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ON THE QUASIVARIETIES GENERATED BY A FINITE GROUP AND LACKING ANY INDEPENDENT BASES OF QUASI-IDENTITIES

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Abstract: Let $\mathscr{R}_{p^k}$ be the variety of 2-nilpotent groups of exponent p^k with commutator subgroup of exponent p (p is a prime). We prove the infinity of the set of the subquasivarieties of $\mathscr{R}_{p^k}$ ($k \geq 2$) generated by a finite group and lacking any independent bases of quasi-identities.

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Introduction

We consider the question of the existence of the independent bases of quasi-identities of groups and address the complexity of the lattices of quasivarieties of groups. Significantly many papers are devoted to studying the independent bases of quasi-identities of groups is the contents. Here are some of them. We proved in [1] that if some quasivariety of groups contains an infinite cyclic group and lacks infinitely many groups of prime order then it has an independent basis of quasi-identities. Therefore, we consider the problem of the existence of an independent basis of quasi-identities in the class of torsion-free groups for quasivarieties of torsion-free groups. In [2], the conditions are found for the existence of an independent basis of quasi-identities in the class of torsion-free groups. In particular, it turned out that some widely studied quasivarieties (for example, the quasivariety generated by a free nonabelian soluble group or the quasivariety of all linearly ordered groups) have independent bases of quasi-identities in the class of torsion-free groups. It is shown in [3] that the set of the quasivarieties of the soluble groups which lack any independent bases of quasi-identities in the class of torsion-free groups has cardinality of the continuum. In [4], some continuous series are constructed of the quasivarieties of nilpotent groups which lack any independent bases of quasi-identities. In [1], we constructed a quasivariety of groups which lacks any independent basis of quasi-identities that can be defined by an independent system of $\forall$-formulas.

Fedorov proved in [5] that a free 2-nilpotent group of rank $n \geq 2$ has no independent basis of quasi-identities in the class of torsion-free groups. He also demonstrated in [6] that the analogous property is possessed by the quasivariety generated by a nonabelian group of order p^3, with p a prime and $p \neq 2$. Let $\mathscr{R}_{p^k}$ be the variety of 2-step nilpotent groups of exponent p^k with commutator subgroup of exponent p (p is a prime and $k \geq 2$). It follows from [7] that the quasivariety generated by a free nonabelian $\mathscr{R}_{p^k}$-group lacks any independent bases of quasi-identities.

In the process of the development of quasivariety theory, it became known rather quickly that the lattices of quasivarieties have a very complicated structure. For example, the articles [8–11] point out the complexity of the lattices of quasivarieties. Information on the complexity of the lattices of quasivarieties of groups can be found in [7, 12–15]. Let us mention the articles [13, 14, 16] in which it is proved that only one nonabelian quasivariety of torsion-free 2-nilpotent groups of exponent p^k with commutator subgroup of prime exponent has a finite lattice of subquasivarieties.

We obtain the theorem that characterizes the complexity of the lattice of the subquasivarieties of the quasivariety $\mathscr{R}_{p^k}$ ($k > 1$); namely, we prove that there exists an infinite set of the subquasivarieties $\mathscr{M} \subseteq \mathscr{R}_{p^k}$ generated by a finite group such that the interval $[\mathscr{M}, \mathscr{N}]$ of the lattice of quasivarieties has cardinality of the continuum for every quasivariety $\mathscr{N}$ ($\mathscr{M} \subsetneq \mathscr{N} \subseteq \mathscr{R}_{p^k}$). Basing on this result, we show that the set of the quasivarieties generated by a finite group which are contained in $\mathscr{R}_{p^k}$ ($k \geq 2$) and lack any independent bases of quasi-identities is infinite.

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1. Preliminaries

Note that the needed information on quasivarieties can be found in [16, 17]. We introduce the following definitions and notations:

$\langle S\rangle$ is the group generated by a set S, $\langle a\rangle$ is the cyclic group generated by an element a, G' is the commutator subgroup of a group G, $Z(G)$ is the center of G, and $\ker\varphi$ is the kernel of a homomorphism φ. If x and y are elements of a group then $[x,y]=x^{-1}y^{-1}xy$. As usual, if A and B are subgroups of G then $[A,B]=\langle [a,b]\mid a\in A, b\in B\rangle$.

$\mathscr{R}_{p^k}$ (p is a prime, $k\geq 2$) is the variety defined by the identities

$$(\forall x)(\forall y)(\forall z)([x,y,z]=1),\quad (\forall x)(x^{p^k}=1),\quad (\forall x)(\forall y)([x,y]^p=1);$$

qG is the quasivariety generated by a group G;

$\mathscr{M}(G)$ is the least normal subgroup of G the quotient group by which belongs to a quasivariety $\mathscr{M}$;

$[\mathscr{M},\mathscr{N}]=\{\mathscr{R}\mid \mathscr{M}\subseteq\mathscr{R}\subseteq\mathscr{N}\}$ is a closed interval in the lattice of quasivarieties of groups;

$\vee$ is the lattice sum;

F_2 is a free $\mathscr{R}_{p^k}$-group with free generators a and b;

and

Z_{p^n} is a cyclic group of order p^n.

We will use the fact that the following identities hold in every nilpotent group of class 2:

$$(\forall x)(\forall y)(\forall z)([xy,z]=[x,z][y,z]),\quad (\forall x)(\forall y)(\forall z)([x,yz]=[x,y][x,z]).$$

We say that commutation on the noncentral elements of a group G is *transitive* if $[x,y]=1$, $[x,z]=1$ implies $[y,z]=1$ for all $x,y,z\in G$ not contained in the center of G.

Suppose that some groups A and B have the representations

$$A=\langle\{x_i\mid i\in I_1\};\{t_j=1\mid j\in J_1\}\rangle,\quad B=\langle\{y_i\mid i\in I_2\};\{r_j=1\mid j\in J_2\}\rangle,$$

in $\mathscr{R}_{p^k}$ with the disjoint sets of generating symbols. Then the group having in $\mathscr{R}_{p^k}$ the representation

$$\langle\{x_i\mid i\in I_1\}\cup\{y_i\mid i\in I_2\};\{t_j=1\mid j\in J_1\}\cup\{r_j=1\mid j\in J_2\}\rangle$$

is called the *free product* in $\mathscr{R}_{p^k}$ of A and B and is denoted by $A*_{\mathscr{R}_{p^k}}B$. We will often write $A*B$ instead of $A*_{\mathscr{R}_{p^k}}B$, omitting the index $\mathscr{R}_{p^k}$. Note that the set of the generating symbols of $A*_{\mathscr{R}_{p^k}}B$ coincides with the union of the sets of the generating symbols of A and B, and the set of the defining relations of $A*_{\mathscr{R}_{p^k}}B$ coincides with the union of the sets of defining relations of A and B.

