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ON THE WEAK π-POTENCY OF SOME GROUPS AND FREE PRODUCTS

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Abstract: Let π be a set of primes. A group G is weakly π-potent if G is residually finite and, for each element x of infinite order in G, there is a positive integer m such that, for every positive π-integer n, there exists a homomorphism of G onto a finite group which sends x to an element of order mn. We obtain a few results about weak π-potency for some groups and generalized free products.

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1. Introduction

Recall that a group G is called *residually finite* if, for every nonidentity element x in G, there exists a homomorphism of G onto a finite group under which the image of x differs from the identity. Here we consider some more subtle approximation properties of groups.

Let π be a set of primes. A positive integer n is called a *π-integer* if all prime divisors of n belong to π. An element x of G is called *π-potent* if one of the following two conditions is fulfilled:

1. The order of x is infinite; and, for every positive π-integer n, there exists a homomorphism of G onto some finite group which sends x to an element of order n.

2. The order of x is finite; and, for every positive π-integer n dividing the order of x, there is a homomorphism of G onto a finite group which sends x to an element of order n.

Call a group G *π-potent* if G is residually finite and all elements of G are π-potent. If π coincides with the set of all primes then the notions of π-potent element and π-potent group coincide with the classical notions of potent element and potent group.

For an element of infinite order, the property of "being potent" (without the corresponding term) arose in Stebe's paper [1] where the potency of free groups and finitely generated torsion-free nilpotent groups was established as an auxiliary result. Later the notion of potency was studied by a few authors and extended to elements of finite order (see, for example, [2, 3]).

Compared to residual finiteness, the property of a group "being potent" is a sufficiently rigid constraint. Examples of residually finite but not potent groups are given by the Baumslag–Solitar groups $G_{1n} = (a, b; b^{-1}ab = a^n)$ for $n \geq 2$. Nevertheless, these groups are π-potent, where π is the set of all primes not dividing n.

As it turned out, many groups lacking the π-potency property are weakly π-potent. The notion of weak π-potency is introduced as follows:

An element x of infinite order in a group G is called *weakly π-potent* if there is a positive integer m such that, for each positive π-integer n, there is a homomorphism of G onto some finite group taking x to an element of order mn.

A group G is called *weakly π-potent* if G is residually finite and all elements of G of infinite order are weakly π-potent. If π coincides with the set of all primes then the notion of weakly π-potent element coincides with the familiar notion of weakly potent element (see, for example, [4]).

Let us formulate the results of the present article:

Theorem 1. *Let G be a residually finite almost soluble minimax group and let π be the set of all primes not belonging to the spectrum of G. Let $X = (x)$ be an infinite cyclic subgroup in G.*

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Then there is a positive integer m such that, for every positive π-integer n, the group G contains a characteristic subgroup W of finite index such that $X \cap W = X^{mn}$.

This theorem immediately implies

Corollary 1.1. *Let G be a residually finite almost soluble minimax group. Then G is weakly π-potent, where π is the set of all primes not belonging to the spectrum of G.*

In connection with the statements of Theorem 1 and Corollary 1.1, recall that a group almost possesses some property whenever it contains a subgroup of finite index with this property. Recall also that, in each almost soluble minimax group G, there exists a subnormal series whose every factor is either a quasicyclic group or an infinite cyclic group or a finite group. The *spectrum* of an almost soluble minimax group G is the set of all primes p for which the corresponding quasicyclic group is present among the terms of the above-mentioned series. Obviously, polycyclic groups are soluble minimax groups with empty spectrum.

Polycyclic groups (even in the absence of torsion) are in general not potent. Some example is constructed in [3]. Since every polycyclic group is residually finite [5] and its spectrum is empty, Corollary 1.1 implies the following well-known assertion (see, for example, [4, 6]):

Corollary 1.2. *Every polycyclic group is weakly potent.*

Soluble minimax groups constitute an important subclass of the class of all soluble groups of finite rank (see, for example, [7]). Recall that a group G has finite rank if there exists a positive integer r such that each finitely generated group in G is generated by at most r elements. Since each finitely generated residually finite group of finite rank is an almost soluble minimax group (see [8]); as another consequence of Corollary 1.1, we obtain

Corollary 1.3. *Let G be a finitely generated residually finite group of finite rank. Then G is weakly π-potent for some set π consisting of almost all primes.*

Let us turn to free products of groups. Observe first of all that the question remains open of the closedness of the class of all potent groups under free products (*The Kourovka Notebook*, Problem 9.1). On the other hand, Theorems 2–7 of the present article show that the weak potency property (the weak π-potency property) behaves sufficiently "well" under free constructions:

