Theorems A, 2, and 3, we obtain

Theorem 8. *Let Ω be an open convex set in $\mathbb{R}^n$ of finite inner radius $\delta_0=\delta_0(\Omega)$. Suppose that $p\ge 1$, $r\in[1,p]$ and $f\in C_0^1(\Omega)$. If $s>0$, $q>s$, $\mu\in\big(0;\frac{2s}{q}\big]$, and $\nu\in\big[0,\frac{s}{q}\big]$, then*

$$(s^2-r\nu^2q^2)\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s+1}}\,dx+\frac{qr}{\delta_0^q}\left(q\lambda_\nu^2(2s/q)+\mu(q-s)\left(2-\frac{q\mu}{s}\right)\right)\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,dx$$
$$\le p^r s^{2(1-r)}\left(2s+2\nu q-\frac{q^2\mu^2}{s}\right)^r\int\limits_\Omega \frac{|f(x)|^{p-r}\cdot|\nabla f(x)|^r}{\delta(x)^{s-r+1}}\,dx;$$

if $s>0$, $q>s$, $\mu\in\big(0;\frac{2s}{q}\big]$, and $\nu\in\big[0,\frac{s}{q}\big]$, then

$$\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s+1}}\,dx+\frac{qr}{\delta_0^q}\left(q\lambda_\nu^2(2s/q)+\mu(q-s)\left(2-\frac{q\mu}{s}\right)\right)\int\limits_\Omega \frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,dx$$
$$\le\frac{p^r}{s^r}\left(\frac{2s}{s-\nu q}-\frac{q^2\mu^2}{s^2-\nu^2q^2}\right)^r\int\limits_\Omega \frac{|f(x)|^{p-r}\cdot|\nabla f(x)|^r}{\delta(x)^{s-r+1}}\,dx,$$

where the constant $z=\lambda_\nu(r)$ is the first positive solution to the equation

$$(r-2\mu)J_\nu(z)+2zJ_\nu'(z)=0,\quad z\in(0,j_\nu).$$

To prove the inequalities in L_p, we can use the L_p-Lemma from [48] (see also [17, 18]).

References

1. Hardy G. H., Littlewood J. E., and Pólya G., *Inequalities*, Cambridge University, Cambridge (1973).
2. Sobolev S. L., *Some Applications of Functional Analysis in Mathematical Physics*, Amer. Math. Soc., Providence (1991).
3. Mazya V. G., *Sobolev Spaces*, Springer, Berlin and Heidelberg (1985).
4. Avkhadiev F. G., "Solution of the generalized Saint Venant problem," Sb. Math., vol. 189, no. 12, 1739–1748 (1998).
5. Balinsky A. A., Evans W. D., and Lewis R. T., *The Analysis and Geometry of Hardy's Inequality*, Springer, Heidelberg, New York, Dordrecht, and London (2015).
6. Dubinskii Yu. A., "Bilateral scales of Hardy inequalities and their applications to some problems of mathematical physics," J. Math. Sci., vol. 201, no. 6, 751–795 (2014).
7. Dubinskii Yu. A., "A Hardy-type inequality and its applications," Proc. Steklov Inst. Math., vol. 269, 106–126 (2010).
8. Avkhadiev F. G. and Wirths K.-J., "Unified Poincaré and Hardy inequalities with sharp constants for convex domains," Z. Angew. Math. Mech., vol. 87, no. 8–9, 632–642 (2007).
9. Avkhadiev F. G. and Wirths K.-J., "Sharp Hardy-type inequalities with Lamb's constants," Bull. Belg. Math. Soc. Simon Stevin, vol. 18, no. 4, 723–736 (2011).
10. Brezis H. and Marcus M., "Hardy's inequalities revisited," Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), vol. 25, no. 1–2, 217–237 (1998).
11. Hoffmann-Ostenhof M., Hoffmann-Ostenhof T., and Laptev A., "A geometrical version of Hardy's inequality," J. Funct. Anal., vol. 189, 539–548 (2002).
12. Avkhadiev F. G., "Hardy–Rellich integral inequalities in domains satisfying the exterior sphere condition," St. Petersburg Math. J., vol. 30, no. 2, 161–179 (2019).
13. Avkhadiev F. G., "A geometric description of domains whose Hardy constant is equal to 1/4," Izv. Math., vol. 78, no. 5, 855–876 (2014).
14. Avkhadiev F. and Laptev A., "Hardy inequalities for nonconvex domains," in: *Around the Research of Vladimir Maz'ya. V. I. Int. Math. Ser. (N. Y.)*, vol. 11, Springer Science+Business Media, New York (2010), 1–12.
15. Avkhadiev F. G., "Hardy–Rellich inequalities in domains of the Euclidean space," J. Math Anal. Appl., vol. 442, 469–484 (2016).
16. Avkhadiev F. G., "Hardy-type inequalities in higher dimensions with explicit estimate of constants," Lobachevskii J. Math., vol. 21, 3–31 (2006).
17. Avkhadiev F. G., "Hardy-type inequalities on planar and spatial open sets," Proc. Steklov Inst. Math., vol. 255, 2–12 (2006).
18. Avkhadiev F. G. and Nasibullin R. G., "Hardy-type inequalities in arbitrary domains with finite inner radius," Sib. Math. J., vol. 55, no. 2, 191–200 (2014).
19. Avkhadiev F. G., Nasibullin R. G., and Shafigullin I. K., "L_p-Versions of one conformally invariant inequality," Russian Math. (Iz. VUZ), vol. 62, no. 8, 76–79 (2018).
20. Avkhadiev F. G., Nasibullin R. G., and Shafigullin I. K., "Conformal invariants of hyperbolic planar domains," Ufa Math. J., vol. 11, no. 2, 3–18 (2019).
21. Avkhadiev F. G., "Integral inequalities in domains of hyperbolic domains and their applications," Sb. Math., vol. 206, no. 12, 1657–1681 (2015).
22. Fernández J. L., "Domains with strong barrier," Rev. Mat. Iberoam., vol. 5, no. 2, 47–65 (1989).
23. Fernández J. L. and Rodríguez J. M., "The exponent of convergence of Riemann surfaces. Bass Riemann surfaces," Ann. Acad. Sci. Fenn. Ser. A. I. Math., vol. 15, 165–183 (1990).
24. Alvarez V., Pestana D., and Rodríguez J. M., "Isoperimetric inequalities in Riemann surfaces of infinite type," Rev. Mat. Iberoam., vol. 15, no. 2, 353–425 (1999).
25. Tidblom J., "A geometrical version of Hardy's inequality for $W_0^{1,p}(\Omega)$," Proc. Amer. Math. Soc., vol. 132, no. 8, 2265–2271 (2004).
26. Evans W. D. and Lewis R. T., "Hardy and Rellich inequalities with remainders," J. Math. Inequal., vol. 1, no. 4, 473–490 (2007).
27. Burenkov V. I., Senouci A., and Tararykova T. V., "Hardy-type inequality for $0 < p < 1$ and hypodecreasing functions," Eurasian Math. J., vol. 1, no. 3, 27–42 (2010).
28. Burenkov V. I., "On the exact constant in the Hardy inequality with $0 < p < 1$ for monotone functions," Proc. Steklov Inst. Math., vol. 194, no. 4, 59–63 (1993).
29. Burenkov V. I., *Function Spaces. Main Integral Inequalities Related to L_p-Spaces*, Peoples' Friendship University, Moscow (1989).
30. Psaradakis G., "L_1 Hardy inequalities with weights," J. Geom. Anal., vol. 23, no. 4, 1703–1728 (2013).
31. Lamb H., "Note on the induction of electric currents in a cylinder placed across the lines of magnetic force," Proc. Lond. Math. Soc., vol. 1–15, 270–274 (1884).
32. Watson G. N., *A Treatise on the Theory of Bessel Functions*, Cambridge University, Cambridge (1966).