We will use Dyck's Theorem [18, Subsection 11.2, Theorem 5].

Lemma 1. *Suppose that a group A has the representation*

$$A=\langle\{x_i\mid i\in I\};\{r_j(x_{j_1},\dots,x_{j_{l(j)}})=1\mid j\in J\}\rangle$$

in a given quasivariety $\mathscr{M}$. Suppose that $H\in\mathscr{M}$ and H includes some set of elements $\{g_i\mid i\in I\}$ such that $r_j(g_{j_1},\dots,g_{j_{l(j)}})=1$ is true in H for each $j\in J$. Then the mapping $x_i\to g_i$ ($i\in I$) extends to a homomorphism of A to H.

Below we will need the following test for the membership of a finitely-defined group G in the subvariety $q\mathscr{R}$ generated by a class $\mathscr{R}$ [16, Theorem 2.3.9; 17, Corollary 2.1.21].

Lemma 2. *A group G finitely-defined in a quasivariety $\mathscr{N}$ belongs to the quasivariety generated by a class $\mathscr{R}$ ($\mathscr{R}\subseteq\mathscr{N}$) if and only if for every $g\in G$ ($g\neq 1$) there exists a homomorphism φ_g of G into some group of class $\mathscr{R}$ such that $g^{\varphi_g}\neq 1$.*

Lemma 3. *Let $G \in \mathscr{R}_{p^k}$ be a nontrivial finite group. Then G is representable in $\mathscr{R}_{p^k}$ as*

$$G = \big\langle x_1, \dots, x_n; x_1^{p^{n_1}} c_1 = 1, \dots, x_l^{p^{n_l}} c_l = 1, c_{l+1} = 1, \dots, c_s = 1 \big\rangle,$$

where $c_1, \dots, c_s$ are elements in the commutator subgroup, $n_i \geq 1, i = 1, \dots, l$ (the relations $x_1^{p^{n_1}} c_1 = 1, \dots, x_l^{p^{n_l}} c_l = 1$ or $c_{l+1} = 1, \dots, c_s = 1$ can be absent).

PROOF. The group G/G' splits into the direct product of its nontrivial cyclic subgroups: $G/G' = \langle x_1 G' \rangle \times \dots \times \langle x_n G' \rangle$. The elements $x_1, \dots, x_n$, generating G modulo the commutator subgroup, generate G. In these generators, G has the desired representation. The lemma is proved.

Lemma 4. *Let $\mathscr{M} \subseteq \mathscr{R}_{p^k}$ be an arbitrary quasivariety of groups such that $Z_p * Z_p \in \mathscr{M}$. Suppose that groups A and B in $\mathscr{M}$ have representations $\mathscr{R}_{p^k}$ analogous to the representation of Lemma 3 in generators $a_1, \dots, a_n$ and $b_1, \dots, b_m$ respectively. If N is a (possibly trivial) subgroup in $A * B$ generated by some commutators of the form $[a_i, b_j]$ then $(A * B)/N \in \mathscr{M}$.*

PROOF. Put $G = (A * B)/N$ and $[A, B] = \langle [a_i, b_j] \mid i = 1, \dots, n,\ j = 1, \dots, m \rangle$. Since $G/[A, B] \cong A \times B$, we have $G/[A, B] \in \mathscr{M}$. Take an arbitrary nonidentity element $g \in [A, B]$. It suffices (in view of Lemma 2) to show that $g^\varphi \neq 1$ for some homomorphism φ of G to a suitable group in $\mathscr{M}$.

The element g can be written down as

$$g = \prod [a_i, b_j]^{m_{ij}}.$$

Fix a pair of indices u, v such that $m_{uv} \not\equiv 0 \pmod p$. Let $Z_p * Z_p = \langle a, b \rangle$. By Lemma 1, the mapping

$$a_u \to a, \quad b_v \to b,$$

$$a_i \to 1 \text{ for } i \neq u; \quad b_j \to 1 \text{ for } j \neq v$$

extends to a homomorphism $\varphi : G \to Z_p * Z_p \in \mathscr{M}$; moreover, $g^\varphi = [a, b]^{m_{uv}} \neq 1$. The lemma is proved.

Corollary 1. *If $\mathscr{M} \subseteq \mathscr{R}_{p^k}$ is a quasivariety of groups such that $Z_{p^k} \in \mathscr{M}$ and $Z_p * Z_p \in \mathscr{M}$ then $Z_{p^s} * Z_{p^t} \in \mathscr{M}$ for $s, t \leq k$.*

Let A_{rm} $(r, m \geq 1)$ be the group defined in $\mathscr{R}_{p^k}$ by the generators

$$a_{jl}, b_{jt} \quad (j = 1, \dots, 2r; l = 1, \dots, 2m, t = 1, \dots, 2w)$$

and the defining relations

$$\prod_{j=1}^{w} [b_{1j}, b_{1,w+j}] \prod_{j=1}^{m} [a_{1j}, a_{1,m+j}] = \dots = \prod_{j=1}^{w} [b_{2r,j}, b_{2r,w+j}] \prod_{j=1}^{m} [a_{2r,j}, a_{2r,m+j}],$$

$$[a_{lj}, a_{ti}] = 1 \quad (\text{for all } l \neq t).$$

Put $c_1 = \prod_{j=1}^{w} [b_{1j}, b_{1,w+j}] \prod_{j=1}^{m} [a_{1j}, a_{1,m+j}]$.

The groups with the same representation but with respect to other quasivarieties were considered in [7, 15, 19], and for $k = 1$, in [14].

