ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 5, pp. 778–794. © *Pleiades Publishing, Ltd., 2020.*
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NONSOLVABLE FINITE GROUPS WHOSE ALL NONSOLVABLE SUPERLOCALS ARE HALL SUBGROUPS

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Abstract: We describe the nonabelian simple finite groups whose every nonsolvable local maximal subgroup is a Hall subgroup, and the nonsolvable finite groups whose all nonsolvable superlocals are Hall subgroups.

DOI: 10.1134/S003744662005002X

Keywords: finite group, nonsolvable group, local maximal subgroup, superlocal, Hall subgroup

1. Introduction

Only finite groups are considered. Thompson described in [1] the structure of N-groups; i. e., the nonsolvable groups whose every local subgroup is solvable. Monakhov studied in [2] the structure of π-solvable groups with maximal Hall subgroups whose indices in the group are π-numbers. Tikhonenko and Tyutyanov described in [3] all nonabelian simple groups modulo the classification of finite simple groups, and Maslova described in [4] all nonabelian simple composition factors of every nonsolvable group with maximal Hall subgroups. Maslova and Revin obtained in [5] a full description of the structure of finite groups whose every maximal subgroup is a Hall subgroup. Modulo the classification of finite simple groups, the author described in [6] the structure of nonabelian simple groups G whose every maximal subgroup is either a solvable group or a Hall subgroup of G, as well as the structure of nonabelian composition factors of every nonsolvable group whose every nonsolvable subgroup is a Hall subgroup. There are some other articles in this direction; see [7–10] for instance.

A subgroup H of a group G is called a *local* (*p-local*) *subgroup* of G whenever G includes a nonidentity primary subgroup (p-subgroup) P such that $H = N_G(P)$. A subgroup H of a group G is called a *local* (*p-local*) *maximal subgroup* of G whenever H is both local (p-local) and maximal in G. A subgroup H of a group G is called a *nonsolvable maximal subgroup* of G whenever H is a nonsolvable subgroup of G and H is maximal in G. A subgroup H of a group G is called a *nonsolvable local* (*p-local*) *maximal subgroup* of G whenever H is local (p-local) in G and H is a nonsolvable maximal subgroup of G. A subgroup H of a group G is called a maximal local (p-local) subgroup of G whenever H is inclusion-maximal in the set of all local (p-local) subgroups of G. In every nonabelian simple group G each local (p-local) maximal subgroup is a maximal local (p-local) subgroup, but the converse is false. Following [11, 1.5], denote by $\mathrm{Chev}(p)$ the set of all groups of Lie type over finite fields of characteristic p. Each maximal p-local subgroup in $G \in \mathrm{Chev}(p)$ is a parabolic subgroup of G by [11, Theorem 1.41].

Aschbacher introduced [12] the concept of superlocal to generalize parabolic subgroups: A *p-superlocal* in a group G is a p-local subgroup A such that $A = N_G(O_p(A))$, and A is called a *superlocal* in G whenever A is a p-superlocal in G for some $p \in \pi(A)$. As established in [12], each p-local subgroup H of G lies in some p-superlocal A of G such that $O_p(H) \leq O_p(A)$.

Following [13], define the binary relation $\leq_p$ on the set of all subgroups of some group G as follows: Given subgroups A and B of G, write $A \leq_p B$ if and only if $A \leq B$ and $O_p(A) \leq O_p(B)$. Then $\leq_p$ is a partial order. The maximal elements with respect to this order are called [13] *p-maximal subgroups* of G. As established in [13], the concept of p-superlocal in some group G is equivalent to the concept of p-maximal subgroup of G. Thus, we will freely pass between these concepts.

Original article submitted October 7, 2019; revised July 25, 2020; accepted August 10, 2020.

This article continues [6]. Our goal now is to study the structure of nonabelian simple groups G whose every nonsolvable local maximal subgroup is a Hall subgroup and to establish the normal structure of nonsolvable groups G whose every nonsolvable p-superlocal is a Hall subgroup for all $p \in \pi(G)$.

Following [6], we use the notation: $\mathfrak{J}_h$ is the class of nonabelian simple groups whose every maximal subgroup is either a solvable group or a Hall subgroup; $\mathfrak{J}_{lmh}$ is the class of nonabelian simple groups whose every nonsolvable local maximal subgroup is a Hall subgroup; $\mathfrak{J}_{slh}$ is the class of nonabelian simple groups whose every nonsolvable superlocal is a Hall subgroup; $\mathfrak{T}_h$ is the class of groups whose every nonsolvable maximal subgroup is a Hall subgroup; $\mathfrak{T}_{lmh}$ is the class of groups whose every nonsolvable local maximal subgroup is a Hall subgroup; and $\mathfrak{T}_{slh}$ is the class of groups whose every nonsolvable superlocal is a Hall subgroup. The definitions of these classes of groups imply directly that $\mathfrak{J}_{slh} \subseteq \mathfrak{J}_{lmh}$. However, the inductive arguments for $\mathfrak{T}_{slh}$-groups are simpler than for $\mathfrak{T}_{lmh}$-groups.

We obtain the next results:

Theorem 1. *A group G is a $\mathfrak{J}_{lmh}$-group if and only if G is isomorphic to one of the following groups:*
(1) $L_2(q)$ *with* $q > 3$*;*
(2) $Sz(q)$ *with* $q = 2^{2n+1}$ *and* $n \geq 1$*;*
(3) $L_3(q)$ *with* $q = p^s \geq 3$ *and* $s \geq 1$*, where* $q \not\equiv 1 \pmod 3$*;*
(4) $L_5(2)$*;* $L_5(4)$*;* $U_3(3)$*;* ${}^2F_4(2)'$*;* A_7*;* M_{11}*;* M_{23}*; and* J_1.

Theorem 2. *Given a nonsolvable $\mathfrak{T}_{slh}$-group G, denote by $S(G)$ the solvable radical of G and put $\overline{G} := G/S(G)$. Then*

(1) $\mathrm{Inn}(A) \leq \overline{G} \leq \mathrm{Aut}(A)$*, where the group A is isomorphic to one of the $\mathfrak{J}_{lmh}$-groups* (1)–(4) *in Theorem 1;*

(2) $S(G)$ *is a dispersive group.*

2. Definitions, Notations, and Auxiliary Results

For the notations and definitions not explicit in this article; see [14–20]. Given a group G and $p \in \pi(G)$, denote by G_p some Sylow p-subgroup of G; by $S(G)$, the solvable radical of G; while by $H \leq G$, $H \triangleleft G$, and $H \cdot\!\triangleleft G$ the properties that H is a subgroup, a normal subgroup, and a minimal normal subgroup of G.

Given some set $\mathfrak{X}$ of groups, if $\mathfrak{X}$ contains all groups isomorphic to A for every group $A \in \mathfrak{X}$ then $\mathfrak{X}$ is called a *class of groups*. A class $\mathfrak{X}$ closed under homomorphic images is called a *homomorph*. A class $\mathfrak{X}$ is *closed under normal subgroups* or, briefly, S_n*-closed* whenever $A \in \mathfrak{X}$ and $H \triangleleft A$ imply that $H \in \mathfrak{X}$. Every group belonging to $\mathfrak{X}$ is called an $\mathfrak{X}$*-group*. Denote by $\mathfrak{K}(G)$ the class of simple groups isomorphic to the composition factors of G, and by $\mathfrak{K}(\mathfrak{X})$ the union of the classes $\mathfrak{K}(G)$ over all $G \in \mathfrak{X}$ [19, 20].

To prove the main results of this article, we apply the classification theorem for nonabelian simple groups and refer to their list as in [11, Table 2.4]. Proving Theorem 2, we use the main result of [21], while proving Theorem 1, we depend much on the results about the minimal permutation representations of simple classical groups [22–27]. A *minimal permutation representation of a group G* is a faithful permutation representation of G of the least degree n. A group G is called *dispersive* if G has a normal series whose every quotient is isomorphic to some Sylow subgroup of G; see [20, 4.7].

For $G \in \mathrm{Chev}(p)$ the subgroup $B = N_G(G_p)$ is called the *Borel subgroup* of G. A proper subgroup P of G which includes B is called a *parabolic subgroup* of G. There exists a bijective correspondence between the parabolic subgroups P of G and the subsets S of the system $\Pi = \{p_1, p_2, \dots, p_l\}$ of simple roots [11, 2.1]; henceforth we will enumerate the vertices of the Dynkin diagram in accordance with [11, Fig. 2.1]. Given $i = 1, 2, \dots, l$, put $S_i := \Pi \backslash \{p_i\}$ and let the parabolic subgroup P_i of G correspond to S_i. Then P_i is a parabolic maximal subgroup of G for all $i = 1, 2, \dots, l$ [11, 2.1].

Recall the main result of [1]:

Lemma 2.1. *Each nonsolvable N-group is isomorphic to a group G satisfying* $\mathrm{Inn}(A) \leq G \leq \mathrm{Aut}(A)$*, where A is one of the following N-groups:*

(1) $L_2(q)$ *with* $q > 3$*;*
(2) $Sz(q)$ *with* $q = 2^{2n+1}$ *and* $n \geq 1$*;*
(3) $L_3(3)$, M_{11}, A_7, $U_3(3)$, *and* ${}^2F_4(2)'$.

REMARK 2.1. The statement of Lemma 2.1 includes the group ${}^2F_4(2)'$, omitted in the first part of the fundamental article [1]; it is an N-group; see [17, p. 74] for instance. Lemma 2.1 implies that a nonabelian simple group A is an N-group if and only if A is isomorphic to one of the groups in claims (1)–(3) of Lemma 2.1.

Proposition 2.1 [12]. *If H is a p-local subgroup of a group G then G includes a p-superlocal A satisfying $H \leq A$ and $O_p(H) \leq O_p(A)$.*

Proposition 2.2. $\mathfrak{J}_{slh} \subseteq \mathfrak{J}_{lmh}$.

PROOF. Take $G \in \mathfrak{J}_{slh}$. Then for every $p \in \pi(G)$ each nonsolvable p-superlocal is a Hall subgroup of G. Take a nonsolvable p-local maximal subgroup H of G. Proposition 2.1 shows that H lies in some p-superlocal A of G. Since G is a simple group, from $A < G$ and $H \leq A$ it follows that $H = A$ is a Hall subgroup of G. Thus, $G \in \mathfrak{J}_{lmh}$, and so $\mathfrak{J}_{slh} \subseteq \mathfrak{J}_{lmh}$. $\square$

Proposition 2.3. *Every superlocal is solvable in a nonsolvable group G if and only if G is an N-group.*

PROOF. Straightforward from the definition of N-group and Proposition 2.1.