Theorem 2. *Let $G = A * B$ be the free product of groups A and B.*

1. *If A and B are π-potent torsion-free groups then G is a π-potent group.*
2. *If A and B are π-potent groups then G is a weakly π-potent group.*
3. *If A and B are weakly π-potent groups then G is a weakly π-potent group.*

Item 1 of this theorem was proved in [2] for the case when π coincides with the set of all primes. Item 2 is a particular case of item 3, and item 3 is a particular case of the following theorem:

Theorem 3. *Let $G = (A * B; H)$ be the free product of the groups A and B with finite amalgamated subgroup H. If A and B are weakly π-potent groups then G is a weakly π-potent group.*

One more result of the present article concerns with the case when the amalgamated subgroup H is cyclic. This result is formulated as follows:

Theorem 4. *Let $G = (A * B; H)$ be the free product of groups A and B with infinite cyclic amalgamated subgroup H. Suppose that the subgroup H is residually separable in each of the groups A and B and the generator h of H is weakly potent in each of the groups A and B.*

If A and B are weakly π-potent subgroups then G is a weakly π-potent group.

This implies the following result of [4]:

Corollary 4.1. *Let $G = (A * B; H)$ be the free product of groups A and B with infinite cyclic amalgamated subgroup H and suppose that the subgroup H is residually separable in each of the groups A and B. If A and B are weakly potent groups then G is a weakly potent group.*

The following three theorems are proved for the free product $G = (A * B; H)$ of residually finite almost soluble minimax groups A and B with amalgamated subgroup H under various constraints on H. Since every finitely generated residually finite group of finite rank is almost soluble and minimax (see [8]), these theorems are applicable to the free product $G = (A * B; H)$ of finitely generated residually finite groups A and B of finite rank with amalgamation H.

Theorem 5. *Let $G = (A * B; H)$ be the free product of groups A and B with cyclic amalgamated subgroup H, where $H \neq A$ and $H \neq B$, and let A and B be residually finite almost soluble minimax groups.*

The following three assertions are equivalent:

1. *G is residually finite.*
2. *H is residually separable in each of the groups A and B.*
3. *G is weakly π-potent, where π is the set of all primes not belonging to the union of the spectra of A and B.*

The equivalence of assertions 1 and 2 of Theorem 2 is proved in [9].

Corollary 5.1. *A free product of two almost polycyclic groups with cyclic amalgamation is a weakly potent group.*

This well-known assertion stems from Theorem 5 since every polycyclic group has empty spectrum and all subgroups in it are residually finite.

Theorem 6. *Let $G = (A * B; H)$ be the free product of groups A and B with amalgamated subgroup H normal in each of the groups A and B, where $H \neq A$ and $H \neq B$, and let A and B be residually finite almost soluble minimax groups.*

The following three assertions are equivalent:

1. *G is residually finite.*
2. *H is residually separable in each of the groups A and B.*
3. *G is weakly π-potent, where π is the set of all primes not belonging to the union of the spectra of A and B.*

The equivalence of assertions 1 and 2 of Theorem 6 is proved in [10].

Corollary 6.1. *A free product of two almost polycyclic groups with normal amalgamation is a weakly potent group.*

This result supplements Classical Baumslag's Theorem (see [11]), which states the residual finiteness of two polycyclic groups with normal amalgamation.

Theorem 7. *Let $G = (A * B; H)$ be the free product of groups A and B with amalgamated subgroup H having finite index in each of the groups A and B, where $H \neq A$ and $H \neq B$, and let A and B be residually finite almost soluble minimax groups.*

The following three assertions are equivalent:

1. *G is residually finite.*
2. *H contains a subgroup L of finite index normal in G.*
3. *G is weakly π-potent, where π is the set of all primes not belonging to the spectrum of H.*

The equivalence of assertions 1 and 2 in Theorem 7 is proved in [12].

Corollary 7.1. *Let $G = (A * B; H)$ be the free product of almost polycyclic groups A and B with amalgamated subgroup H having finite index in each of the groups A and B, where $H \neq A$ and $H \neq B$.*

Then the following three assertions are equivalent:

1. *G is residually finite.*
2. *H contains a subgroup L of finite index normal in G.*
3. *G is weakly potent.*

2. Auxiliary Assertions

Let G be an arbitrary group and let n be a positive integer. As usual, we denote by G^n the power subgroup; i.e., the subgroup in G generated by the nth powers of all elements of G. If G is an abelian group then G^n coincides with the set of the nth powers of elements of G. If G is an almost soluble minimax group then, as is easy to see, G^n has finite index in G.