Lemma 5. *Let $\mathscr{M} \subseteq \mathscr{R}_{p^k}$ be an arbitrary quasivariety of groups such that $Z_{p^k}, Z_p * Z_p \in \mathscr{M}$. Then $A_{rm}/\langle c_1 \rangle, A_{rm} \in \mathscr{M}$.*

PROOF. Let B_i be the group having the following representation in $\mathscr{R}_{p^k}$:

$$B_i = \left\langle \{a_{il}, b_{it} \mid l = 1, \dots, 2m, t = 1, \dots, 2w\}; \prod_{j=1}^{w} [b_{ij}, b_{i,w+j}] \prod_{j=1}^{m} [a_{ij}, a_{i,m+j}] = 1 \right\rangle.$$

These groups were studied in [16, Theorem 4.2.24; 19] in $\mathscr{R}_p$. The proof of the fact that $B_i \in \mathscr{M}$ (by induction on $m + w$ on applying Lemma 4) is an almost verbatim repetition of the proofs of the analogous assertions in [16, Theorem 4.2.24; 19] and so we omit the proof. The group $A_{rm}/\langle c_1\rangle$ can be constructed from B_i by repeating the construction of Lemma 4 several times. By Lemma 4 $A_{rm}/\langle c_1\rangle \in \mathscr{M}$.

Consider the homomorphism $\varphi : A_{rm} \to F_2$ for which $b_{j1}^{\varphi} = a$, $b_{j,w+1}^{\varphi} = b$ $(j = 1, \dots, 2r)$ and the images of the other generators are equal to the identity. The existence of φ follows from Dyck's Theorem (Lemma 1). We see that $c_1^{\varphi} = [a, b] \neq 1$. Consequently, $\langle c_1\rangle \cap \ker \varphi = 1$. Hence, A_{rm} embeds in the group $A_{rm}/\langle c_1\rangle \times A_{rm}/\ker \varphi$; therefore, $A_{rm} \in \mathscr{M}$. The lemma is proved.

Denote by $C_p^t(w)$ $(1 \le t \le k - 1)$ the group that is represented in $\mathscr{R}_{p^k}$ as

$$C_p^t(w) = \left\langle a_0, a_1, \dots, a_{2w}; a_0^{p^t} \prod_{i=1}^{w} [a_i, a_{w+i}] = 1 \right\rangle.$$

Lemma 6. *Commutation on noncentral elements is transitive in $C_p^t(w)$.*

Proof. It is not hard to see that the commutator subgroup of $C_p^t = C_p^t(w)$ is the free abelian group of exponent p with free generators $[a_i, a_j]$ $(0 \le i < j \le 2w)$. Let $a, b, c \notin Z(C_p^t)$ and $[a, b] = 1$, $[a, c] = 1$. These elements are representable as

$$a = a_0^{k_0} a_1^{k_1} \dots a_{2w}^{k_{2w}} c_1, \quad b = a_0^{l_0} a_1^{l_1} \dots a_{2w}^{l_{2w}} c_2, \quad c = a_0^{m_0} a_1^{m_1} \dots a_{2w}^{m_{2w}} c_3,$$

where c_1, c_2, and c_3 are in the commutator subgroup of the free group. Then

$$[a, b] = \prod_{0 \le i < j \le 2w} [a_i, a_j]^{k_i l_j - k_j l_i} = 1, \quad [a, c] = \prod_{0 \le i < j \le 2w} [a_i, a_j]^{k_i m_j - k_j m_i} = 1,$$

whence $k_i l_j - k_j l_i \equiv 0 \pmod p$ and $k_i m_j - k_j m_i \equiv 0 \pmod p$ for all $0 \le i < j \le 2w$. It follows that the rank of the matrix

$$\begin{pmatrix} k_0 & k_1 & k_2 & \dots & k_{2w} \\ l_0 & l_1 & l_2 & \dots & l_{2w} \\ m_0 & m_1 & m_2 & \dots & m_{2w} \end{pmatrix}$$

is equal to 1. Consequently, $l_i m_j - l_j m_i \equiv 0 \pmod p$ for all $0 \le i < j \le 2w$. This means that $[b, c] = 1$. The lemma is proved.

Lemma 7. *If $w_1 < w_2$ then $\left(a_0^{p^t}\right)^{\varphi} = 1$ for every homomorphism $\varphi : C_p^t(w_1) \to C_p^s(w_2)$.*

Proof. Suppose that $\left(a_0^{p^t}\right)^{\varphi} \neq 1$. Let F be the $\mathscr{R}_{p^k}$-free group with free generators $a_0, a_1, \dots, a_{2w_2}$ and let b_i be the inverse image of a_i^{φ} under the natural homomorphism of F onto $C_p^s(w_2)$. Since

$$\left(a_0^{\varphi}\right)^{p^t} \prod_{i=1}^{w_1} [a_i^{\varphi}, a_{w_1+i}^{\varphi}] = 1;$$

in F we have

$$b_0^{p^t} \prod_{i=1}^{w_1} [b_i, b_{w_1+i}] = a_0^{mp^s} \prod_{i=1}^{w_2} [a_i, a_{w_2+i}]^m$$

for some m $(0 \le m < p)$. If $m = 0$ then $b_0^{p^t} \in F'$, whence $b_0^{p^t} = 1$. Hence, $\left(a_0^{p^t}\right)^{\varphi} = 1$, which fails. Thus, we suppose that $m \neq 0$.

Consider the group $\overline{F} = F/F^{p^2}\left\langle a_0^F \right\rangle$ free in the manifold $\mathscr{R}_{p^2}$ with free generators $\bar{a}_1, \dots, \bar{a}_{2w_2}$. Let $\bar{b}_i$ be the image of b_i under the natural homomorphism of F onto $\overline{F}$. We have

$$\bar{b}_i^{p^t} \prod_{i=1}^{w_1} [\bar{b}_i, \bar{b}_{w_1+i}] = \prod_{i=1}^{w_2} [\bar{a}_i, \bar{a}_{w_2+i}]^m.$$

Since $\bar{b}_i^{p^t} \in \overline{F}'(t \ge 1)$, we have $\bar{b}_i^{p^t} = 1$. We get a contradiction to the fact that, as was proved in [16, Theorem 4.2.3] (see also [6, Lemma 3]), the element $\prod_{i=1}^{w_2} [\bar{a}_i, \bar{a}_{w_2+i}]$ and so its every power other than the identity cannot be written down as a product of less than w_2 commutators. The lemma is proved.

Corollary 2. *If $w_1 < w_2$ then the quasi-identity*

$$\Phi_{t,w_1} = (\forall x_0)\dots(\forall x_{2w_1})\Big(x_0^{p^t}\prod[x_i, x_{w_1+i}] = 1 \to x_0^{p^t} = 1\Big)$$

holds in $C_p^s(w_2)$.