Lemma 2.2 [13, Proposition 1]. *If a p-superlocal N of a group G normalizes a p-subgroup Q then $Q \leq O_p(N)$. In particular, $O_p(G) \leq O_p(N)$.*

Lemma 2.3 [13, Proposition 3]. *Consider two superlocals N_1 and N_2 of some group G and the corresponding radicals $P_1 = O_p(N_1)$ and $P_2 = O_p(N_2)$. If $N_1 \leq N_2$ then $P_1 \geq P_2$ and $P_2 \lhd N_1$. Furthermore, if $N_1 \leq_p N_2$ then $N_1 = N_2$. In particular, each superlocal is a p-maximal subgroup.*

Lemma 2.4 [13, Proposition 4]. *Consider a normal subgroup H of some group G. The following hold:*

(1) *If P is a p-radical in G then $P \cap H$ is a p-radical in H.*

(2) *If P_H is a p-radical in H and $N_H = N_H(P_H)$ is the corresponding p-superlocal of H then $N = N_G(P_H)$ is a p-superlocal in G; moreover, $N \cap H = N_H$ and $P = O_p(N)$ is a p-radical in G satisfying $P \cap H = P_H$.*

Lemma 2.5. *The class $\mathfrak{T}_{slh}$ is an S_n-closed homomorph.*

PROOF. Given $G \in \mathfrak{T}_{slh}$, take $A \lhd G$ and a p-superlocal L in A for some $p \in \pi(A)$. Verify that $A \in \mathfrak{T}_{slh}$. The definition of p-superlocal yields $O_p(L) \neq 1$ and $L = N_A(O_p(L))$. By Lemma 2.4 L lies in a p-superlocal U of G; furthermore, $L = U \cap A$ and $O_p(L) = A \cap O_p(U)$. By assumption, either U is a solvable group or U is a Hall subgroup of G. If U is solvable then $L \leq U$ implies that L is a solvable subgroup of A. If U is a Hall subgroup of G then $L = U \cap A$ is a Hall subgroup of A. Therefore, the superlocal L in A is either solvable or Hall, and so $A \in \mathfrak{T}_{slh}$. Consequently, $\mathfrak{T}_{slh}$ is S_n-closed.

Given $G \in \mathfrak{T}_{slh}$, take $N \lhd G$ and verify that $G/N \in \mathfrak{T}_{slh}$. Take a nonsolvable p-superlocal K/N in G/N and put $B/N := O_p(K/N) \neq 1$. Then $K/N = N_{G/N}(B/N) = N_G(B)/N$, and so $K = N_G(B)$; furthermore, $B = B_pN$. Frattini's argument yields $K = BN_K(B_p) = NN_K(B_p)$.

Take $g \in N_G(B_p) := T$. Then $(B_p)^g = B_p$ and $B^g = (B_p)^g N^g = B_pN = B$. Consequently, $g \in N_G(B) = K$ and $T \leq K$. Thus, $T = N_G(B_p) = N_K(B_p)$. Since $K/N = NT/N \cong T/(N \cap T)$ is a nonsolvable group, T is a nonsolvable p-local subgroup of G. Then by Proposition 2.1 we see that T lies in a p-superlocal W of G; furthermore, $O_p(T) \leq O_p(W)$. By assumption, W is a Hall subgroup of G. Since $K = NT$; therefore, $K \leq NW$. This implies that $K/N \leq NW/N$. Since $O_p(K/N) = B/N = B_pN/N$ and $B_p \leq O_p(T) \leq O_p(W)$, it follows that $O_p(K/N) = B_pN/N \leq O_p(W)N/N$. Now the p-maximality of K/N in G/N implies that $K/N = NW/N$, and so $K = NW$. Since W is a Hall subgroup of G, infer that $K/N = NW/N$ is a Hall subgroup of G/N. Hence, $G/N \in \mathfrak{T}_{slh}$, and so $\mathfrak{T}_{slh}$ is a homomorph. The proof of Lemma 2.5 is complete. □

Lemma 2.6. *If each superlocal in a solvable group G is a Hall subgroup then G is a dispersive group.*

PROOF. For a counterexample G of minimal order, take a nonidentity p-group $M \triangleleft G$. Consider the quotient G/M and take a q-superlocal L/M in G/M. Then $B/M = O_q(L/M) \neq 1$ and $L/M = N_{G/M}(B/M) = N_G(B)/M$. If $q = p$ then $B/M = O_p(L)/M$ and $L/M = N_G(O_p(L))/M$. Therefore, $L = N_G(O_p(L))$, and so L is a p-superlocal in G. By assumption, L is a Hall subgroup of G. Then L/M is a Hall subgroup of G/M.

Assume that $q \neq p$. Then $B = [M]B_q$ and Frattini's argument yields $L = BN_L(B_q) = MN_L(B_q)$. Take $g \in N_G(B_q)$. Then $B^g = M^g(B_q)^g = MB_q = B$, and so $g \in N_G(B)$. Since $L/M = N_G(B)/M$, it follows that $g \in N_G(B) = L$, and so $g \in N_L(B_q)$. Consequently, $N_L(B_q) = N_G(B_q) := T$. Since T is a q-local subgroup of G, Proposition 2.1 shows that T lies in a q-superlocal U of G; moreover, $B_q \leq O_q(T) \leq O_q(U)$. Thus, $B \leq O_q(T)M \leq O_q(U)M$, and so

$$B/M = O_q(L/M) \leq O_q(T)M/M \leq O_q(U)M/M.$$

Moreover, $L = MT \leq MU$ and $L/M = MT/M \leq MU/M$. Since L/M is a q-superlocal in G/M, it follows that L/M is a q-maximal subgroup of G/M. The definition of q-maximal subgroup implies that $L/M = MU/M$. By assumption, U is a Hall subgroup of G. Therefore, L/M is a Hall subgroup of G/M.

By induction, G/M is dispersive. In case $M = G_p$, G would be dispersive as well. Consequently, G does not have normal Sylow subgroups and $M < G_p$. Take a normal Sylow r-subgroup C/M of G/M. Then C_r is a Sylow r-subgroup of G. Since $C \triangleleft G$, it follows that $G = MN_G(C_r)$. So, $N := N_G(C_r)$ is an r-superlocal in G, and by assumption N is a Hall subgroup of G. Since $p|(|N|, |G : N|)$; infer that N is not a Hall subgroup of G; a contradiction. □

REMARK 2.2. The converse to Lemma 2.6 is false. Consider $G := SL_2(3)$. Then G is dispersive, but the 3-superlocal $N_G(G_3)$ is not a Hall subgroup of G.

Lemma 2.7. *If G is a nonsolvable $\mathfrak{T}_{slh}$-group then the solvable radical $S(G)$ of G is a dispersive group.*

PROOF. Given a nonsolvable $\mathfrak{T}_{slh}$-group G, suppose that $R := S(G)$ is not a dispersive group and G is a group of minimal order with these properties. Take a minimal normal subgroup M of G included into R. Then M is an elementary abelian p-group for some $p \in \pi(R)$. Since by Lemma 2.5 $\mathfrak{T}_{slh}$ is a homomorph, $G/M \in \mathfrak{T}_{slh}$. Then $R/M := S(G/M)$, and by induction R/M is a dispersive group. Consequently, R/M is q-closed for some $q \in \pi(R/M)$. Suppose that $q = p$. Then $R_p \triangleleft R$, and so $R_p \triangleleft G$. By induction, R/R_p is dispersive; hence, so is R; a contradiction. Thus, $q \neq p$ and R does not have normal Sylow subgroups. Suppose that $M \leq \Phi(R)$. Then from [20, Theorem 3.24] it follows that R is a q-closed group; a contradiction.

Consequently, $M \cap \Phi(R) = 1$ and $M < R_p$. The normal subgroup M is complemented in R. Suppose that $R = [M]H$. Since $R/M \cong H$ is q-closed, $H = N_R(H_q)$, and furthermore H_q is a Sylow q-subgroup of R. Frattini's argument yields $G = RN$, where $N := N_G(H_q)$. Since H_q is a q-radical in R, while $H = N_R(H_q)$ is the corresponding q-superlocal in R, Lemma 2.4 shows that $N = N_G(H_q)$ is a q-superlocal in G satisfying $N \cap R = H$, while $Q = O_q(N)$ is a q-radical in G satisfying $Q \cap H = H_q$. Since $G/R = RN/R \cong N/(N \cap R)$; therefore, N is nonsolvable, and by assumption N is a Hall subgroup of G. Then $H = N \cap R$ is a Hall subgroup of R, which contradicts the property that p divides $(|H|, |R : H|)$. Thus, R is dispersive. □

Lemma 2.8 [21]. *A group G is a 2-nilpotent group if and only if the index of the normalizer of an arbitrary Sylow subgroup in G is odd.*

3. Proof of Theorem 1

Necessity: Consider a $\mathfrak{J}_{lmh}$-group G; i. e., a nonabelian simple group such that for every $p \in \pi(G)$ each p-local maximal subgroup of G is either a solvable group or a Hall subgroup of G. Verify that G is isomorphic to one of the groups (1)–(4) in Theorem 1. If G is an N-group then $G \in \mathfrak{J}_{lmh}$, and Lemma 2.1 shows that G is a group of one of the types (1)–(4) in Theorem 1. Assume now that G is not an N-group.

Applying the results of [17; 22–27] on the minimal permutation representations of simple groups of Lie type, we determine the cases in which the simple groups of Lie type, the alternating, and the sporadic groups include a p-local maximal subgroup which is nonsolvable and is not a Hall subgroup of G for some $p \in \pi(G)$.

1. Suppose that $G = PSL_{l+1}(q) = L_{l+1}(q) \cong A_l(q) \in \mathfrak{J}_{lmh}$ with $l \geq 1$ and $q = p^s$ for some prime p and $s \geq 1$. By Lemma 2.1, we may assume that $l > 1$.

Applying Theorem 1 of [23] on the minimal permutation representation of $L_{l+1}(q)$ for $l > 1$, we have separately to consider the case of $L_4(2)$ and the remaining groups with $l > 1$ of [23, Theorem 1].

Assume that $G := L_4(2)$. Then by [17, p. 22] G includes nonsolvable maximal subgroups of type $2^3 : L_3(2)$ of order 1344 and index 15, which are not Hall subgroups of G, and so G is not a $\mathfrak{J}_{lmh}$-group; a contradiction.