Lemma 1. *Let π be a set of primes and let A be a torsion-free abelian group containing subgroups isomorphic to the group Q_p of p-adic fractions for no $p \in \pi$. Let $H = (h)$ be a nonidentity subgroup in A.*

Then for each positive π-integer n there exists a positive π-integer k such that $H \cap A^k = H^n$.

Proof. Let $n = {p_1}^{s_1} \dots {p_r}^{s_r}$ be the decomposition of n into prime factors from π. By hypothesis, for each $i = 1, \dots, r$ the group A has no subgroups isomorphic to Q_{p_i}. Therefore, for each $i = 1, \dots, r$ there exists a greatest nonnegative integer t_i for which the equation $x^{{p_i}^{t_i}} = h$ is solvable in A. Then

$$H \cap A^{{p_i}^{s_i+t_i}} = H^{{p_i}^{s_i}}.$$

Taking intersection over of the subgroups on the left-hand (right-hand) side of this equality over all $i = 1, \dots, r$, we obtain $H \cap A^k = H^n$, where $k = {p_1}^{s_1+t_1} \dots {p_r}^{s_r+t_r}$ is a desired π-integer.

Lemma 2. *Suppose that G is a finitely generated almost free group, F is a normal free subgroup of finite index in G, while $X = (x)$ is an infinite cyclic subgroup in G and m is a positive integer such that $X \cap F = X^m$.*

For each positive integer n there exists a homomorphism ψ of G onto a finite group such that $|x\psi| = mn$.

In particular, the following assertions hold:

1. *If $x \in F$ (i.e., if $m = 1$) then x is a potent element of G.*
2. *G is weakly potent.*
3. *In particular, a generalized free product of two finite groups is a weakly potent group (because it is almost free [11]).*

Proof. By the Magnus Theorem, in a finitely generated group F, the intersection of all terms in the lower central series $F = \gamma_1(F) \geq \gamma_2(F) \geq \cdots$ is trivial and all factors in this series are free abelian groups of finite ranks. Since $x^m \in F$, there are adjacent terms $A = \gamma_{i-1}(F)$ and $B = \gamma_i(F)$ such that $x^m \in A$ and $x^m \notin B$. Then $x^m B$ is a nonidentity element of the free abelian group A/B; and, by Lemma 1, for each positive integer n, there exists a positive integer k such that $x^m B$ has order n modulo the subgroup $(A/B)^k = A^k B/B$. Then $|x^m A^k B| = n$; i.e.,

$$X^m \cap A^k B = X^{mn},$$

whence $X \cap A^k B = X^{mn}$; i.e., the order of the element xA^kB of G/A^kB is equal to mn. This and the fact that the group G/A^kB is residually finite (as a finitely generated almost nilpotent group) imply that there is a homomorphism φ of G/A^kB onto a finite group taking xA^kB to an element of order mn. Now, as a desired homomorphism ψ, we must take the product $\varepsilon\varphi$, where ε is the natural homomorphism of G onto G/A^kB.

3. Proofs of Theorems 1–7

Proof of Theorem 1. Let G be a residually finite almost soluble minimax group, and let π be the set of all primes not belonging to the spectrum of G. Let $X = (x)$ be an infinite cyclic subgroup in G.

Show that there exists a positive integer m such that, for every positive π-integer n, the group G contains a characteristic subgroup W of finite index such that $X \cap W = X^{mn}$.

By hypothesis, G has a soluble minimax subgroup H of finite index. Since H contains a power subgroup of G and all power groups in G are characteristic subgroups of finite index, we may assume that H is a characteristic soluble minimax subgroup of finite index in G.

Let $\tau(H)$ be the periodic radical of H; i.e., the greatest normal periodic subgroup. Then $\tau(H)$ is a soluble group with the minimality condition; i.e., a finite extension of a direct product of finitely many (or zero) quasicyclic groups. On the other hand, G is residually finite and, therefore, it (and so $\tau(H)$) contains no quasicyclic subgroups. Consequently, $\tau(H)$ is a finite subgroup. This and the residual finiteness of G imply that G contains a normal subgroup N of finite index with $N \cap \tau(H) = 1$; moreover, we may assume additionally that N is characteristic in G.

The properties of H and N imply that $S = H \cap N$ is a characteristic soluble minimax subgroup of finite index in G and $S \cap \tau(H) = 1$. This and the fact that $\tau(S) \subseteq \tau(H)$ imply that $\tau(S) = 1$.