2. The Group H_{rm}

Fix quasivarieties $\mathscr{M}$ and $\mathscr{N}$ such that $\mathscr{M} \subsetneq \mathscr{N} \subseteq \mathscr{R}_{p^k}$. We assume that $\mathscr{M}$ is generated by a set of finite groups in each of which commutation on noncentral elements is transitive. Moreover, we assume that the groups Z_{p^k} and $Z_p * Z_p$ belong to $\mathscr{M}$.

If there is a finite group in $\mathscr{N} \setminus \mathscr{M}$ defined in $\mathscr{R}_{p^k}$ by a representation in which all defining words are elements of the commutator subgroup of a free group (i.e., commutator words), then we fix a group $G \in \mathscr{N} \setminus \mathscr{M}$ with the least number of these relations. If there is no such group in $\mathscr{N} \setminus \mathscr{M}$; then, as the fixed group G, we take a finite group in $\mathscr{N} \setminus \mathscr{M}$ with the least number of relations in its representations (with respect to $\mathscr{R}_{p^k}$) in the statement of Lemma 3.

Thus, G is generated by $x_1, \dots, x_n$ and defined in $\mathscr{R}_{p^k}$ by the defining relations $r_1 = 1, \dots, r_s = 1$, where

$$r_1 = x_1^{p^{n_1}} w_1, \dots, r_d = x_d^{p^{n_d}} w_d,$$

$$r_{d+1} = [x_{f(d+1)}, x_{g(d+1)}]w_{d+1}, \dots, r_s = [x_{f(s)}, x_{g(s)}]w_s,$$

and $n_1, \dots, n_d$ are naturals different from zero and less than k, while $w_1, \dots, w_s$ are elements of the commutator subgroup of the free group.

We say that the group G *possesses property* (P_1) if r_1 does not belong to the commutator subgroup of the free group, and G *possesses property* (P_2) if all its defining relations are elements of the commutator subgroup of the free group (i.e., commutator words). Note that if G possesses property (P_2) then its defining relations have the form

$$[x_{f(1)}, x_{g(1)}]w_1 = 1, [x_{f(2)}, x_{g(2)}]w_2 = 1, \dots, [x_{f(s)}, x_{g(s)}]w_s = 1.$$

We may and will assume that the commutator $[x_{f(i)}, x_{g(i)}]$ occurs in a nonzero degree only in the word r_i.

Take an arbitrary nonidentity element $v \in \mathscr{M}(G)$. Since $G/G' \in \mathscr{M}$, we have $\mathscr{M}(G) \subseteq G'$. If some commutator $[x_{f(i)}, x_{g(i)}]$ occurs in v in a nonzero degree then it can be excluded from the representation of v with the use of the defining relation $r_i = 1$. Thus, fix $v \in \mathscr{M}(G)$ such that $v \neq 1$, $v = \prod_{i=1}^{q}[x_{h(i)}, x_{h(q+i)}]^{\gamma_i}$, and $\gamma_i \not\equiv 0 \pmod p$ $(i = 1, \dots, q)$ whose representation involves the commutators $[x_{f(i)}, x_{g(i)}]$ in a nonzero degree $(i = d+1, \dots, s)$.

Let $H_{rm}(w)$ be the group (denoted by H_{rm} to be as a rule) defined in $\mathscr{R}_{p^k}$ by the generators $x_1, \dots, x_n$, a_{jl}, b_{jt} $(j = 1, \dots, 2r;\ l = 1, \dots, 2m,\ t = 1, \dots, 2w)$ and the following defining relations:

(i) all relations of A_{rm};

(ii) $r_1 c_1 = 1$;

(iii) if G possesses property (P_1) then, in each commutator $[x_{h(i)}, x_{h(q+i)}]$ $(i = 1, \dots q)$, fix exactly one element (this is $x_{h(i)}$ or $x_{h(q+i)}$) different from x_1 (denote it by $\bar{x}_i$) and introduce the relations: $[\bar{x}_i, b_{tj}] = 1$ and $[\bar{x}_i, b_{t,w+j}] = 1$ for all $j = 1, \dots, w$ and all t such that $t \equiv i \pmod q$; if G possesses property (P_2) then we introduce the relations $[x_{h(i)}, b_{tj}] = 1$ and $[x_{h(q+i)}, b_{tj}] = 1$ for all $j = 1, \dots, w$ and all t such that $t \equiv i \pmod q$;

(iv) $r_2 = 1, \dots, r_s = 1$.

Recall that $c_1 = \prod_{j=1}^{w}[b_{1j}, b_{1,w+j}]\prod_{j=1}^{m}[a_{1j}, a_{1,m+j}]$ is the word from the definition of A_{rm}. The definition of H_{rm} depends on which of the properties, (P_1) or (P_2), is possessed by G.

Denote $\prod_{i=1}^{q}[x_{h(i)}, x_{h(q+i)}]^{\gamma_i} \in H_{rm}$ again by v.

Throughout the article we denote by H the group that has in $\mathscr{R}_{p^k}$ the representation $H = \langle x_1, \dots, x_n;$ $r_2 = 1, \dots, r_s = 1\rangle$.

Lemma 8. *$H \in \mathscr{M}$.*

Proof. Clearly, $H/\langle r_1\rangle \cong G$; therefore, $H/\langle r_1\rangle \in \mathscr{N}$. Let $\langle a\rangle$ be a cyclic group of order p^k. By Lemma 1, the mapping $x_1 \to a$, $x_i \to 1$ $(i \geq 2)$ extends to some homomorphism $\varphi : H \to \langle a\rangle \in \mathscr{N}$; moreover, $r_1^\varphi = a^{p^{n_1}}$. The groups $\langle r_1\rangle$ and $\langle a^{p^{n_1}}\rangle$ have identical orders equal to p^{k-n_1}; therefore, φ sends $\langle r_1\rangle$ onto $\langle a^{p^{n_1}}\rangle$ isomorphically. (If G possesses property (P_2) then consider the mapping φ under which $x_{f(1)} \to a \in F_2$, $x_{g(1)} \to b \in F_2$, and $x_i \to 1$ for $i \notin \{x_{f(1)}, x_{g(1)}\}$.) We see that $\langle r_1\rangle \cap \ker\varphi = 1$. Hence, H embeds in $H/\langle r_1\rangle \times H/\ker\varphi \in \mathscr{N}$. Since H has less defining relations than G, the choice of G implies that $H \in \mathscr{M}$. The lemma is proved.