Suppose that $G = L_{l+1}(q)$ for $l > 1$ such that the pair $(l+1, q)$ is distinct from $(4, 2)$. By [17, p. xv], $L_3(2) \cong L_2(7)$. Lemma 2.1 shows that $L_3(2)$ and $L_3(3)$ are N-groups; thus, we may assume that $(l+1, q) \notin \{(3,2),(3,3)\}$. Theorem 1 of [23] implies that G includes a nonsolvable p-local maximal subgroup $P_1 \cong p^{sl} \cdot SL_l(q) \cdot t$, where $t = (q-1)/(q-1, l+1)$,

$$|P_1| = q^{l(l+1)/2} \frac{(q-1)}{d} \prod_{i=1}^{l-1} (q^{i+1} - 1),$$

and $n = |G : P_1| = \frac{q^{l+1}-1}{q-1}$.

Assume that $l = 2$ and $(l+1, q) \notin \{(3,2),(3,3)\}$. We have

$$|G| = |L_3(q)| = (1/(q-1,3)) q^3 (q^3 - 1)(q^2 - 1), \quad |P_1| = q^3 \frac{(q-1)}{d} (q^2 - 1),$$

where $d = (3, q-1)$ and $|G : P_1| = (q^3 - 1)/(q-1) = q^2 + q + 1$. Then P_1 is a nonsolvable p-local maximal subgroup of G. Suppose that $q \equiv 1 \,(\mathrm{mod}\, 3)$. Then $|G : P_1| \equiv 0 \,(\mathrm{mod}\, 3)$ and $q - 1 \equiv 0 \,(\mathrm{mod}\, 3)$. Consequently, $(q-1, 3) = 3$. Therefore, $|P_1|$ and $|G : P_1|$ are divisible by 3, and so P_1 is not a Hall subgroup of G; a contradiction. Thus, $q \not\equiv 1 \,(\mathrm{mod}\, 3)$ and G is a group of type (3) in Theorem 1.

Assume that $l \geq 3$ is odd. Then $l + 1 = 2k$ with $k > 1$, and $q^{2k} - 1$ is divisible by $q^2 - 1$. Hence, $(q+1) | (|G : P_1|, |P_1|)$. Furthermore, for $l \geq 3$ the subgroup P_1 is a nonsolvable p-local maximal subgroup of G. Consequently, $G \notin \mathfrak{J}_{lmh}$; a contradiction.

Assume that $l \geq 4$ is even. Proposition 1 of [24] shows that G includes the parabolic subgroup P_2 corresponding to $S_2 = \Pi \backslash \{p_2\}$. Furthermore,

$$|P_2| = q^{l(l+1)/2} \frac{(q-1)}{d} (q^2 - 1) \prod_{i=1}^{l-2} (q^{i+1} - 1)$$

and $|G : P_2| = \frac{(q^{l+1}-1)(q^l-1)}{(q-1)(q^2-1)}$, where $d = (l+1, q-1)$. Then [14, § 2] implies that P_2 is a nonsolvable p-local maximal subgroup of G; also see [16, Proposition 4.1.17]).

Assume that $l = 4$. Then $|G : P_2| = \frac{(q^5-1)(q^4-1)}{(q-1)(q^2-1)}$. If q is odd then $2 | (|G : P_2|, |P_2|)$. Hence, $G \notin \mathfrak{J}_{lmh}$; a contradiction. Suppose that q is even. If $q = 2$ then $G \cong L_5(2)$. By [17, p. 70], all maximal subgroups of G are Hall subgroups of G; moreover, each nonsolvable group among them is 2-local and G is a group of type (4) of Theorem 1. Suppose that $q = 2^s$ with $s > 1$ and $q \equiv 1 \,(\mathrm{mod}\, 5)$. Then

$|G:P_2| = (q^4+q^3+q^2+q+1)(q^2+1) \equiv 0 \,(\mathrm{mod}\, 5)$. Since $d = (5, q-1) = 5$ and $|P_2|$ is divisible by 5; therefore, $(|G:P_2|, |P_2|)$ is divisible by 5. Thus, P_2 is not a Hall subgroup of G; and, furthermore, P_2 is a nonsolvable 2-local maximal subgroup of G; a contradiction. Consequently, $q \not\equiv 1 \,(\mathrm{mod}\, 5)$. According to [18, Table 8.18], for $q \geq 5$ the group $SL_5(q)$ includes a maximal subgroup $A \cong (q-1)^4 : S_5$ which is not a Hall subgroup of $SL_5(q)$ for $q \geq 5$. Since for $q \not\equiv 1 \,(\mathrm{mod}\, 5)$ we have $d := |Z(SL_5(q))| = (5, q-1) = 1$, it follows that $SL_5(q) \cong L_5(q) = G$. This yields $q = 2^s < 5$ and $G \cong L_5(2^s)$ for $s = 1, 2$. So, G is a group of type (4) in Theorem 1.

Suppose that $l \geq 6$ is even. Then $l = 2m$ with $m \geq 3$, and $q^l - 1 = q^{2m} - 1 = (q^m-1)(q^m+1)$. Define $d_1 = (q^m - 1, q^2 - 1)$, then $q^m - 1 = d_1 t$, and finally $q^2 - 1 = d_1 t_1$. Put $s := \frac{q^{l+1}-1}{q-1}$. Since $m \geq 3$; infer that $t > 1$, and moreover, $(t, t_1) = 1$. Then

$$|G:P_2| = \frac{(q^{l+1}-1)(q^l-1)}{(q-1)(q^2-1)} = \frac{s(q^m+1)t}{t_1}.$$

Since $(t, t_1) = 1$, it follows that $s(q^m+1)$ is divisible by t_1. This implies that $|G:P_2|$ is divisible by t. Since $|P_2|$ is divisible by $q^k - 1$ for every $2 \leq k \leq l-1$ and $3 \leq m = l/2 \leq l-1$, we have $(q^m-1)||P_2|$ and so $t||P_2|$. Then $t|(|G:P_2|, |P_2|)$; furthermore, by [14, § 2], for $l \geq 6$ the subgroup P_2 is a nonsolvable p-local maximal subgroup of G; also see [16, Proposition 4.1.17]. Thus, $G \notin \mathfrak{J}_{lmh}$; a contradiction.

2. Consider $G = PSp_{2l}(q) = S_{2l}(q) \cong C_l(q) \in \mathfrak{J}_{lmh}$ with $l \geq 1$ and $q = p^s$ for some prime p and $s \geq 1$. Thus, G is a projective symplectic group and $|G| = (1/d)q^{l^2}(q^2-1)(q^4-1)\cdots(q^{2l}-1)$, where $d := (2, q-1)$. Then G is a simple group with the exception of $S_2(2) = L_2(2) \cong S_3$, $S_2(3) = L_2(3) \cong A_4$, and $S_4(2) \cong S_6$. Since $S_2(q) = L_2(q)$, we may assume that $l > 1$ and the pair $(2l, q)$ is distinct from $(4, 2)$. The minimal permutation representations of $S_{2l}(q)$, where $2l \geq 4$, with the point stabilizer H and degree $n = |G:H|$ are described in [23]. Applying [23, Theorem 2], we notice that H is a nonsolvable local maximal subgroup of G in all but last cases; furthermore, in the first case for $2l = 4$ and $q = 3$ we have $3|(n, |H|)$, and in all but last remaining cases we obtain $(q+1)|(n, |H|)$.

Assume that $2l \geq 6$ and $q = 2$. Since $d := (2, q-1) = 1$, it follows that $G = S_{2l}(2) \cong Sp_{2l}(2)$. For $l \in \{3, 4\}$, by [17, p. 46 and p. 123] the group G includes a subgroup H of type $2^5 : S_4(2)$ and $2^7 : S_6(2)$ respectively and, moreover, $3|(|H|, |G:H|)$. For $l \in \{5, 6\}$ by [18, p. 413 and p. 424], the group G includes a subgroup H of type $2^9 : S_8(2)$ and $2^{11} : S_{10}(2)$ respectively.

Assume that $l > 6$. Then by [16, Proposition 4.1.19] for $m = 1$ the group G includes a subgroup H of type $2^a.S_{2l-2}(2)$, where $a = (1/2) - (3/2) + 2l = 2l - 1$. Since $|H| = 2^{l^2}(2^2-1)(2^4-1)\cdots(2^{2(l-1)}-1)$ and $|G| = 2^{l^2}(2^2-1)(2^4-1)\cdots(2^{2(l-1)}-1)(2^{2l}-1)$, we find that $|G:H| = 2^{2l} - 1 \equiv 0 \,(\mathrm{mod}\, 3)$, and so $3|(|H|, |G:H|)$ for every $l \geq 3$. Thus, $G \notin \mathfrak{J}_{lmh}$ for $l > 1$; a contradiction.

3. Consider $G = PSU_m(q) = U_m(q) \cong {}^2A_{m-1}(q)$, where $m \geq 2$ and $q = p^s$ for some prime p and $s \geq 1$. Therefore, G is a special projective unitary group and

$$|G| = (1/d)q^{(m-1)m/2}(q^m - (-1)^m)(q^{m-1} - (-1)^{m-1})\cdots(q^2-1),$$

where $d = (m, q+1)$. Then G is a simple group with the exception of $U_2(2) = L_2(2) \cong S_3$, $U_2(3) = L_2(3) \cong A_4$, and $U_3(2) \cong 3^2 \cdot 2 \cdot 2^2$. Since $U_2(q) = L_2(q)$, we may assume that $m > 2$. Observe that $U_4(2) \cong S_4(3)$. Lemma 2.1 implies that $U_3(3)$ is an N-group, and so $U_3(3) \in \mathfrak{J}_{lmh}$. Thus, we may assume that $m \geq 3$ and $(m, q) \notin \{(3,2), (3,3), (4,2)\}$.

The minimal permutation representations of $U_m(q)$, where $m \geq 3$, with the point stabilizer H and degree $n = |G:H|$, are studied in [23]. Applying [23, Theorem 3], we consider the following cases:

3.1. Assume that $m = 3$ and $q = p^s \geq 4$. According to [18, Table 8.5], $S := SU_3(q)$ has a maximal subgroup $M := GU_2(q)$. Furthermore, $|S| = q^3(q^3+1)(q^2-1)$, $|M| = q(q+1)(q^2-1)$, and $|Z(M)| = q+1 \geq 5$. Put $Z := Z(S)$. Since M is maximal in S, infer that $Z \leq M$; furthermore, $|Z| = (q+1, 3) \in \{1; 3\}$. Then $|S/Z : M/Z| = |S:M| = q^2(q^2-q+1)$ and $(|S/Z : M/Z|, |M/Z|) > 1$. This implies that M/Z is maximal in S/Z; also, M/Z is neither a solvable group nor a Hall subgroup of the group $S/Z \cong U_3(q)$.