By Maltsev's Theorem [7, Subsection 5.2.1], the quotient group of every soluble group of finite rank modulo its periodic radical admits a characteristic series whose all infinite factors are torsion-free abelian groups of finite rank. This and the fact that $\tau(S) = 1$ imply that such a series also exists in S. Thus, we obtain a characteristic series of the group G:

$$G \geq S = S_1 \geq S_2 \geq \cdots \geq S_r = 1,$$

in which all infinite factors are torsion-free abelian groups of finite rank.

Obviously, there exist adjacent terms of this series $A = S_{i-1}$ and $B = S_i$ such that $X \cap A = X^m \neq 1$ and $X \cap B = 1$. Then $x^m B$ is an element of infinite order of A/B. Therefore, A/B is an infinite group; hence, A/B is a torsion-free abelian group of finite rank.

For applying Lemma 1 to A/B and π, we must check that A/B contains subgroups isomorphic to Q_p for no $p \in \pi$. If we assume that for some $p \in \pi$ the subgroup Q_p lies in A/B then the spectrum of Q_p (consisting of the only number p) is contained in the spectrum of A/B which is in turn included in the spectrum of G contradicting the definition of π.

Let n be a positive π-integer. Show that G has a characteristic subgroup W of finite index such that $X \cap W = X^{mn}$.

Applying Lemma 1 to the infinite cyclic subgroup $(x^m B)$ of A/B, we conclude that there exists a positive π-integer k such that $x^m B$ has order n modulo $(A/B)^k = A^k B/B$. Therefore, the element x^m of A has order n modulo $A^k B$; i.e., $X^m \cap A^k B = X^{mn}$. Then

$$X \cap A^k B = X \cap A \cap A^k B = X^m \cap A^k B = X^{mn};$$

i.e., the order of the element $xA^k B$ of $G/A^k B$ is equal to mn.

Show that $G/A^k B$ is residually finite. The general criterion for the residual finiteness of an almost soluble group of finite rank was obtained by Robinson and says that this group is reduced; i.e., contains no quasicyclic subgroups and subgroups isomorphic to the additive group Q of rationals [7, Subsection 5.3.2]. Since $G/A^k B$ inherits the almost solubility and minimaxness from G; it contains no subgroups isomorphic to Q. The absence of quasicyclic subgroups in $G/A^k B$ follows from the fact that all infinite factors in the series $G \geq S = S_1 \geq S_2 \geq \cdots \geq S_{i-1} = A \geq A^k B$ are torsion-free abelian groups.

Thus, $xA^k B$ is an element of finite order mn in the residually finite group $G/A^k B$. Therefore, $G/A^k B$ contains the power subgroup $(G/A^k B)^l = G^l A^k B/A^k B$ modulo which the order of the element $xA^k B$ is equal to mn. Then the order of x modulo $W = G^l A^k B$ equals to mn; i.e., $X \cap W = X^{mn}$. Moreover, W is a characteristic subgroup of finite index in G.

Proof of Theorem 2. As was observed, assertions 2 and 3 of Theorem 2 are consequences of Theorem 3. Prove assertion 1 of Theorem 2; i.e., show that the free product $G = A * B$ of torsion-free π-potent groups A and B is a π-potent group.

Denote the cartesian subgroup of G by D. The quotient group G/D is isomorphic to the direct product of the π-potent groups A and B; therefore, G/D is a π-potent group.

Show that every nonidentity element x in G is π-potent.

This is obvious when $x \notin D$. In this case (due to the absence of torsion in A and B), xD has infinite order in $G/D \cong A \times B$ and is a π-potent element of G/D. Then x is a π-potent element of G as a homomorphic preimage of a π-potent element of infinite order.

Consider the case when $x \in D$. In this case x has an irreducible representation $x = x_1x_2\dots x_r$ of length $r > 1$. We may assume without loss of generality that $x_1 \in A$, $x_2 \in B$, $x_3 \in A, \dots$. Owing to the residual finiteness of A and B, there exist normal subgroups M and N of finite index in A and B such that $x_1 \notin M$, $x_2 \notin N$, $x_3 \notin M, \dots$. Consider the free product $G_{MN} = A/M * B/N$ of the finite groups A/M and B/N and the homomorphism $\rho_{MN} : G \to G_{MN}$ extending the natural homomorphisms $A \to A/M$ and $B \to B/N$. Then $x\rho_{MN}$ has in G_{MN} the irreducible representation $x\rho_{MN} = x_1M \cdot x_2N \cdot x_3M \dots$ of length $r > 1$; therefore, $x\rho_{MN} \neq 1$. This and the fact that x belongs to the cartesian subgroup D of G imply that $x\rho_{MN}$ is a nonidentity element of the cartesian subgroup F of G_{MN}; moreover (as any cartesian subgroup) F is a normal free subgroup in G_{MN} and its index in G_{MN} is finite (since $G_{MN}/F \cong A/M \times B/N$ is a finite group). Therefore, by Lemma 2(1), $x\rho_{MN}$ is a potent element (of infinite order) in G_{MN}. It follows that x is a potent (and, in particular, π-potent) in G.