Lemma 9. *If, for some w_1 $(1 \leq w_1 < w)$, the groups $C_p^1(w_1), C_p^2(w_1), \dots, C_p^{k-1}(w_1)$ belong to $\mathscr{M}$ then $H_{rm}(w) \in \mathscr{M} \vee qG$.*

Proof. Clearly,

$$H_{rm}/\langle c_1\rangle \cong (G * A_{rm}/\langle c_1\rangle)/N,$$

where $N = \langle [\bar{x}_i, b_{tj}], [\bar{x}_i, b_{t,w+j}] \mid i = 1, \dots, q,\ \text{all } t \equiv i \pmod q, j = 1, \dots, w\rangle$ or $N = \langle [x_{h(i)}, b_{tj}], [x_{h(q+i)}, b_{tj}] \mid i = 1, \dots, q,\ j = 1, \dots, w, \text{all } t \equiv i \pmod q\rangle$ if G possesses property (P_1) or (P_2) respectively.

By Lemma 4 (applied to the quasivariety $\mathscr{M} \vee qG$), $H_{rm}/\langle c_1\rangle \in \mathscr{M} \vee qG$. It suffices now to construct a homomorphism $\varphi : H_{rm} \to M$ into a suitable group $M \in \mathscr{M}$ such that $c_1^\varphi \neq 1$.

If G possesses property (P_1) then we take $M = C_p^{n_1}(w_1) \in \mathscr{M}$. Put $x_1^\varphi = a_0$ and $(b_{tj})^\varphi = a_j$, $(b_{t,w+j})^\varphi = a_{w_1+j}$ for $j = 1, \dots, w_1$ and all t. The images of the remaining generators are assumed to be equal to the identity. Since $\bar{x}_i^{\,\varphi} = 1$ for every i, we see that we can apply Lemma 11, by which φ extends to a homomorphism (which we again denote by φ) of H_{rm} onto $C_p^{n_1}(w_1)$; moreover, $c_1^\varphi = \prod_{i=1}^{w_1}[a_i, a_{w_1+i}] \neq 1$.

If G possesses property (P_2) then we take $M = F_2 = \langle a, b\rangle \in \mathscr{M}$ and put $x_{f(1)}^\varphi = a$, $x_{g(1)}^\varphi = b$. If the relation $[x_{f(1)}, b_{l1}] = 1$ or the relation $[x_{g(1)}, b_{u1}] = 1$ occurs among the defining relations of H_{rm} then we put $b_{l1}^\varphi = a$, $b_{l,w+1}^\varphi = b^{-1}$ or $b_{u1}^\varphi = b$, $b_{u,w+1}^\varphi = a$ respectively. If none of the relations $[x_{f(1)}, b_{t1}] = 1$ and $[x_{g(1)}, b_{t1}] = 1$ occurs in list (iii) then we put $b_{t1}^\varphi = a$ and $b_{t,w+1}^\varphi = b^{-1}$ for all t. The images of the remaining generators are assumed equal to the identity. Since the commutator $[x_{f(1)}, x_{g(1)}]$ does not occur in the representation of v, the relations $[x_{f(1)}, b_{l1}] = 1$ and $[x_{g(1)}, b_{l1}] = 1$ cannot simultaneously occur in the list of the defining relations of H_{rm}. This means that φ is well defined. By Lemma 1, this mapping extends to a homomorphism (which we still denote by φ) onto F_2; moreover, $c_1^\varphi = [a, b]^{-1} \neq 1$. The lemma is proved.

The proof of Lemma 9 implies that if a group G is defined only by commutation defining relations then the assumption of the presence in $\mathscr{M}$ of the groups $C_p^1(w_1), C_p^2(w_1), \dots, C_p^{k-1}(w_1)$ is unnecessary. In this case, we have

Corollary 3. *If G is defined in $\mathscr{R}_{p^k}$ only by commutation defining relations and $Z_{p^k}, Z_p * Z_p \in \mathscr{M}$ then $H_{rm} \in \mathscr{M} \vee qG$.*

Lemma 10. *Suppose that R is a group in $\mathscr{M}$, while $\varphi : H_{rm} \to R$ is an arbitrary homomorphism and $r \geq q$. Then $v^\varphi = 1$, where $v = \prod_{i=1}^q [x_{h(i)}, x_{h(q+i)}]^{\gamma_i} \in H_{rm}$. In particular, $H_{rm} \notin \mathscr{M}$.*

Proof. Assume that $v^\varphi \neq 1$. Since $\mathscr{M}$ is generated by the groups in which commutation on noncentral elements is transitive, by the membership test (Lemma 2), there exists a homomorphism ψ of R into some such group for which $v^{\varphi\psi} \neq 1$. So, we may and will assume that commutation on noncentral elements of R is transitive.

Since $v^\varphi \neq 1$, there exists a number i $(1 \leq i \leq q)$ such that $[x_{h(i)}, x_{h(q+i)}]^\varphi \neq 1$. In particular, $\bar{x}_i^{\,\varphi} \notin Z(R)$. Since in H_{rm} we have the relations

$$[\bar{x}_i, b_{tj}] = 1, \quad [\bar{x}_i, b_{t,w+j}] = 1 \quad (t \equiv i \pmod q,\ j = 1, \dots, w)$$

and commutation is transitive on noncentral elements, we obtain

$$[b_{tj}, b_{t,w+j}]^\varphi = 1 \quad (t \equiv i \pmod q,\ j = 1, \dots, w).$$

In the case when G possesses property (P_2), the relations

$$[x_{h(i)}, x_{h(q+i)}]^{\varphi} \neq 1,\ [x_{h(i)}, b_{tj}]^{\varphi} = 1,$$

$$[x_{h(q+i)}, b_{tj}]^{\varphi} = 1 \quad (t \equiv i \ (\mathrm{mod}\ q),\ j = 1, \dots, w)$$

and the transitivity of commutation on noncentral elements implies that $b_{tj}^{\varphi} \in Z(R)$, whence also

$$[b_{tj}, b_{t,w+j}]^{\varphi} = 1 \quad (t \equiv i \ (\mathrm{mod}\ q),\ j = 1, \dots, w).$$

Suppose that there exists l with $[a_{il}, a_{i,m+l}]^{\varphi} \neq 1$. Take the commutator $[a_{tf}, a_{t,m+f}]^{\varphi}$ $(t \neq i)$. The defining relations of A_{rm} imply that the elements in different commutators commute. The transitivity of commutation implies that $[a_{tf}, a_{t,m+f}]^{\varphi} = 1$. The above yields

$$\prod_{j=1}^{m} [a_{tj}, a_{t,m+j}]^{\varphi} = 1 \quad \text{for } t \equiv i \ (\mathrm{mod}\ q) \quad (t \neq i);$$

i.e., $c_1^{\varphi} = 1$.