Since $Z(M)/Z \neq 1$; therefore, the group M/Z is a nonsolvable local maximal subgroup of S/Z, which contradicts the hypotheses of Theorem 1.

3.2. Assume that $m = 4$ and $q = p^s$ for some prime p. By [23, Theorem 3] $G = U_4(q)$ includes a nonsolvable local maximal subgroup $H \cong q^4 \cdot SL_2(q^2) : ((q+1)/(q+1,4))$ with $|G : H| = (q^3+1)(q+1)$. Since $|SL_2(q^2)| = q^2(q^4-1)$, we have $(q+1)|(|H|, |G : H|)$; and so $G = U_4(q) \notin \mathfrak{I}_{lmh}$; a contradiction.

3.3. Assume that $m > 4$ and $(m, q) \neq (2s, 2)$, where $q = p^s$ for some prime p. By [23, Theorem 3] $G = U_m(q)$ includes a maximal subgroup $H \cong q \cdot q^{2(m-2)} : SU_{m-2}(q) : ((q^2-1)/(m, q+1))$ with $|G : H| = (q^m - (-1)^m)(q^{m-1} - (-1)^{m-1})/(q^2-1)$. Suppose that $(m, q) = (5, 2)$. By [17, p. 73] $G = U_5(2)$ includes a nonsolvable local maximal subgroup $M \cong 3^4 : S_5$. Therefore, $2|(|M|, |G : M|)$, and so $G = U_5(2) \notin \mathfrak{I}_{lmh}$. Consequently, we may assume that $m \geq 7$ is odd for $q = 2$. Then H is a nonsolvable local maximal subgroup of G. Since for $m = 2k$ or $m = 2k-1$ the index $|G : H|$ equals $(q^{2k}-1)(q^{2k-1}+1)/(q^2-1)$ or $(q^{2k-1}+1)(q^{2k-2}-1)/(q^2-1)$ respectively, and

$$|SU_{m-2}| = q^{(m-2)(m-3)/2}(q^{m-2} - (-1)^{m-2})(q^{m-3} - (-1)^{m-3}) \cdots (q^2-1),$$

it follows that $(q+1)|(|H|, |G : H|)$. Thus, for $m > 4$ with $(m, q) \neq (2s, 2)$ we see that $G = U_m(q) \notin \mathfrak{I}_{lmh}$; a contradiction.

3.4. Assume that $m \geq 6$ is even, $m = 2k$, and $q = 2$. By [17, p. 115] $G = U_6(2)$ includes a nonsolvable local maximal subgroup $K \cong 2^9 : U_4(2)$, and furthermore $3|(|K|, |G : K|)$. Consequently, $G = U_6(2) \notin \mathfrak{I}_{lmh}$. Suppose that $m \in \{8, 10, 12\}$. According to [18, Tables 8.46, 8.62, and 8.72] $SU_m(2)$ includes a nonsolvable local maximal subgroup K of type $2^{13} : U_6(2) : (2^2-1)$, $2^{17} : U_8(2) : (2^2-1)$, and $2^{21} : U_{10}(2) : (2^2-1)$ respectively; and, furthermore, $|Z(SU_m(2))| \in \{1, 3\}$. By [16, Proposition 4.1.18], $G = U_m(2)$, for even $m = 2k \geq 8$ includes a nonsolvable local maximal subgroup $K \cong 2^{2m-3} : (a/(2+1, m) \cdot U_{m-2}(2)) \cdot b$, where $b = (2^2-1)(2^2-1, 1)(2+1, m-2)/a$. Since

$$|G| = 2^{k(2k-1)}(2^{2k}-1)(2^{2k-1}+1)(2^{2k-2}-1) \cdots (2^3+1)(2^2-1)$$

and

$$|K| = 2^{4k-3} \cdot 2^{(k-1)(2k-3)}(2^{2k-2}-1)(2^{2k-3}+1))$$
$$\cdots (2^3+1)(2^2-1) \cdot 3(3, 2k-2)/(3, 2k),$$

we infer that

$$|G : K| = (2^{2k}-1)(2^{2k-1}+1)(3, 2k)/3(3, 2k).$$

Observe that both factors $2^{2k}-1$ and $2^{2k-1}+1$ are divisible by 3. Since $2^{2k}-1 = (2^k-1)(2^k+1)$ is divisible by 3, one of the factors is divisible by 3. If $3|(2^k-1)$ then $((2^k-1)/3)|(|K|, |G : K|)$. If $3|(2^k+1)$ then $((2^k+1)/3)|(|K|, |G : K|)$. Consequently, $G = U_m(2) \notin \mathfrak{I}_{lmh}$ if $m \geq 6$ is even; a contradiction.

4. Consider $G = P\Omega_{2l+1}(q) = \Omega_{2l+1}(q) = O_{2l+1}(q) \cong B_l(q)$ with $l \geq 1$ and $q = p^s$ for some prime p and $s \geq 1$. By [17, p. xii], we see that $|G| = (1/d)q^{l^2}(q^{2l}-1)(q^{2l-2}-1) \cdots (q^2-1)$, where $d = (2, q-1)$. Following [24], we have

$$O_3(q) \cong L_2(q), \quad O_4^+(q) \cong L_2(q) \times L_2(q), \quad O_4^-(q) \cong L_2(q^2),$$
$$O_5(q) \cong S_4(q), \quad O_6^+(q) \cong L_4(q), \quad O_6^-(q) \cong U_4(q).$$

Therefore, we assume that $m = 2l+1 \geq 7$.

The minimal permutation representations of the simple orthogonal group $G = O_{2l+1}(q)$, where $m = 2l+1 \geq 7$ with the point stabilizer H of degree $n = |G : H|$ are described in [24]. Applying [24, Theorem], we notice that the following two cases must be considered:

4.1. Assume that $q = 3$. According to [24], in the group $G = O_{2l+1}(3)$ with $2l+1 \geq 7$ the maximal subgroup of least index is not local. Verify that G includes a nonsolvable 3-local maximal subgroup. Indeed, for $l = 3, 4, 5$, according to [18, Tables 8.39, 8.58, and 8.74], each of the groups $\Omega_7(3)$, $\Omega_9(3)$, and $\Omega_{11}(3)$ includes a nonsolvable 3-local maximal subgroup of type $E_3^{3+3} : (1/2)GL_3(3)$, $E_3^{6+4} : (1/2)GL_4(3)$, and $E_3^{10+5} : (1/2)GL_5(3)$ respectively. For $l > 5$ by [16, Proposition 4.1.20] $G = \Omega_{2l+1}(3)$ includes a nonsolvable 3-local maximal subgroup $H \cong [3^a] : (1/2)GL_l(3)$, where $a = l(2l+1) - (1/2)l(3l+1) = (1/2)l(l+1)$. Since

$$|G| = (1/2)3^{l^2}(3^{2l} - 1)(3^{2(l-1)} - 1)\cdots(3^4 - 1)(3^2 - 1),$$

$$|H| = 3^{l^2}(3^l - 1)(3^{(l-1)} - 1)\cdots(3^3 - 1)(3^2 - 1),$$

it follows that

$$|G : H| = (1/2)(3^l + 1)(3^{(l-1)} + 1)\cdots(3^3 + 1)(3^2 + 1)(3^2 - 1).$$

Then $2|(|H|, |G : H|)$. This implies that $G = O_{2l+1}(3)$ for $2l+1 \geq 7$ is not a $\mathfrak{J}_{lmh}$-group; a contradiction.

4.2. Assume that $q = p^s \neq 3$ and p is some odd prime. By [24, Theorem] $G = O_{2l+1}(q)$ for $2l + 1 \geq 7$ includes a nonsolvable p-local maximal subgroup $H \cong q^{2l-1} \cdot ((\Omega_{2l-1}(q) \times (q-1)/2) \cdot 2)$; and, furthermore, $|G : H| = (q^{2l} - 1)/(q - 1) = (q^l - 1)(q^l + 1)/(q - 1)$. Since $2l - 1 = 2(l - 1) + 1$, we see that $|H| = q^{2l-1}((1/d)q^{(l-1)^2}(q^{2(l-1)} - 1)(q^{2(l-1)-2} - 1)\cdots(q^2 - 1))$, where $d = (2, q - 1)$. If l is odd then $(q+1)|(|H|, |G : H|)$. Suppose that $l = 2r$ is even. Since $2(l-1) > l$ for $l \geq 3$, it follows that $|H|$ includes the factor $(q^l - 1) = (q^r - 1)(q^r + 1)$; and, furthermore, $l \geq 4$ and $r \geq 2$. Then $(q^r - 1)/(q - 1)$ divides $(|H|, |G : H|)$. Consequently G is not a $\mathfrak{J}_{lmh}$-group; a contradiction.

5. Consider $G = P\Omega^+_{2l}(q) = O^+_{2l}(q) \cong D_l(q)$ with $l \geq 1$ and $q = p^s$ for some prime p and $s \geq 1$. By [17, p. xii],

$$|G| = (1/d)q^{l(l-1)}(q^l - 1)(q^{2l-2} - 1)(q^{2l-4} - 1)\cdots(q^4 - 1)(q^2 - 1),$$

where $d = (4, q^l - 1)$. Appreciating the isomorphism of the groups in Subsection 4 of the proof, we may assume that $2l \geq 8$. Verify that $G = O^+_{2l}(q)$ for $l \geq 4$ includes a nonsolvable local maximal subgroup of G which is not a Hall subgroup of G.

5.1. Assume that $l = 4$. According to [18, Table 8.50], $O := \Omega^+_8(q)$ includes a nonsolvable local maximal subgroup $A \cong q^6 : (1/(q-1, 2))GL_4(q)$. Put $Z := Z(O)$. Then $|Z| = (q - 1, 2)$. Since

$$|O| = |\Omega^+_8(q)| = (1/(2, q^4 - 1))q^{12}(q^4 - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1),$$

$$|A| = q^6 \cdot (1/(q-1, 2))(q - 1)q^6(q^4 - 1)(q^3 - 1)(q^2 - 1),$$

it follows that

$$|O : A| = (q^3 + 1)(q^2 + 1)(q^2 - 1)(q - 1, 2)/(q - 1)(2, q^4 - 1) = (q^3 + 1)(q^2 + 1)(q + 1).$$

5.1.1. Suppose that $p = 2$. Then $Z = 1$ and $|A| = q^{12}(q - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1)$, and so $(q + 1)|(|A|, |O : A|)$. Since $G \cong O$, we see that G is not a $\mathfrak{J}_{lmh}$-group.