Proof of Theorem 3. Let $G = (A * B; H = K)$ be the free product of weakly π-potent groups A and B with finite amalgamated subgroups H and K. Prove that G is a weakly π-potent group. Since the free product of two residually finite groups with finite amalgamation is residually finite (see [11]), the group G is residually finite. Therefore, for proving the weak π-potency of G, we must check that every element x of infinite order in G is weakly π-potent.

Let us first consider the case when $x \in A$. Let $X = (x)$. Since A is a weakly π-potent group, there exists a positive integer m such that, for every positive π-integer n, the group A contains a normal subgroup P_n of finite index such that $X \cap P_n = X^{mn}$. In particular, $X \cap P_1 = X^m$.

Since H is a finite subgroup in the residually finite group A; therefore, A contains a normal subgroup T of finite index such that $H \cap T = 1$.

Put $S = P_1 \cap T$. Then S is a normal subgroup of finite index in A, while $H \cap S = 1$ and

$$X \cap S = X \cap P_1 \cap T = X^m \cap T = X^{ml}$$

for some positive integer l.

The proof of the weak π-potency of x in G will be as follows: we will show that, for every positive π-integer k, there exists a homomorphism of G onto a finite group under which the image of x has order mlk.

Write down l as $l = l_1l_2$, where l_1 is a π-integer, l_2 is a π'-integer, and π' is the complement to π in the set of all primes. For the π-integer $n = l_1k$, the equality $X \cap P_n = X^{mn}$ takes the form $X \cap P_{l_1k} = X^{ml_1k}$. Let $U = P_{l_1k} \cap S$. Then U is a normal subgroup of finite index in A, $H \cap U = 1$, $X \cap U = X \cap P_{l_1k} \cap X \cap S = X^{ml_1k} \cap X^{ml} = X^d$, where $d = \mathrm{LCM}(ml_1k, ml_1l_2) = ml_1kl_2 = mlk$ (since l_2 and k are coprime). Thus, $X \cap U = X^{mlk}$; i.e., the order of the element x modulo U is equal to mlk.

Denote by V a normal subgroup of finite index in B which trivially intersects K. The condition $H \cap U = 1 = K \cap V$ makes it possible to consider the free product with amalgamation $G_{UV} = (A/U * B/V; HU/U = KV/V)$ of the finite groups A/U and B/V and also the homomorphism $\rho_{UV} : G \to G_{UV}$ extending the natural homomorphisms $A \to A/U$ and $B \to B/V$. Since the order of the element $x\rho_{UV} = xU$ of G_{UV} is equal to mlk and G_{UV} is residually finite (see [11]), there exists a homomorphism ρ of G_{UV} onto a finite group preserving the order of $x\rho_{UV}$; i.e., such that $|x\rho_{UV}\rho| = |x\rho_{UV}| = mlk$. This finishes the proof of the weak π-potency of x for the case of $x \in A$. The case of $x \in B$ is treated similarly.

Consider the general case when x is an arbitrary element of infinite order in G. Since the orders of conjugate elements of a group are equal and each element in G is conjugate to some cyclically irreducible element of the group, we must prove weak π-potency for a cyclically irreducible element x of infinite order in G. Consider the irreducible representation $x = x_1x_2\dots x_r$. If $r = 1$ then $x \in A$ or $x \in B$; and so, by the particular case studied above, the element x is weakly π-potent in G.

Let $r > 1$. We may assume without loss of generality that $x_1 \in A \setminus H$, $x_2 \in B \setminus K$, $x_3 \in A \setminus H, \dots$. Since H and K are finite subgroups in the residually finite groups A and B, H and K are residually separate in A and B. Therefore, there exist normal subgroups M and N of finite index in A and B such that $H \cap M = 1 = K \cap N$ and $x_1 \notin HM$, $x_2 \notin KN$, $x_3 \notin HM, \dots$. The condition $H \cap M = 1 = K \cap N$ makes it possible to consider the generalized free product $G_{MN} = (A/M * B/N; HM/M = KM/N)$ of the finite groups A/M and B/N and also the homomorphism $\rho_{_{MN}} : G \to G_{_{MN}}$ extending the natural homomorphisms $A \to A/M$ and $B \to B/N$. Since the representation $x = x_1x_2 \dots x_r$ is cyclically irreducible, by the choice of M and N, the representation $x\rho_{_{MN}} = x_1M \cdot x_2N \cdot x_3M \dots$ is also cyclically irreducible, and its length r is greater than 1. Therefore, $x\rho_{_{MN}}$ has infinite order in the group $G_{_{MN}}$ which is weakly potent by Lemma 2(3). Hence, $x\rho_{_{MN}}$ is a weakly potent element of $G_{_{MN}}$. It follows that x is a weakly potent (and, in particular, weakly π-potent) element of G.