If $[a_{il}, a_{i,m+l}]^{\varphi} = 1$ for every l then we also find that $c_1^{\varphi} = 1$. Thus, $c_1^{\varphi} = 1$.

Let $\psi : H_{rm} \to H_{rm}/\langle c_1 \rangle$ be a natural homomorphism. Since $\langle c_1 \rangle \subseteq \ker \varphi$, there exists a homomorphism $\xi : H_{rm}/\langle c_1 \rangle \to R$ such that $\varphi = \psi\xi$; in particular, $(g\langle c_1 \rangle)^{\xi} = g^{\varphi}$ for every $g\langle c_1 \rangle \in H_{rm}/\langle c_1 \rangle$. In proving Lemma 9, we noticed that $H_{rm}/\langle c_1 \rangle \cong (G * A_{rm}/\langle c_1 \rangle)/N$. This means in particular that $\langle x_1, \dots, x_n \rangle^{\psi} \cong G$. Since $v^{\psi} \in \mathscr{M}(G)$, we conclude that $v^{\psi\xi} = 1$; i.e., $v^{\varphi} = 1$. The lemma is proved.

Denote by B_{rm} the group defined in $\mathscr{R}_{p^k}$ by the generators

$$a_{jl}, b_{jt}\ (j = 1, \dots, 2r;\ l = 1, \dots, 2m,\ t = 1, \dots, 2w), \quad x_1, \dots, x_n$$

and the defining relations (presenting a part of the defining relations of H_{rm})

(i') $[a_{li}, a_{tj}] = 1$ for all l, j, t, and i such that $l \neq t$;

(ii') $[\bar{x}_i, b_{tj}] = 1$, $[\bar{x}_i, b_{t,w+j}] = 1$ for all t such that $t \equiv i$ (mod q) ($j = 1, \dots, w$, $i = 1, \dots, q$) if G possesses property (P_1) and $[x_{h(i)}, b_{tj}] = 1$, $[x_{h(q+i)}, b_{tj}] = 1$ for all t such that $t \equiv i$ (mod q) ($j = 1, \dots, w$, $i = 1, \dots, q$) if G possesses property (P_2);

(iii') $r_2 = 1, \dots, r_s = 1$.

Let $c_t = \prod_{j=1}^{w} [b_{tj}, b_{t,w+j}] \prod_{j=1}^{m} [a_{tj}, a_{t,m+j}] \in B_{rm}$ and let $N_{rm} = \left\langle r_1 c_1, c_1 c_2^{-1}, \dots, c_1 c_{2r}^{-1} \right\rangle$ be a subgroup in B_{rm}. Clearly, $H_{rm} \cong B_{rm}/N_{rm}$.

Lemma 11. $B_{rm} \in \mathscr{M}$.

Proof. Let a group B be defined in $\mathscr{R}_{p^k}$ by the generators a_{jl} and b_{jt} ($j = 1, \dots, 2r$; $l = 1, \dots, 2m$, $t = 1, \dots, 2w$) and the defining relations $[a_{lj}, a_{ti}] = 1$ for all l, j, t, and i such that $l \neq t$ (this is a part of the defining relations of B_{rm}). By Lemma 4, $B \in \mathscr{M}$. As is easy to observe, $B_{rm} = (H * B)/N$, where

$$N = \langle [\bar{x}_i, b_{tj}], [\bar{x}_i, b_{t,w+j}] \mid t \equiv i \ (\mathrm{mod}\ q)\ (i = 1, \dots, q,\ j = 1, \dots, w) \rangle$$

if G possesses property (P_1) and

$$N = \langle [x_{h(i)}, b_{tj}], [x_{h(q+i)}, b_{tj}] \mid t \equiv i \ (\mathrm{mod}\ q)\ (i = 1, \dots, q,\ j = 1, \dots, w) \rangle$$

if G possesses property (P_2).

By Lemma 8, $H \in \mathscr{M}$; hence, by Lemma 4 $B_{rm} \in \mathscr{M}$. The lemma is proved.

Lemma 12. *Let B be an l-generated subgroup in B_{rm}. If $m > C_l^2 = \frac{n(n-1)}{2}$ then $\Phi(B) \cap N_{rm} = 1$, where $\Phi(B)$ is the Frattini subgroup of B.*

Proof. Let $g \in \Phi(B) \cap N_{rm},\ g \neq 1$. Since $\Phi(B) = B^p B'$, the element g is representable as $g = g_1^p c$ for some $g_1 \in B$ and $c \in B'$. Since B is an l-generated subgroup, B' is a C_l^2-generated group; hence, c is representable as the product of C_l^2 commutators. Thus, g has the form

$$g = g_1^p \prod_{i=1}^{C_l^2} [f_i, f_{C_l^2+i}].$$

Since $g \in N_{rm}$, g is representable as

$$g = (r_1 c_1)^{t_1} \prod_{i=2}^{2r} \left(c_1 c_i^{-1}\right)^{t_i}.$$

Therefore, g is representable as

$$g = r_1^{t_1} c_1^{t_1+t_2+\cdots} \prod_{i=2}^{2r} c_i^{-t_i}.$$

Since $g \neq 1$, we see that some element c_i (suppose for convenience that this is c_1) occurs in g in a nonzero degree. Let $F_{2m} = \langle y_1, \dots, y_{2m} \rangle$ be a free $\mathscr{R}_{p^k}$-group of rank $2m$ and let $\pi : B_{rm} \to F_{2m}$ be a homomorphism for which $(a_{11})^\pi = y_1, \dots, (a_{1,2m})^\pi = y_{2m}$ and the images of the remaining generators of B_{rm} are equal to 1 (the existence of such a homomorphism follows from Lemma 1). We infer

$$(g_1^\pi)^p \prod_{i=1}^{C_l^2} \left[f_i^\pi, f_{C_l^2+i}^\pi\right] = \prod_{j=1}^{m} [(a_{1j})^\pi, (a_{1,m+j})^\pi]^{t_1+t_2+\cdots}.$$