5.1.2. Suppose that $p > 2$. Then $|Z| = 2$, while $G = O/Z$ and $Z < A$. Since

$$|A/Z| = (1/4)q^{12}(q - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1),$$

we have $|O/Z : A/Z| = |O : A| = (q^3 + 1)(q^2 + 1)(q + 1)$, and so $(q + 1)|(|A/Z|, |O/Z : A/Z|)$. Since $G \cong O/Z$, we see that $G = O^+_8(q)$ is not a $\mathfrak{J}_{lmh}$-group.

5.2. Assume that $l = 5$. According to [18, Table 8.66], $O := \Omega^+_{10}(q)$ includes a nonsolvable local maximal subgroup $B \cong q^{10} : (1/(q-1, 2))GL_5(q)$. Put $Z := Z(O)$. Then $|Z| \leq 2$. Since

$$|O| = |\Omega^+_{10}(q)| = (1/(2, q^5 - 1))q^{20}(q^5 - 1)(q^8 - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1),$$
$$|B| = q^{10} \cdot (1/(q-1, 2))(q - 1)q^{10}(q^5 - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1),$$

we see that

$$|O:B| = (q^4+1)(q^3+1)(q^2+1)(q^2-1)(q-1,2)/(q-1)(2,q^5-1)$$
$$= (q^4+1)(q^3+1)(q^2+1)(q+1).$$

5.2.1. Suppose that $p = 2$. Then $Z = 1$, while $|B| = q^{20}(q-1)(q^5-1)(q^4-1)(q^3-1)(q^2-1)$, and $|O:B| = (q^4+1)(q^3+1)(q^2+1)(q+1)$, and so $(q+1)|(|B|, |O:B|)$. Since $G \cong O$, we see that G is not a $\mathfrak{J}_{lmh}$-group.

5.2.2. Suppose that $p > 2$. Then $|Z| \le 2$, while $G = O/Z$ and $Z < B$. We have $|B/Z| = (1/4)q^{20}(q-1)(q^5-1)(q^4-1)(q^3-1)(q^2-1)$, and $|O/Z : B/Z| = |O:B| = (q^4+1)(q^3+1)(q^2+1)(q+1)$. Hence, $(q+1)|(|A/Z|, |O/Z : A/Z|)$. Since $G \cong O/Z$, we see that $G = O^+_{10}(q)$ is not a $\mathfrak{J}_{lmh}$-group.

5.3. Assume that $l = 6$. According to [18, Table 8.82], the group $O := \Omega^+_{12}(q)$ includes a nonsolvable local maximal subgroup $C \cong q^{15} : (1/(q-1,2))GL_6(q)$. Put $Z := Z(O)$. Then $|Z| = (2, q-1)$. Since

$$|O| = |\Omega^+_{12}(q)| = (1/(2,q^6-1))q^{30}(q^6-1)(q^{10}-1)(q^8-1)(q^6-1)(q^4-1)(q^2-1),$$
$$|C| = q^{15} \cdot (1/(q-1,2))(q-1)q^{15}(q^6-1)(q^5-1)(q^4-1)(q^3-1)(q^2-1),$$

it follows that

$$|O:C| = (q^5+1)(q^4+1)(q^3+1)(q^2+1)(q+1).$$

5.3.1. Suppose that $p = 2$. Then $Z = 1$,

$$|C| = q^{30}(q-1)(q^6-1)(q^5-1)(q^4-1)(q^3-1)(q^2-1),$$
$$|O:C| = (q^5+1)(q^4+1)(q^3+1)(q^2+1)(q+1).$$

Hence, $(q+1)|(|C|, |O:C|)$. Since $G \cong O$, we see that G is not a $\mathfrak{J}_{lmh}$-group.

5.3.2. Suppose that $p > 2$. Then $|Z| = 2$, while $G = O/Z$, and $Z < C$. We have

$$|C/Z| = (1/4)q^{30}(q-1)(q^6-1)(q^5-1)(q^4-1)(q^3-1)(q^2-1),$$
$$|O/Z : C/Z| = |O:C| = (q^5+1)(q^4+1)(q^3+1)(q^2+1)(q+1).$$

Hence, $(q+1)|(|A/Z|, |O/Z : A/Z|)$. Since $G \cong O/Z$, we see that $G = O^+_{12}(q)$ is not a $\mathfrak{J}_{lmh}$-group.

5.4. Assume that $l > 6$ and $p > 2$. By [16, Proposition 4.1.20], $G = O^+_{2l}(q)$ includes a nonsolvable p-local maximal subgroup of the following type.

5.4.1. Suppose that $(1/2)l(q-1)$ is odd. It follows that $[q^a] : (1/2)(GL_l(q)) \cong D \le G$, where $a = l(2l) - (l/2)(3l+1) = (l/2)(l-1)$. Since

$$|D| = q^{(l/2)(l-1)}(1/2)(q-1)q^{(l/2)(l-1)}(q^l-1)(q^{l-1}-1)\cdots(q^3-1)(q^2-1),$$

we have

$$|G:D| = 2(q^{l-1}+1)(q^{l-2}+1)\cdots(q^3+1)(q^2+1)(q^2-1)/(q-1)(4,q^l-1).$$

Since $(1/2)l(q-1)$ is odd and $p > 2$, we find that the 2-part of $(q-1)$ satisfies $(q-1)_2 = 2$. Consequently, $(4, q^l-1) = 2$ and $(q+1)|(|D|, |G:D|)$; thus, $G = O^+_{2l}(q) \notin J_{lmh}$.

5.4.2. Suppose that $(1/2)l(q-1)$ is even. Then $[q^a] : J \cong F \le G$, where $a = l(2l) - (l/2)(3l+1) = (l/2)(l-1)$ and $|F| = q^{(l/2)(l-1)}|J|$.

5.4.2.1. Suppose that l is even. Then $J \cong (1/2)(q-1).L_l(q) \cdot ((1/2)(q-1,l))$, and so

$$|F| = q^{(l/2)(l-1)}(1/2)(q-1)(1/(q-1,l))q^{(l/2)(l-1)}(q^l-1)(q^{l-1}-1)$$
$$\cdots(q^3-1)(q^2-1)((1/2)(q-1,l)).$$

Thus,

$$|G:F| = (q^{l-1}+1)(q^{l-2}+1)\cdots(q^3+1)(q^2+1)(q^2-1)/(q-1)(4,q^l-1).$$

Since l is even, we see that $(4,q^l-1)=4$; and, furthermore, $4|(q^{l-1}+1)(q^{l-2}+1)$. Then $(q+1)|(|F|,|G:F|)$ and $G=O^+_{2l}(q)\notin J_{lmh}$.

5.4.2.2. Suppose that l is odd. Then $J \cong (1/4)(q-1).L_l(q)\cdot(q-1,l)$, and so

$$|F| = q^{(l/2)(l-1)}(1/4)(q-1)(1/(q-1,l))q^{(l/2)(l-1)}$$
$$\cdot(q^l-1)(q^{l-1}-1)\cdots(q^3-1)(q^2-1)(q-1,l).$$

Consequently,

$$|G:F| = 4(q^{l-1}+1)(q^{l-2}+1)\cdots(q^3+1)(q^2+1)(q^2-1)/(q-1)(4,q^l-1).$$

Since l is odd, we see that $(4,q^l-1)=2$. Then $(q+1)|(|F|,|G:F|)$ and $G=O^+_{2l}(q)\notin J_{lmh}$.

5.5. Assume that $l>6$ and $p=2$. By [16, Proposition 4.1.20] $G=O^+_{2l}(q)$ includes a nonsolvable 2-local maximal subgroup $K\cong[q^a]:(GL_l(q)\times 1)$, where $a=l(2l)-(l/2)(3l+1)=(l/2)(l-1)$. Since

$$|K| = q^{(l/2)(l-1)}(q-1)q^{(l/2)(l-1)}(q^l-1)(q^{l-1}-1)\cdots(q^3-1)(q^2-1),$$

we obtain

$$|G:K| = (q^{l-1}+1)(q^{l-2}+1)\cdots(q^3+1)(q^2+1)(q^2-1)/(q-1)(4,q^l-1).$$

Since $p=2$, we see that $(4,q^l-1)=1$. Hence, $(q+1)|(|K|,|G:K|)$, and so $G=O^+_{2l}(q)\notin J_{lmh}$.

6. Consider $G=P\Omega^-_{2l}(q)=O^-_{2l}(q)\cong {}^2D_l(q)$ with $l\geq 1$ and $q=p^s$ for some prime p and $s\geq 1$. From [17, p. xii] we infer that

$$|G| = (1/d)q^{l(l-1)}(q^l+1)(q^{2l-2}-1)(q^{2l-4}-1)\cdots(q^4-1)(q^2-1),$$

where $d=(4,q^l+1)$. Using the group isomorphism of Subsection 4 of the proof, we may assume that $2l\geq 8$. Let us verify that $G=O^-_{2l}(q)$ with $l\geq 4$ includes a nonsolvable local maximal subgroup that is not a Hall subgroup of G.

6.1. Assume that $l=4$. According to [18, Table 8.52], $O:=\Omega^-_8(q)$ includes a nonsolvable local maximal subgroup $A\cong q^9:((1/(q-1,2))GL_2(q)\times\Omega^-_4(q))\cdot(q-1,2)$ and $Z(O)=1$. Since

$$|O| = |\Omega^-_8(q)| = (1/(2,q^4+1))q^{12}(q^4+1)(q^6-1)(q^4-1)(q^2-1),$$

$$|A| = q^9\cdot(1/(q-1,2))(q-1)q(q^2-1)(1/(2,q^2+1))q^2(q^2+1)(q^2-1)(q-1,2),$$

we have

$$|O:A| = (q^4+1)(q^6-1)(q-1,2)(2,q^2+1)/(q-1)(2,q^4+1)(q-1,2)$$
$$= (q^4+1)(q^6-1)/(q-1).$$

Consequently, $(q+1)|(|A|,|O:A|)$. Since $G\cong O$, we see that G is not a $\mathfrak{J}_{lmh}$-group.