Proof of Theorem 4. Let $G = (A * B; h = k)$ be the free product of weakly π-potent groups A and B with infinite amalgamated cyclic subgroups $H = (h)$ and $K = (k)$ residually separable in A and B respectively. Suppose that elements h and k are weakly potent in A and B. Since G is residually finite [9, Lemma 5], for proving the weak π-potency of G, we must check that any element x of infinite order in G is weakly π-potent.

Since h and k are weakly potent in A and B, there exists a positive integer m such that for every positive integer n the groups A and B contain normal subgroups S_n and T_n of finite index such that $H \cap S_n = H^{mn}$ and $K \cap T_n = K^{mn}$. In particular, $H \cap S_1 = H^m$.

Let us first consider the particular case when $x \in A$. Let $X = (x)$. Since A is a weakly π-potent group, there exists a positive integer μ such that, for every positive π-integer ν, the group A has a normal subgroup P_ν of finite index such that $X \cap P_\nu = X^{\mu\nu}$. In particular, $X \cap P_1 = X^\mu$.

Let $S = S_1 \cap P_1$. Then S is a normal subgroup of finite index in A; moreover, $H \cap S = H \cap S_1 \cap P_1 = H^m \cap P_1 = H^{ml}$ and $X \cap S = X \cap P_1 \cap S_1 = X^\mu \cap S_1 = X^{\mu\lambda}$ for some positive l and λ.

The proof of the weak π-potency of the element x in G will be as follows: For each positive π-integer κ, we will present a homomorphism of G onto a finite group under which the image of x has order $\mu\lambda\kappa$.

Write down λ as $\lambda = \lambda_1\lambda_2$, where λ_1 is a π-integer, λ_2 is a π'-integer. For the π-integer $\nu = \lambda_1\kappa$, the equality $X \cap P_\nu = X^{\mu\nu}$ takes the form $X \cap P_{\lambda_1\kappa} = X^{\mu\lambda_1\kappa}$.

Let $U = P_{\lambda_1\kappa} \cap S$. Then U is a normal subgroup of finite index in A, $H \cap U = H \cap S \cap P_{\lambda_1\kappa} = H^{ml} \cap P_{\lambda_1\kappa} = H^{mll_1}$ for some positive integer l_1 and

$$X \cap U = X \cap P_{\lambda_1\kappa} \cap X \cap S = X^{\mu\lambda_1\kappa} \cap X^{\mu\lambda} = X^\delta,$$

where $\delta = \mathrm{LCM}(\mu\lambda_1\kappa, \mu\lambda_1\lambda_2) = \mu\lambda_1\kappa\lambda_2 = \mu\lambda\kappa$ (since λ_2 and κ are coprime). Thus, $X \cap U = X^{\mu\lambda\kappa}$ and $H \cap U = H^{mll_1}$; i.e., $|xU| = \mu\lambda\kappa$ and $|hU| = mll_1$.

For $n = ll_1$, the equality $K \cap T_n = K^{mn}$ takes the form $K \cap T_{ll_1} = K^{mll_1}$. The subgroup $V = T_{ll_1}$ is a normal subgroup of finite index in B and $|kV| = mll_1 = |hU|$.

The last equality makes it possible to consider the free product with cyclic amalgamation $G_{_{UV}} = (A/U * B/V; hU = kV)$ of the finite groups A/U and B/V and also the homomorphism $\rho_{_{UV}} : G \to G_{_{UV}}$ extending the natural homomorphisms $A \to A/U$ and $B \to B/V$. Since $|x\rho_{_{UV}}| = |xU| = \mu\lambda\kappa$ and the group $G_{_{UV}}$ is residually finite (see [11]), there exists a homomorphism ρ of $G_{_{UV}}$ onto a finite group such that $|x\rho_{_{UV}}\rho| = |x\rho_{_{UV}}| = \mu\lambda\kappa$. This finishes the proof of the weak π-potency of x for the case of $x \in A$. The case of $x \in B$ is treated similarly.