33. Nasibullin R. G., “Brezis–Marcus type inequalities with Lamb constant,” Sib. Electr. Math. Reports, vol. 16, 449–464 (2019).
34. Nasibullin R. G., “Sharp Hardy type inequalities with weights depending on Bessel function,” Ufa Math. J., vol. 9, no. 1, 89–97 (2017).
35. Nasibullin R. G., “Hardy type inequalities with weights dependent on the Bessel functions,” Lobachevskii J. Math., vol. 37, no. 3, 274–283 (2016).
36. Nasibullin R. G., “A geometrical version of Hardy–Rellich type inequalities,” Math. Slovaca, vol. 69, no. 4, 785–800 (2019).
37. Levin V., “Notes on inequalities. II. On a class of integral inequalities,” Rec. Math. [Mat. Sbornik] N.S., vol. 4, no. 2, 309–325 (1938).
38. Shum D. T., “On a class of new inequalities,” Trans. Amer. Math. Soc., vol. 204, 299–341 (1975).
39. Filippas S., Maz'ya V. G., and Tertikas A., “On a question of Brezis and Marcus,” Calc. Var. Partial Differential Equations, vol. 25, no. 4, 491–501 (2006).
40. Prokhorov D. V., Stepanov V. D., and Ushakova E. P., “Hardy–Steklov integral operators,” Proc. Steklov Inst. Math., vol. 300, no. suppl. 2, 1–112 (2018).
41. Stepanov V. D. and Shambilova G. E., “On weighted iterated Hardy-type operators,” Anal. Math., vol. 44, no. 2, 273–283 (2018).
42. Prokhorov D. V., “On a weighted inequality for a Hardy-type operator,” Proc. Steklov Inst. Math., vol. 284, no. 1, 208–215 (2014).
43. Bandaliev R. A., “On Hardy-type inequalities in weighted variable Lebesgue space $L_{p(x),\omega}$ for $0 < p(x) < 1$,” Eurasian Math. J., vol. 4, 5–16 (2013).
44. Bandaliyev R. A., Serbetci A., and Hasanov S. G., “On Hardy inequality in variable Lebesgue spaces with mixed norm,” Indian J. Pure Appl. Math., vol. 49, 765–782 (2018).
45. Nasibullin R. G., “Generalizations of Hardy-type inequalities in the form of Dubinskii,” Math. Notes, vol. 95, no. 1, 98–110 (2014).
46. Avkhadiev F. G. and Makarov R. V., “Hardy type inequalities on domains with convex complement and uncertainty principle of Heisenberg,” Lobachevskii J. Math., vol. 40, no. 9, 1250–1259 (2019).
47. Kamke E., *Differentialgleichungen. Lösungsmethoden und Lösungen*, Teubner, Stuttgart (1977).
48. Avkhadiev F. G., Nasibullin R. G., and Shafigullin I. K., “Hardy-type inequalities with power and logarithmic weights in domains of the Euclidean space,” Russian Math. (Iz. VUZ), vol. 55, no. 9, 76–79 (2011).

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