6.2. Assume that $l=5$. According to [18, Table 8.68], $O:=\Omega^-_{10}(q)$ includes a nonsolvable local maximal subgroup $B\cong q^{15}:((1/(q-1,2))GL_3(q)\times\Omega^-_4(q))\cdot(q-1,2)$. Put $Z:=Z(O)$. Then $|Z|\leq 2$. Since

$$|O| = |\Omega^-_{10}(q)| = (1/(2,q^5+1))q^{20}(q^5+1)(q^8-1)(q^6-1)(q^4-1)(q^2-1),$$

$$|B| = q^{15}\cdot(1/(q-1,2))(q-1)q^3(q^3-1)(q^2-1)(1/(2,q^2+1))q^2(q^2+1)(q^2-1)(q-1,2);$$

therefore,

$$|O : B| = (q^5 + 1)(q^8 - 1)(q^3 + 1)(q - 1, 2)(2, q^2 + 1)/(q - 1)(2, q^5 + 1)(q - 1, 2)$$
$$= (q^5 + 1)(q^8 - 1)(q^3 + 1)/(q - 1).$$

Since $(q^2 - 1)|(q^8 - 1)$, it follows that $(q + 1)||O : B|$ and $(q + 1)|(|B|, |O : B|)$.

6.2.1. Suppose that $p = 2$. Then $Z = 1$. Since $G \cong O$, we see that G is not a $\mathfrak{J}_{lmh}$-group.

6.2.2. Suppose that $p > 2$. Then $|Z| \leq 2$, while $G = O/Z$ and $Z < B$. We have $|B/Z| = (1/4)q^{20}(q-1)(q^3-1)(q^2-1)(q^2+1)(q^2-1)/|Z|$, and $|O/Z : B/Z| = |O : B| = (q^5+1)(q^8-1)(q^3+1)/(q-1)$. Hence, $(q + 1)|(|B/Z|, |O/Z : B/Z|)$. Since $G \cong O/Z$, we see that $G = O^+_{10}(q)$ is not a $\mathfrak{J}_{lmh}$-group.

6.3. Assume that $l = 6$. According to [18, Table 8.84, p. 428], $O := \Omega^-_{12}(q)$ includes a nonsolvable local maximal subgroup $C \cong q^{21} : ((1/(q - 1, 2))GL_3(q) \times \Omega^-_6(q)) \cdot (q - 1, 2)$ and $Z := Z(O) = 1$. Since

$$|O| = |\Omega^-_{12}(q)| = (1/(2, q^6 + 1))q^{30}(q^6 + 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1),$$

$$|C| = q^{21} \cdot (1/(q - 1, 2))(q - 1)q^3(q^3 - 1)(q^2 - 1)$$
$$\cdot(1/(2, q^3 + 1))q^6(q^3 + 1)(q^4 - 1)(q^2 - 1)(q - 1, 2),$$

we have

$$|O : C| = (q^6 + 1)(q^{10} - 1)(q^8 - 1)(q - 1, 2)(2, q^3 + 1)/(q - 1)(2, q^6 + 1)(q - 1, 2).$$

Since $(q^2 - 1)|(q^8 - 1)$, it follows that $(q + 1)||O : C|$ and $(q + 1)|(|C|, |O : C|)$. Since $G \cong O$, we see that G is not a $\mathfrak{J}_{lmh}$-group.

6.4. Assume that $l > 6$ and $p > 2$. By [16, Proposition 4.1.20], $G = O^-_{2l}(q)$ includes a nonsolvable p-local maximal subgroup of the following type.

6.4.1. Suppose that $-1 \in \Omega$ and $(1/2)m(q - 1)$ is odd. Suppose also that $m = 3$. It follows that $[q^a] : ((1/2)GL_3(q) \times \Omega^-_{2l-6}(q)) \cong D \leq G$, where $a = 3(2l) - (3/2)(9 + 1) = 6l - 15$. Since $2l - 6 = 2(l - 3)$, we have

$$|D| = q^{6l-15}(1/2)(q - 1)q^3(q^3 - 1)(q^2 - 1)(1/(2, q^{l-3} + 1))$$
$$\cdot q^{(l-3)(l-4)}(q^{l-3} + 1)(q^{2l-8} - 1)(q^{2l-10} - 1) \cdots (q^4 - 1)(q^2 - 1),$$

$$|G : D| = 2(q^l + 1)(q^{2l-2} - 1)(q^{2l-4} - 1)$$
$$\cdot(q^{l-3} - 1)(2, q^3 + 1)/(q - 1)(q^2 - 1)(q^3 - 1)(4, q^l + 1).$$

Observe that $(4, q^l + 1)|(q^l + 1)$. Since $(q^2 - 1)|(q^{2l-2} - 1)$ and $(q^2 - 1)|(q^{2l-4} - 1)$, the numerator is divisible by $(q + 1)^2$, while the denominator, only by $q + 1$. Treating the numerator and denominator as polynomials in q over the field of rationals, we find that $(q+1)||G : D|$ and $(q+1)|(|G : D|, |D|)$. Thus, G is not a $\mathfrak{J}_{lmh}$-group.

6.4.2. Suppose that $-1 \in \Omega$, $(1/2)m(q - 1)$ is even, and $l > m$. Suppose also that $m = 3$. Then

$$[q^a] : 2.(J \times P\Omega^-_{2l-6}(q)) \cong F \leq G,$$

where $a = 3(2l) - (3/2)(9 + 1) = 6l - 15$. Since $2l - 6 = 2(l - 3)$, we have

$$|F| = q^{6l-15}2(1/4)(q - 1)(1/(q - 1, 3))q^3(q^3 - 1)(q^2 - 1)(q - 1, 3)(1/(4, q^{l-3} + 1))$$
$$\cdot q^{(l-3)(l-4)}(q^{l-3} + 1)(q^{2l-8} - 1)(q^{2l-10} - 1) \cdots (q^4 - 1)(q^2 - 1)2,$$
$$|G : F| = (q^l + 1)(q^{2l-2} - 1)(q^{2l-4} - 1)(q^{l-3} - 1)$$
$$\cdot(4, q^{l-3} + 1)/(q - 1)(q^3 - 1)(q^2 - 1)(4, q^l + 1).$$

As in Subsection 6.4.1, it is not difficult to show that $(q+1)||G:F|$ and $(q+1)|(|G:F|,|F|)$. Thus, G is not a $\mathfrak{J}_{lmh}$-group.

6.4.3. Suppose that $-1 \notin \Omega$ and $2l-2m \geq 2$. Suppose also that $m=3$. It follows that

$$[q^a]:((1/2)GL_3(q) \times \Omega^-_{2l-6}(q)).2 \cong K \leq G,$$

where $a = 3(2l) - (3/2)(9+1) = 6l - 15$. Since $2l-6 = 2(l-3)$, we have

$$|K| = q^{6l-15}(1/2)(q-1)q^3(q^3-1)(q^2-1)(1/(2,q^{l-3}+1))$$
$$\cdot q^{(l-3)(l-4)}(q^{l-3}+1)(q^{2l-8}-1)(q^{2l-10}-1)\cdots(q^4-1)(q^2-1),$$

$$|G:K| = 2(q^l+1)(q^{2l-2}-1)(q^{2l-4}-1)(q^{l-3}-1)$$
$$\cdot(2,q^3+1)/(q-1)(q^2-1)(q^3-1)(4,q^l+1).$$

As in Subsection 6.4.1, it is not difficult to show that $(q+1)||G:D|$ and $(q+1)|(|G:D|,|D|)$. Thus, G is not a $\mathfrak{J}_{lmh}$-group.

6.5. Assume that $l > 6$ and $p = 2$. By [16, Proposition 4.1.20], $G = O^-_{2l}(q)$ for $m = 3$ includes a nonsolvable 2-local maximal subgroup $L \cong [q^a]:(GL_3(q) \times \Omega^-_{2l-6}(q))$, where $a = 3(2l) - (3/2)(9+1) = 6l-15$. Since $2l-6 = 2(l-3)$, we have

$$|L| = q^{6l-15}(1/2)(q-1)q^3(q^3-1)(q^2-1)(1/(2,q^{l-3}+1))$$
$$\cdot q^{(l-3)(l-4)}(q^{l-3}+1)(q^{2l-8}-1)(q^{2l-10}-1)\cdots(q^4-1)(q^2-1),$$

$$|G:L| = (q^l+1)(q^{2l-2}-1)(q^{2l-4}-1)(q^{l-3}-1)$$
$$\cdot(2,q^3+1)/(q-1)(q^2-1)(q^3-1)(4,q^l+1).$$

As in Subsection 6.4.1, it is not difficult to show that $(q+1)||G:L|$ and $(q+1)|(|G:L|,|L|)$. Thus, G is not a $\mathfrak{J}_{lmh}$-group.

7. Consider $G = G_2(q)$ with $q = p^s$ for some prime p and $s \geq 1$. Then $|G| = q^6(q^6-1)(q^2-1)$. Observe that $G_2(2)$ is not simple; and, furthermore, $G_2(2)' \cong U_3(3)$ is a nonabelian simple group. For $q \geq 3$ this G is a nonabelian simple group. Assume that $q = 3$. By [17, p. 60], G includes a nonsolvable maximal subgroup $H \cong 2^3 \cdot L_3(2)$ which is not a Hall subgroup of G. Assume that $q = 4$. By [17, p. 97], G includes a nonsolvable maximal subgroup $M \cong 2^{2+8}:(3 \times A_5)$ which is not a Hall subgroup of G; a contradiction with the hypotheses of Theorem 1.

Assume that $q > 4$. The degrees $n = |G:P|$ of the minimal permutation representations of G and the corresponding point stabilizers P are listed in [25, Theorem 1]. It is not difficult to verify that for $q > 4$ in all cases P is a nonsolvable local maximal subgroup of G; furthermore, $(n,|P|) > 1$; i. e., P is not a Hall subgroup of G; a contradiction with the hypotheses of Theorem 1.

8. Consider $G = F_4(q)$, with $q = p^s$ for some prime p and $s \geq 1$. We have $|G| = q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. By [25, Theorem 2], the group G admits the minimal permutation representation of degree $n = |G:P| = \frac{(q^{12}-1)(q^4+1)}{q-1}$ with point stabilizer P; and, furthermore, P is a nonsolvable local maximal subgroup of G. Let us verify that P is not a Hall subgroup of G. Suppose that $q = 2^s$. Then

$$P \cong (2^s \cdot 2^{8s} \times 2^{6s}):(C_3(q) \times (q-1)).$$

Since

$$|C_3(q)| = (1/d)q^9(q^6-1)(q^4-1)(q^2-1), \quad d = (2,q-1),$$

it follows that $(q+1)|(n,|P|)$. Suppose that $q=p^s$ for some prime $p>2$. Then

$$P \cong (p^s \cdot p^{14s}) : (2 \cdot (C_3(q) \times (q-1)/2) \cdot 2)$$

or

$$P \cong (p^{7s} \cdot p^{8s}) : (2 \cdot (B_3(q) \times (q-1)/2) \cdot 2),$$

again $(q+1)|(n,|P|)$, and so P is not a Hall subgroup of G; a contradiction with the hypotheses of Theorem 1.