Consider the general case when x is an arbitrary element of infinite order in G. As in the proof of Theorem 3, we may assume that x is cyclically irreducible. Consider the irreducible representation $x = x_1x_2 \dots x_r$. If $r = 1$ then $x \in A$ or $x \in B$, and then, by the particular case considered above, the element x is weakly π-potent in G.

Suppose now that $r > 1$. We may assume that $x_1 \in A \setminus H$, $x_2 \in B \setminus K$, $x_3 \in A \setminus H, \dots$. Since by hypothesis H and K are residually finite in A and B, there exist normal subgroups Q and R of finite index in A and B such that $x_1 \notin HQ$, $x_2 \notin KR$, $x_3 \notin HQ, \dots$.

Obviously, there exist positive integers i and j such that $H \cap Q = H^i$ and $K \cap R = K^j$. For $n = ij$, the equalities $H \cap S_n = H^{mn}$ and $K \cap T_n = K^{mn}$ look as $H \cap S_{ij} = H^{mij}$ and $K \cap T_{ij} = K^{mij}$. On the form, as was observed above, $H \cap Q = H^i$ and $K \cap R = K^j$. Therefore, $M = Q \cap S_{ij}$ and $N = R \cap T_{ij}$ are such that $H \cap M = H^{mij}$ and $K \cap N = K^{mij}$; i.e., $|hM| = mij = |kN|$.

The last equality enables us to consider the generalized free product $G_{MN} = (A/M * B/N; hM = kN)$ of the finite groups A/M and B/N and also the homomorphism $\rho_{MN} : G \to G_{MN}$ extending the natural homomorphisms $A \to A/M$ and $B \to B/N$.

Since $M \subseteq Q$, $N \subseteq R$, $x_1 \notin HQ$, $x_2 \notin KR$, $x_3 \notin HQ, \dots$; therefore, $x_1 \notin HM$, $x_2 \notin KN$, $x_3 \notin HM, \dots$. This means that the expression $x\rho_{MN} = x_1M \cdot x_2N \cdot x_3M \dots$ is irreducible (and even cyclically irreducible) and its length r is greater than 1. Therefore, $x\rho_{MN}$ has infinite order in the group G_{MN} which is weakly potent by Lemma 2(3). Consequently, $x\rho_{MN}$ is a weakly potent element of G_{MN}. It follows that x is a weakly potent (and, in particular, weakly π-potent) element of G.

Proof of Theorem 5. Let A and B be residually finite almost soluble minimax groups and let π be the set of all primes not belonging to the union of the spectra of A and B. By Corollary 1.1, the groups A and B are weakly π-potent.

Let $G = (A * B; H)$ be the free product of A and B with cyclic amalgamated subgroup $H = (h)$; moreover, $H \neq A$ and $H \neq B$. Theorem 5 proved here states the equivalence of the following conditions:

1. G is residually finite.
2. H is residually separable in each of the groups A and B.
3. G is weakly π-potent.

The equivalence of conditions 1 and 2 is proved in [9]. Here we will prove that these conditions are equivalent to condition 3. Observe first of all that if H is finite then all three conditions are fulfilled (the weak π-potency of the group G is guaranteed by Theorem 3, and the residual finiteness of G is ensured by Baumslag's Theorem (see [11]).

We may now assume that H is an infinite cyclic group. It is proved in [9, Lemma 3] that if, in a residually finite almost soluble minimax group A, an infinite cyclic subgroup H is residually separable then its generating element h is weakly potent in A. Therefore, if condition 2 is fulfilled then h is weakly potent in A and B, and then by Theorem 4 the group G inherits weak π-potency from A and B. Thus, we have the implication $2 \Rightarrow 3$.

For finishing the proof, it remains to notice that the implication $3 \Rightarrow 1$ is obvious.

Proof of Theorem 6. Let A and B be residually finite almost soluble minimax groups and let π be the set of all primes not belonging to the union of the spectra of A and B. Let $G = (A * B; H)$ be the free product of A and B with normal amalgamated subgroup H; moreover, $H \neq A$ and $H \neq B$.

Theorem 6 asserts the equivalence of the following conditions:

1. G is residually finite.
2. H is residually finite in each of the groups A and B; i.e., the quotient groups A/H and B/H are residually finite.
3. G is weakly π-potent.

The equivalence of conditions 1 and 2 is proved in [10]. The implication $3 \Rightarrow 1$ is obvious, and it remains to check that $2 \Rightarrow 3$.

Suppose the fulfillment of condition 2; i.e., that the quotient groups A/H and B/H are residually finite. We have the following collection of residually finite almost soluble minimax groups: A, B, A/H, and B/H, and the spectrum of each of these groups is disjoint from the set π. Therefore, by Corollary 1.1 A, B, A/H, and B/H are weakly π-potent. Hence, by Theorem 2 the free product $A/H * B/H \cong G/H$ is a weakly π-potent group.