Since the quasi-identities

$$(\forall x)(\forall x_1)\dots(\forall x_{2n}) \left(x^p \prod_{i=1}^{n} [x_i, x_{n+i}] = 1 \to x^p = 1 \right), \quad n = 1, 2, \dots,$$

hold in F_{2m}, we obtain the following equality in F_{2m}:

$$\prod_{i=1}^{C_l^2} \left[f_i^\pi, f_{C_l^2+i}^\pi\right] = \prod_{j=1}^{m} [y_j, y_{m+j}]^{t_1+t_2+\cdots}.$$

This contradicts [16, Theorem 4.2.3] (see also [6, Lemma 3]) the fact that the element $\prod_{j=1}^m [y_j, y_{m+j}]$, and hence its any nonidentity degree cannot be written down in F_{2m} as the product of at most m commutators. The lemma is proved.

Lemma 13. *Let A be an l-generated subgroup in H_{rm}. If $m > C_l^2$ then $A \in \mathscr{M}$.*

Proof. Let $\varphi : B_{rm} \to H_{rm}$ be the natural homomorphism with kernel N_{rm} and let B be an arbitrary minimal preimage of A under φ. (An analogous trick was used in [20].) Show that $B \cap N_{rm} \leq \Phi(B)$.

Suppose that this fails. Then $B \cap N_{rm} \not\leq M$ for some maximal subgroup M in B. Hence, $B = (B \cap N_{rm})M$. Since $(B \cap N_{rm})^\varphi = 1$, we conclude that $M^\varphi = A$. This contradicts the minimality of the preimage. Thus, $B \cap N_{rm} \leq \Phi(B)$.

By Lemma 12, $B \cap N_{rm} = 1$, and hence $B \cong A$. It remains to use Lemma 11, which implies that $B \in \mathscr{M}$. The lemma is proved.

The meaning of the rather cumbersome inequalities in Lemma 14 is as follows: The number occurring in it are "sufficiently large" and "one is much larger than the other," the number of generators in H_{rm}, while $l = n + 4rm + 4rw$ and $n + 4wq + 4q$ is the number of generators in H_{q1}.

Lemma 14. *Let $\varphi : H_{rm} \to H_{xy}$ be a homomorphism where $y > C^2_{n+4wq+4q}$, $f = 2^{|H_{xy}|}$. Put $l = n + 4rm + 4rw$. If $r > 4mqf$ or $y > C^2_l$ then $v^\varphi = 1$, where $v = \prod_{i=1}^{q} [x_{h(i)}, x_{h(q+i)}]^{\gamma_i}$ and q is the number of commutators in this decomposition.*

PROOF. Suppose first that $y > C^2_l$. Since H_{rm} is l-generated, H^φ_{rm} is generated by l elements too. Lemma 13 implies that $H^\varphi_{rm} \in \mathscr{M}$. By Lemma 10, $v^\varphi = 1$.

Suppose now that $r > 4mqf$. Our goal is to construct the homomorphism $\psi : H_{q1} \to H_{xy}$ such that $v^\psi = v^\varphi$. Note that f is greater than the number of subgroups in H_{xy}. For each i $(1 \le i \le q)$, consider the following subgroups in H_{rm}:

$$B_i = \langle a_{i,1}, a_{i,2}, \dots, a_{i,2m} \rangle.$$

Note that the subgroups B_i and B_j for $i \ne j$ commute.

Let

$$C_t = B_t B_{t+q}, B_{t+2q} \dots B_{t+(2f-1)q} \quad (t = 1, 2, \dots, q).$$

The existence of all these subgroups B_i and C_t follows from the inequality $r > 4mqf$.

Study C^φ_t. Since the number of the groups $B^\varphi_t, B^\varphi_{t+q}, B^\varphi_{t+2q}, \dots, B^\varphi_{t+(f-1)q}$ is equal to f and the number of subgroups in H_{xy} is less than f, some of these subgroups coincide. Let $B^\varphi_{t+qi} = B^\varphi_{t+qj}$ $(i < j)$. The equality $[B_{t+qi}, B_{t+qj}] = 1$ implies

$$1 = \big[B^\varphi_{t+qi}, B^\varphi_{t+qj}\big] = \big[B^\varphi_{t+qi}, B^\varphi_{t+qi}\big];$$

i.e., the group B^φ_{t+qi} is abelian. Therefore,

$$c^\varphi_1 = \prod_{u=1}^{w} [b_{t+qi,u}, b_{t+qi,w+u}]^\varphi \prod_{u=1}^{m} [a_{t+qi,u}, a_{t+qi,m+u}]^\varphi = \prod_{u=1}^{w} [b_{t+qi,u}, b_{t+qi,w+u}]^\varphi.$$

Put $d_{tj} = b^\varphi_{t+qi,j}$ $(j = 1, \dots, 2w)$.

Let us first assume that G possesses property (P_1). Since

$$[\bar{x}_t, b_{t+qi,1}] = 1, \quad [\bar{x}_t, b_{t+qi,w+1}] = 1,$$

we have

$$\big[\bar{x}^\varphi_t, d_{tj}\big] = 1 \quad (j = 1, \dots, 2w).$$

Similarly, considering the subgroups $B^\varphi_{t+fq}, B^\varphi_{t+(f+1)q}, B^\varphi_{t+(f+2)q}, \dots, B^\varphi_{t+(2f-1)q}$ in C^φ_t; find the elements, denoted by $d_{t+q,j}$ $(j = 1, \dots, 2w)$, in some of these groups such that $\big[\bar{x}_t{}^\varphi, d_{t+q,j}\big] = 1$ $(j = 1, \dots, 2w)$.