9. Consider $G = E_6(q)$, where $q=p^s$ for some prime p and $s \geq 1$. According to [17, Table 6], we have

$$|G| = (1/d)q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1),$$

where $d=(3,q-1)$, and by [26, Theorem 1] the group G includes a nonsolvable local maximal subgroup $P \cong p^{16s} : (e \cdot (D_5(q) \times (q-1)/e') \cdot e)$ with $n = |G:P| = \frac{(q^9-1)(q^8+q^4+1)}{q-1}$, where $e=(q-1,4)$ and $e'=ed$. Since $D_5(q)$ is a nonabelian simple group, infer that P is a nonsolvable group; moreover, [17, Table 6] yields

$$|D_5(q)| = (1/d_1)q^{20}(q^5-1)(q^8-1)(q^6-1)(q^4-1)(q^2-1),$$

where $d_1 = (4,q^5-1)$. Then $(q^2+q+1)|(n,|P|)$. Thus, P is not a Hall subgroup of G; a contradiction with the hypotheses of Theorem 1.

10. Consider $G = E_7(q)$, with $q=p^s$ for some prime p and $s \geq 1$. According to [17, Table 6], we have

$$|G| = (1/d)q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1),$$

where $d=(2,q-1)$, and by [26, Theorem 2] G includes some local maximal subgroup

$$P \cong p^{27s} : (d' \cdot (E_6(q) \times (q-1)/c) \cdot d'),$$

where $d'=(q-1,3)$, $e=(q-1,4)$, $c=d \cdot d'$, and $n=|G:P| = \frac{(q^{14}-1)(q^9+1)(q^5+1)}{q-1}$. Applying Subsection 9, we find that $(q^3+1)|(n,|P|)$. Thus, P is neither a solvable group nor a Hall subgroup of G; a contradiction with the hypotheses of Theorem 1.

11. Consider $G = E_8(q)$, with $q=p^s$ for some prime p and $s \geq 1$. According to [17, Table 6], we have

$$|G| = q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1),$$

and by [26, Theorem 3] G includes a local maximal subgroup

$$P \cong p^s \cdot p^{56s} : (d \cdot (E_7(q) \times (q-1)/d) \cdot d),$$

where $d=(q-1,2)$, and we have

$$n = |G:P| = \frac{(q^{30}-1)(q^{12}+1)(q^{10}+1)(q^6+1)}{q-1}.$$

Applying Subsection 10, we find that $(q^6+1)|(n,|P|)$. Thus, P is a nonsolvable local group and is not a Hall subgroup of G; a contradiction with the hypotheses of Theorem 1.

12. Consider $G = Sz(2^{2m+1}) \cong {}^2B_2(q)$ with $q=2^{2m+1}$ and $m \geq 1$. By Lemma 2.1, G is an N-group, and it is a group of type (2) of Theorem 1.

13. Consider $G = {}^3D_4(q)$ with $q=p^s$ for some prime p and $s \geq 1$. According to [17, Table 6], we have $|G| = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$ and by [27, Theorem 3] G includes the maximal subgroup $P \cong (p^s \cdot p^{8s}) : (d \cdot (A_1(q^3) \times (q-1)/d) \cdot d)$, where $d=(q-1,2)$, and so $n=|G:P| = (q^8+q^4+1)(q+1)$. Since $|A_1(q^3)| = q^3(q^6-1)$ and $A_1(q^3) \cong L_2(q^3)$ is a nonsolvable group, infer that P is a nonsolvable local maximal subgroup of G; and, furthermore, $(q+1)|(n,|P|)$ in contradiction with the hypotheses of Theorem 1.

14. Consider $G = Re(q) \cong {}^2G_2(q)$ with $q = 3^{2n+1}$ and $n \geq 1$. By [11, Theorem 3.33], $|G| = q^3(q^3+1)(q-1)$ and G includes some nonsolvable local maximal subgroup $H \cong 2 \times L_2(q)$. Since $q|(|G:H|,|H|)$, it follows that H is not a Hall subgroup of G in contradiction with the hypotheses of Theorem 1.

15. Consider $G = {}^2F_4(q)$ with $q = 2^s$ and an odd integer $s > 1$. According to [17, Table 6], we have $|G| = q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$. By [27, Theorem 5] G includes a nonsolvable local maximal subgroup $P \cong (2^s \cdot 2^{4s} \cdot 2^{5s}) : ({}^2B_2(q) \times (q-1))$. Since $|{}^2B_2(q)| = q^2(q^2+1)(q-1)$, we see that $(q-1)|(|G:P|,|P|)$. Thus, P is not a Hall subgroup of G in contradiction with the hypotheses of Theorem 1.

16. Consider $G = {}^2E_6(q)$ with $q = p^s$ for some prime p and $s \geq 1$. According to [17, Table 6], we have

$$|G| = (1/d)q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1),$$

where $d = (3, q-1)$, and by [27, Theorem 4] G includes a nonsolvable local maximal subgroup $P \cong (p^s \cdot p^{20s}) : (d_+ \cdot {}^2A_5(q) \times (q-1)/c) \cdot c$, where $d_+ = (q+1, 2)$, and $c = (q+1, 3)$, and so we find that

$$n = |G:P| = \frac{(q^{12}-1)(q^6-q^3+1)(q^4+1)}{q-1}.$$

Since

$$|{}^2A_5(q)| = q^{15}(q^2-1)(q^3+1)(q^4-1)(q^5+1)(q^6-1),$$

it follows that $(|G:P|,|P|) > 1$; hence, P is not a Hall subgroup of G in contradiction with the hypotheses of Theorem 1.

17. Consider $G = A_n$ with $n \geq 5$. Since $A_5 \cong L_2(5)$, $A_6 \cong L_2(9)$, and A_7 are N-groups by Lemma 2.1, we may assume that $n \geq 8$. Denote by A the set of all permutations in S_n keeping the first $n-3$ symbols unmoved; and by B, the set of all permutations in S_n leaving the last three symbols unmoved. Then $A \cong S_3$ and $B \cong S_{n-3}$, while $A \leq S_n$, $B \leq S_n$, and $A \times B \leq S_n$. Since $(n/2) > 3$, using [28], we see that $H = G \cap (A \times B)$ is a maximal subgroup of G. Since $A \not\subseteq G$, it follows that $S_n = A \cdot G$; and, furthermore, $D = A \cap G \leq H$ with $D \lhd A$ and $|A:D| = 2$. This implies that $D \cong A_3$. Since $D \lhd A$, we have $D \lhd A \times B$, and so $D \lhd H$. Similarly, $B \not\subseteq G$, and so $S_n = B \cdot G$; furthermore, $F = B \cap G \leq H$ with $F \lhd B$ and $|B:F| = 2$. Therefore, $F \cong A_{n-3}$. Since $F \lhd B$, infer that $F \lhd A \times B$, and so $F \lhd H$. Since $n-3 \geq 5$, it follows that F is a nonsolvable group. Hence, H is a nonsolvable subgroup of G; furthermore, $H = N_G(D)$, where $|D| = 3$. We can show that $H \cong (3 \times A_{n-3}) : 2$. Since

$$|G:H| = (n!/2)/(3 \cdot (n-3)!) = \frac{(n-2)(n-1)n}{2 \cdot 3},$$

we see that $(|G:H|,|H|) > 1$ for $n \geq 8$. Thus, H is not a Hall subgroup of G, and so G is not a $\mathfrak{J}_{lmh}$-group for $n \geq 8$; a contradiction.

18. Assume that G is one of the 26 sporadic groups, not belonging to $\{M_{11}; M_{23}; J_1\}$. Applying [17], it is not difficult to verify that G includes a nonsolvable local maximal subgroup H, and H is not a Hall subgroup of G.

Applying Subsections 1–18, we conclude that G is isomorphic to one of the groups of Theorem 1.

Sufficiency: Suppose that G is isomorphic to one of the groups listed in Theorem 1 and verify that G is a $\mathfrak{J}_{lmh}$-group; i. e., each nonsolvable local maximal subgroup of G is a Hall subgroup. By Lemma 2.1, the groups listed in items (1), (2), and (4) of Theorem 1, with the exception of the groups $L_5(2)$, $L_5(4)$, J_1, and M_{23} are N-groups, and so they are $\mathfrak{J}_{lmh}$-groups. By [17], $\{L_5(2), J_1, M_{23}\} \subseteq \mathfrak{J}_{lmh}$. According to [18, Table 8.18], $SL_5(4) \cong L_5(4)$ is simple. In the "Notes" column the subgroups of $G = L_5(4)$ marked by the letter N for a "newbie" are not maximal subgroups of G. The subgroups $A \cong E_4^4 : GL_4(4)$ and $B \cong E_4^6 : (SL_2(4) \times SL_3(4)) : 3$ are nonsolvable 2-local maximal Hall subgroups of G. The subgroup $C \cong SL_5(2)$ is a simple group and so it is not a local subgroup of G.

The subgroup $D \cong SU_5(2)$ is a simple group and hence it is not a local subgroup of G. Consequently, $G = L_5(4) \in \mathfrak{J}_{lmh}$.

Consider $G := L_3(q)$ with $q = p^s \geq 3$ and $s \geq 1$ such that $q \not\equiv 1 \pmod 3$. For $q = 3$ Lemma 2.1 shows that G is an N-group, and so $G \in \mathfrak{J}_{lmh}$. Assume that $q > 3$. Up to isomorphism, all maximal subgroups of geometric type of the group $SL_3(q)$ with $q \geq 2$ are listed in [18, Table 8.3]; furthermore, $|SL_3(q)| = q^3(q^3 - 1)(q^2 - 1) = N$, and $|L_3(q)| = N/d$ with $d := |Z(SL_3(q))| = (q - 1, 3)$. Since $q \not\equiv 1 \pmod 3$, infer that $d := 1$, and so $L_3(q) \cong SL_3(q)$.