Since by condition 1 G is residually finite, for proving the weak π-potency of G, we must check the weak π-potency for an arbitrary element x of infinite order in G.

If the order of xH is infinite then, by the weak π-potency of G/H, the element xH (and hence x) is weakly π-potent.

Suppose that xH has finite order l; i.e., $X \cap H = X^l$, where $X = (x)$. Then X^l is an infinite cyclic subgroup in the residually finite almost soluble minimax group H and the spectrum of H is contained in the spectra of A and B and hence is disjoint from π.

Therefore, by Theorem 1 there exists a positive integer m such that, for each positive π-integer n, the group H has a characteristic subgroup W_n of finite index such that $X^l \cap W_n = X^{lmn}$. Then

$$X \cap W_n = X \cap H \cap W_n = X^l \cap W_n = X^{lmn};$$

i.e., $|xW_n| = lmn$. Note that W_n is normal in G (since W_n is characteristic in H and H is normal in G). Let $\varepsilon_n : G \to G/W_n$ be the natural homomorphism. Then $|x\varepsilon_n| = |xW_n| = lmn$.

The groups A/W_n and B/W_n are extensions of the finite group H/W_n by the residually finite groups A/H and B/H. Therefore, A/W_n and B/W_n are residually finite, as it is easy to verify by using Robinson's result [7, Subsection 5.3.2] which states that, for a soluble (almost soluble) group of finite rank, residual finiteness is equivalent to reducedness, i.e., to the absence of quasicyclic subgroups and subgroups isomorphic to the additive group of rationals.

The quotient group $G/W_n = (A/W_n * B/W_n; H/W_n)$ is the free product of residually finite groups A/W_n and B/W_n with finite amalgamated subgroup H/W_n. Therefore, G/W_n is residually finite (see [11]). This and the fact that $x\varepsilon_n$ is an element of order lmn in G/W_n imply that there exists a homomorphism φ_n of G/W_n onto a finite group such that the order of $x\varepsilon_n\varphi_n$ is equal to lmn. This finishes the proof of the weak π-potency of x.

Proof of Theorem 7. Let $G = (A * B; H)$ be the free product of residually finite almost soluble minimax groups A and B with amalgamated subgroup H, where H is a proper subgroup of finite index in each of the groups A and B.

Theorem 7 states the equivalence of the following conditions:

1. G is residually finite.
2. H has a subgroup L of finite index normal in G.
3. G is weakly π-potent, where π is the set of all primes not lying in the spectrum of H.

The equivalence of conditions 1 and 2 is proved in [12]. The implication $3 \Rightarrow 1$ is obvious, and we must only check that $2 \Rightarrow 3$.

Suppose the fulfillment of condition 2; i.e., that H has no subgroup L of finite index normal in G. Since by condition 1 G is residually finite, for checking condition 3, it remains to prove weak π-potency for any element x of infinite order in G.

Let us first consider the case when xL is an element of infinite order in $G/L = (A/L * B/L; H/L)$ (which is the free product of the finite groups A/L and B/L with amalgamation H/L and hence is weakly potent by Lemma 2). In this case, the element xL (and hence x) is weakly potent and, in particular, weakly π-potent.

Suppose that xL has finite order l; i.e., $X \cap L = X^l$, where $X = (x)$. Then X^l is an infinite cyclic subgroup in the residually finite almost soluble minimax group L. Note also that the spectrum of L coincides with the spectrum of H and hence is disjoint from π.

Consequently, by Theorem 1 there exists a positive integer m such that for each positive π-integer n the group L contains a characteristic subgroup W_n of finite index such that $X^l \cap W_n = X^{lmn}$. Then

$$X \cap W_n = X \cap L \cap W_n = X^l \cap W_n = X^{lmn};$$

i.e., $|xW_n| = lmn$. Note that W_n is normal in G (since W_n is characteristic in L and L is normal in G). Let $\varepsilon_n : G \to G/W_n$ be the natural homomorphism. Then

$$|x\varepsilon_n| = |xW_n| = lmn.$$

The quotient group $G/W_n = (A/W_n * B/W_n; H/W_n)$ is the free product of the finite groups A/W_n and B/W_n with amalgamated subgroup H/W_n. Therefore, G/W_n is residually finite (see [11]). This and the fact that $|x\varepsilon_n| = lmn$ imply that there exists a homomorphism φ_n of G/W_n onto a finite group such that $|x\varepsilon_n\varphi_n| = lmn$. This finishes the proof of the weak π-potency of x.

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