Suppose that G possesses property (P_2). Since

$$[x_{h(i)}, b_{t+qi,1}] = 1, \quad [x_{h(q+i)}, b_{t+qi,1}] = 1,$$

we have

$$\big[x^\varphi_{h(i)}, d_{tj}\big] = 1, \quad \big[x^\varphi_{h(q+i)}, d_{tj}\big] = 1 \quad (j = 1, \dots, 2w).$$

Similarly,

$$\big[x^\varphi_{h(i)}, d_{t+q,j}\big] = 1, \quad \big[x^\varphi_{h(q+i)}, d_{t+q,j}\big] = 1 \quad (j = 1, \dots, 2w).$$

In both cases,

$$c^\varphi_1 = \prod_{u=1}^{w} [d_{t+q,u}, d_{t+q,w+u}].$$

Hence,

$$\prod_{u=1}^{w} [d_{t,u}, d_{t,w+u}] = \prod_{u=1}^{w} [d_{t+q,u}, d_{t+q,w+u}].$$

In both cases, use Lemma 1, by which the mapping

$$b_{tj} \to d_{tj},\ b_{t+q,j} \to d_{t+q,j} \quad (t = 1, \dots, q, j = 1, \dots, 2w),$$

$$x_i \to x_i^{\varphi}\ (i = 1, \dots, n), \quad a_{ij} \to 1$$

extends to a homomorphism $\psi : H_{q1} \to H_{xy}$. We see that $v^{\psi} = v^{\varphi}$. Since H_{q1} is generated by $n+4wq+4q$ elements and $y > C^2_{n+4wq+4q}$, by Lemma 13 $H^{\psi}_{q1} \in \mathscr{M}$. By Lemma 10 $v^{\psi} = 1$, whence $v^{\varphi} = 1$. The lemma is proved.

Theorem 1. *Let $\mathscr{M}$ and $\mathscr{N}$ be quasivarieties of groups, and let $\mathscr{M}$ be generated by the finite groups such that commutation on their noncentral elements is transitive, $\mathscr{M} \subsetneq \mathscr{N} \subseteq \mathscr{R}_{p^k}$. Suppose that $Z_p *_{\mathscr{R}_{p^k}} Z_p \in \mathscr{M}$ and $C^1_p(w_1), C^2_p(w_1), \dots, C^{k-1}_p(w_1)$ belong to $\mathscr{M}$ for some w_1. Then the interval $[\mathscr{M}, \mathscr{N}]$ in the lattice of quasivarieties has cardinality of the continuum.*

Proof. Take a finite group $G \notin \mathscr{M}$ such that $G \in \mathscr{N}$. Assume that the group G is defined among the groups in $\mathscr{N} \setminus \mathscr{M}$ by the least number of defining relations (with respect to $\mathscr{R}_{p^k}$) such as in Lemma 3.

We will construct a countable sequence of groups $H_1, H_2, \dots,$ contained in $\mathscr{N}$. Fix a number w ($w \geq w_1$). All groups H_{rm} in what follows contain this w. As H_1, take an arbitrary group H_{rm} in $\mathscr{M} \vee qG$. Suppose that H_{i-1} is already constructed. As H_i, take the group $H_{r_i m_i}$ such that

(1) the image of v is equal to the identity under every homomorphism of H_i into H_j ($j < i$);

(2) if q_{i-1} is the number of generators in H_{i-1} then every q_{i-1}-generated subgroup in H_i belongs to $\mathscr{M}$ (note that H_i exists due to Lemmas 13 and 14).

Let $\mathbf{N}$ be the set of naturals. Given a subset $I \subseteq \mathbf{N}$, let $\mathscr{M}_I = q\{H_i \mid i \in I\}$ be the quasivariety generated by all groups H_i ($i \in I$). If $H_i \in \mathscr{M}_I$ ($i \notin I$); then, by the membership test (Lemma 2), H_i is approximated by groups from $\{H_i \mid i \in I\}$, which contradicts Lemmas 13 and 14. Therefore, $H_i \notin \mathscr{M}_I$ for $i \notin I$, which yields the desired assertion. The theorem is proved.

The presence of $C^1_p(w_1), C^2_p(w_1), \dots, C^{k-1}_p(w_1)$ in $\mathscr{M}$ is necessary only in Lemma 9. If the group G from the proof of Theorem 1 is defined only by commutation defining relations; then, instead of Lemma 9, we can use Corollary 3. The following assertion is a consequence of the proof of Theorem 1:

Theorem 2. *Suppose that a quasivariety $\mathscr{M}$ is generated by finite groups in which commutation on noncentral elements is transitive and $\mathscr{M} \subseteq \mathscr{R}_{p^k}$. Suppose that $Z_{p^k}, Z_p *_{\mathscr{R}_{p^k}} Z_p \in \mathscr{M}$. If $G \notin \mathscr{M}$ can be defined in $\mathscr{R}_{p^k}$ only by commutation defining relations then the interval $[\mathscr{M}, \mathscr{M} \vee qG]$ in the lattice of quasivarieties of groups has cardinality of the continuum.*

Theorem 3. *The set of subquasivarieties $\mathscr{R}_{p^k}$ ($k \geq 2$) generated by a finite group and lacking any independent bases of quasi-identities is infinite.*

Proof. Let $\mathscr{M}_w$ be the quasivariety generated by the groups $C^t_p(w)$ ($t = 1, \dots, k-1$) and $Z_p * Z_p$; i.e.,

$$\mathscr{M}_w = q\bigg(\bigg(\prod_{t=1}^{k-1} C^t_p(w)\bigg) \times (Z_p * Z_p)\bigg).$$

For $w_1 < w_2$, the quasi-identity Φ_{t,w_1} is false in $C^t_p(w_1)$ and, by Corollary 2, is true in each $C^s_p(w_2)$; therefore, $\mathscr{M}_{w_1} \neq \mathscr{M}_{w_2}$ for $w_1 < w_2$.

If we assume that $\mathscr{M}_w$ has an independent basis of quasi-identities; then, as is known [17, Proposition 6.3.1], $\mathscr{M}_w$ has a covering in the lattice $L_q(\mathscr{K})$ of quasivarieties contained in $\mathscr{K}$ for every finitely axiomatizable quasivariety $\mathscr{K}$ containing $\mathscr{M}_w$. But, by Theorem 1, every interval $[\mathscr{M}_w, \mathscr{N}]$ has cardinality of the continuum for $\mathscr{M}_w \subsetneq \mathscr{N} \subseteq \mathscr{R}_{p^k}$; consequently, $\mathscr{M}_w$ has no coverings in $L_q(\mathscr{R}_{p^k})$. The theorem is proved.

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