Take a local subgroup $A \cong E_q^2 : GL_2(q)$ of the group G in the first row of [18, Table 8.3], which has a misprint, as it gives E_q^3 instead of E_q^2. Then A is a nonsolvable local maximal subgroup of G. Verify that A is a Hall subgroup of G. Since $|A| = q^3(q - 1)(q^2 - 1)$, we have $|G : A| = q^2 + q + 1$. If r is a prime divisor of $q^2 + q + 1$ then $(q, r) = 1$. Suppose that $r|(q - 1)$. Then $q = rk + 1$ with $k \in Z$ and $q^2 + q + 1 = r^2k^2 + 3rk + 3$; hence, 3 is divisible by r. Consequently, $r = 3$. Therefore, $3|(q - 1)$, which is impossible because $q \not\equiv 1 \pmod 3$. Thus, r does not divide $(q - 1)$. Suppose that $r|(q^2 - 1)$. Then $r|(q + 1)$. From $r|(q^2 + q + 1)$ we infer that $r|q^2$ and $r|q$. Hence, $(q, r) = r \neq 1$; a contradiction. Thus, $(|A|, |G : A|) = 1$ and A is a Hall subgroup of G.

Since the subgroup isomorphic to $GL_2(q)$ is a newbie, it is not a maximal subgroup of G.

Suppose that $B \cong SL_3(q_0) \cdot (q - 1/q_0 - 1, 3)$, where $q = q_0^r$ for odd r, is a subgroup of the group G in row 6 of [18, Table 8.3]. Suppose that $q_0 \equiv 1 \pmod 3$. Then $q = q_0^r \equiv 1 \pmod 3$, which is impossible. Consequently, $q_0 \not\equiv 1 \pmod{3}$. Then $Z(B) = 1$ and B is not a local subgroup of G.

Suppose that $C \cong d \times SO_3(q)$, where q is odd, is a subgroup of the group G in row 8 of [18, Table 8.3]. Since $d = 1$, it follows that $C \cong SO_3(q)$, where q is odd. Then $(2, q - 1) = 2$ and the second row from the bottom in [17, Table 2, p. xii] implies that C with $q = 3$ is a solvable group, while for $q \geq 5$ it is not local.

Suppose that $D \cong (q_0 - 1, 3) \times SU_3(q_0)$, where $q = q_0^2$, is a subgroup of the group G in row 9 of [18, Table 8.3]. Then $q_0 \not\equiv 1 \pmod 3$ and $D \cong SU_3(q_0)$. Suppose that $q_0 \equiv \pm 1 \pmod 3$. Then $q = q_0^2 \equiv 1 \pmod 3$, which is impossible. Consequently, $q_0 \not\equiv \pm 1 \pmod 3$. Therefore, $(3, q_0 + 1) = 1$ and $D \cong U_3(q_0)$. Then D is not a local subgroup of G. Thus, the groups of item (3) of Theorem 1 are $\mathfrak{J}_{lmh}$-groups.

We have now established that each group of Theorem 1 is a $\mathfrak{J}_{lmh}$-group. The proof of Theorem 1 is complete.

4. Proofs of Theorem 2 and the Corollary to It

Consider a nonsolvable $\mathfrak{T}_{slh}$-group G. Then in G each nonsolvable superlocal is a Hall subgroup. Suppose that G violates the claim of Theorem 2 and G is a group of minimal order with this property. Denote the solvable radical of G by $S(G)$. Lemma 2.7 shows that $S(G)$ is a dispersive group.

Suppose that $S(G) \neq 1$ and consider the quotient group $G/S(G)$. By Lemma 2.5 $G/S(G) \in \mathfrak{T}_{slh}$. Since $|G/S(G)| < |G|$, by induction $G/S(G)$ includes a normal nonabelian simple subgroup $A/S(G)$ satisfying $A/S(G) \leq G/S(G) \leq \mathrm{Aut}(A/S(G))$ and isomorphic to one of the groups of items (1)–(4) of Theorem 1. Hence, the claim of Theorem 2 holds for G; a contradiction.

Thus, $S(G) = 1$. If $M \cdot\!\lhd G$ then M is a direct product of pairwise isomorphic nonabelian simple groups P_i, for $i = 1, 2, \dots, n$. Since $G \in \mathfrak{T}_{slh}$ and by Lemma 2.5 the class $\mathfrak{T}_{slh}$ is S_n-closed, it follows that $M \in \mathfrak{T}_{slh}$, and so $P_i \in \mathfrak{T}_{slh}$ for every $i = 1, 2, \dots, n$. Suppose that $n > 1$. Take a nonidentity p-subgroup N of P_1. Then $H = N_G(N)$ is a p-local subgroup of G; furthermore, $P_2 \times P_3 \times \dots \times P_n < H$, and so H is a nonsolvable group. Proposition 2.1 shows that H lies in a p-superlocal B of G with $O_p(H) \leq O_p(B)$. The assumptions of Theorem 2 imply that B is a Hall subgroup of G. Since $N \leq O_p(H) \leq O_p(B)$ and $O_p(B) \lhd B$, it follows that $P_1 \not\leq B$. Indeed, otherwise we would obtain $N \leq O_p(B) \cap P_1 \lhd P_1$, which is impossible. Since B is a Hall subgroup of G and $M \lhd G$, it follows that $M \cap B$ is a Hall subgroup of M; furthermore, the modular identity yields $B \cap M = (B \cap P_1) \times P_2 \times P_3 \times \dots \times P_n$. Suppose that $q||P_1 : (B \cap P_1)|$. Then $q||P_2|$, and so $q||B \cap M|$. Thus, $B \cap M$ is not a Hall subgroup of M; a contradiction.

Hence, $n = 1$ and $M = P_1$ is a nonabelian simple $\mathfrak{J}_{slh}$-group. Since Proposition 2.2 yields $\mathfrak{J}_{slh} \subseteq \mathfrak{J}_{lmh}$, infer that M is a nonabelian simple $\mathfrak{J}_{lmh}$-group. By Theorem 1, the group M is isomorphic to one of the simple groups in Theorem 1. By [20, Lemma 1.53] we see that $C_G(M) \lhd G$, while [20, Theorem 2.8] implies that the quotient $G/C_G(M)$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$; moreover, $C_G(M) \cap M = 1$. Suppose that $C_G(M) = 1$. Then $M \leq G \leq \mathrm{Aut}(M)$ and G is isomorphic to a group in Theorem 2; a contradiction.

Suppose that $C_G(M) \neq 1$. Since $S = 1$, it follows that $C := C_G(M)$ is a nonsolvable normal subgroup of G, while $S(C) = 1$ and $M \times C \lhd G$. Since $G \in \mathfrak{X}_{slh}$ and by Lemma 2.5 the class $\mathfrak{X}_{slh}$ is S_n-closed, we infer that $M \times C \in \mathfrak{X}_{slh}$. Then $G = M \times C$ by induction.

By Lemma 2.8, M includes a Sylow subgroup whose normalizer in M is of even index. Suppose that $N_M(M_p)$ is of even index in M. Then $L := N_G(M_p) \geq C$; and furthermore $L \cap M = N_M(M_p)$. Since L is a p-local subgroup of G, Proposition 2.1 shows that L lies in a superlocal V of G with $O_p(L) \leq O_p(V)$. Then $C \leq V$, and so V is nonsolvable. The modular identity yields $V := C \times (V \cap M)$. The assumption of Theorem 2 implies that V is a Hall subgroup of G. Since $M_p \leq O_p(L) \leq O_p(V)$ and $O_p(V) \lhd V$, it follows that $M \not\subseteq V$. Indeed, otherwise we would obtain $M_p \leq O_p(V) \cap M \lhd M$, which is impossible. Since V is a Hall subgroup of G and $M \lhd G$, infer that $V \cap M$ is a Hall subgroup of M. On the other hand, $V \cap M < M$, $N_M(M_p) \leq (V \cap M)$, $M_p \leq O_p(V)$, and $M_p \leq (O_p(V) \cap M)$; hence, $M_p = O_p(V) \cap M$. Then $M_p \lhd (V \cap M)$, and so $(V \cap M) = N_M(M_p)$. This implies that $V \cap M$ is of even index in M, and so V is of even index in G. Since $2 | |V|$, it follows that V is not a Hall subgroup of G; a contradiction.

The proof of Theorem 2 is complete.

Corollary 2.1. *If each superlocal in a nonidentity group G is a Hall subgroup then G is a dispersive group.*

Proof. Suppose that each superlocal in a nonidentity group G is a Hall subgroup. Suppose also that G is a nonsolvable group and G is a group of minimal order with these properties. Take a minimal normal subgroup M of G and denote the solvable radical of the group by $S := S(G)$. Suppose that $S \neq 1$ and M is a p-group. Consider the quotient G/M. Suppose further that A/M is a q-superlocal in G/M and put $B/M := O_q(A/M)$. Assume that $q = p$. Then $B/M := O_p(A/M) = O_p(A)/M$ and $B = O_p(A)$. Consequently, $A = N_G(O_p(A))$ is a p-superlocal in the group G. By assumption, A is a Hall subgroup of G; therefore, A/M is a Hall subgroup of G/M.

Assume that $q \neq p$. Then $B \lhd A$ and $B = MB_q$, while Frattini's argument yields $A = BN_A(B_q) = MN_A(B_q)$. Take $g \in N_G(B_q)$. Then $B^g = M^g(B_q)^g = MB_q = B$, and so $g \in N_G(B)$. Since $A/M = N_{G/M}(B/M) = N_G(B)/M$, it follows that $g \in N_G(B) = A$ and so $g \in N_A(B_q)$. Consequently, $N_A(B_q) = N_G(B_q) := T$. Since T is a q-local subgroup of G, by Proposition 2.1 T lies in a q-superlocal U of G. Then $A = MT \leq MU$ and $A/M = MT/M \leq MU/M$. Since A/M is a q-superlocal in G/M, we see that A/M is a q-maximal subgroup of G/M. The definition of q-maximal subgroup yields $A/M = MU/M$. By assumption, U is a Hall subgroup of G. Therefore, A/M is a Hall subgroup of G/M. By induction, the group G/M is solvable; therefore, so is the group G; a contradiction. Consequently, $S = 1$. Then Theorem 2 shows that M is the nonabelian simple group isomorphic to a simple group as in Theorem 1, and $M \leq G \leq \mathrm{Aut}(M)$. It is not difficult to verify that M includes a Sylow subgroup whose normalizer is not a Hall subgroup of M. As in the proof of Theorem 2, it is not difficult to show that G includes a superlocal that is not a Hall subgroup of G and arrive at a contradiction. Thus, G is a solvable group. Then Lemma 2.6 shows that G is a dispersive group. $\square$

The main results of this article were announced in [29].

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