ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 5, pp. 763–777. ⓒ *Pleiades Publishing, Ltd., 2020.*
Russian Text ⓒ *The Author(s), 2020, published in Sibirskii Matematicheskii Zhurnal, 2020, Vol. 61, No. 5, pp. 961–978.*

PRIVILEGED COORDINATES FOR CARNOT–CARATHÉODORY SPACES OF LOWER SMOOTHNESS

S. G. Basalaev

UDC 514.77:517.28

Abstract: We describe classes of local coordinates on the Carnot–Carathéodory spaces of lower smoothness which permit the homogeneous approximation of quasimetrics and basis vector fields. We establish the minimal smoothness that is required for these classes to coincide with the class of the already-described privileged coordinates in the infinite smoothness case. Moreover, we apply these results to prove the analogs of the available theorems in the case of the canonical coordinates of the second kind. Also, we prove some convergence theorems in quasimetric spaces.

DOI: 10.1134/S0037446620050018

Keywords: sub-Riemannian geometry, nilpotent tangent cone, privileged coordinates

1. Introduction

Consider an N-dimensional connected C^∞-smooth Riemannian manifold $\mathbb{M}$ with a fixed distribution $H \subset T\mathbb{M}$ and some inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ on H. The vector field $[X, Y] = XY - YX$ is known as the *commutator* (*Lie bracket*) of two vector fields X and Y. Taking successive commutators of vector fields in H, we obtain the family of subbundles $H_1 = H$ and $H_{k+1} = H_k + [H_k, H]$. It is known [1, 2] that if H is totally nonholonomic, meaning that $H_m = T\mathbb{M}$ for some $m > 0$; then every two points in $\mathbb{M}$ can be connected by a *horizontal curve*, i. e., an absolutely continuous curve γ with $\dot\gamma \in H$ almost everywhere.

Definition 1. The metric $d_{cc}(x, y)$ on $\mathbb{M}$, defined as the greatest lower bound of the lengths of horizontal curves connecting x and y, is called the *Carnot–Carathéodory metric*; while the corresponding metric space, a *Carnot–Carathéodory space* or a *sub-Riemannian space*. The precise definitions of these terms may differ across the sources.

Equiregular Carnot–Carathéodory spaces constitute an important subclass. In these spaces, the dimensions of $H_k(x)$ in the filtration

$$H = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_m = T\mathbb{M} \tag{1}$$

are independent of x, and so H_k is a distribution on $\mathbb{M}$. As [3] shows, we can locally lift each Carnot–Carathéodory space to an equiregular space of higher dimension.

Since we will study the local properties of equiregular spaces, it is convenient to choose a basis for $T\mathbb{M}$ subordinate to the structure in (1); i. e., in a neighborhood of $p \in \mathbb{M}$ we can choose some tuple of vector fields $X_1, \dots, X_N$ such that

$$H_k(x) = \operatorname{span}\{X_1(x), \dots, X_{\dim H_k}(x)\}.$$

Associate the formal weight $\sigma_j = \min\{k : X_j \in H_k\}$ to each X_j.

Nilpotent approximations are essential for studying the local geometry of Carnot–Carathéodory spaces. Nilpotent approximation methods stem from the research on hypoelliptic operators, which involves the *canonical coordinates of the first kind*

$$\theta_p(x_1, \dots, x_N) = \exp(x_1X_1 + \dots + x_NX_N)(p). \tag{2}$$

The author was supported by the Ministry of Science and Education of the Russian Federation (Grant 1.3087.2017/4.6).

Original article submitted July 30, 2018; revised September 11, 2018; accepted October 17, 2018.

Definition 2. Using (2) and considering a neighborhood of $p \in \mathbb{M}$, introduce the family

$$\Delta^p_\varepsilon : \theta_p(x_1, \dots, x_N) \mapsto \theta_p(\varepsilon^{\sigma_1} x_1, \dots, \varepsilon^{\sigma_N} x_N)$$

of *anisotropic dilations*, and the *(quasi)distance function*

$$d_\infty(x, y) = \max_{i=1,\dots,N} |u_i|^{\frac{1}{\sigma_i}} \quad \text{in case } y = \theta_x(u_1, \dots, u_N).$$

Let us state the key assertions as the following theorem whose items may differ from how they were formulated by the authors; see the comparison of various formulations below.

Theorem 1 (of nilpotent approximation). *Let $\mathbb{M}$ be an equiregular Carnot–Carathéodory space $\mathbb{M}$ and $p \in \mathbb{M}$.*

(1) (The Rothschild–Stein Local Approximation Theorem [3, 4].) *The limits*

$$\widehat{X}^p_k(x) = \lim_{\varepsilon \to 0} \big(\Delta^p_\varepsilon\big)^{-1}_* \varepsilon^{\sigma_k} X_k(\Delta_\varepsilon x), \quad k = 1, \dots, N,$$

exist and are uniform on some neighborhood of p; moreover, the homogeneous vector fields $\widehat{X}^p_1, \dots, \widehat{X}^p_N$ constitute a basis for the Lie algebra of some Carnot group $\mathbb{G}^p$ (a graded stratified nilpotent Lie group).

(2) (The Nagel–Stein–Wainger Ball–Box Theorem [5].) *There exist a neighborhood U of p and constants $0 < C_1 \le C_2 < \infty$ such that*

$$C_1 d_\infty(x, y) \le d_{cc}(x, y) \le C_2 d_\infty(x, y)$$

for all $x, y \in U$.

(3) (The Gromov Local Approximation Theorem [6].) *In some neighborhood of p the uniform limit*

$$\hat{d}^p_{cc}(x, y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d_{cc}\big(\Delta^p_\varepsilon x, \Delta^p_\varepsilon y\big)$$

exists, where $\hat{d}^p_{cc}$ is the Carnot–Carathéodory metric for the group $\mathbb{G}^p$ (meaning that $\hat{d}^p_{cc}$ is formed by the homogeneous vector fields $\widehat{X}^p_1, \dots, \widehat{X}^p_N$).

In their original article Rothschild and Stein [3] prove that in the special case of *free* vector fields we have the expansion

$$X_k(x) = \widehat{X}^p_k(x) + R_k(p, x),$$

where $\widehat{X}^p_k$ is homogeneous, while the residues of R_k are small as $x \to p$. The convergence of vector fields to homogeneous fields in the equiregular smooth case is proved in [4] as presented in item (1). In the recent article [7] the convergence of vector fields is proved in the special *regularized coordinates of the first kind* in the case that $H \in C^{m-1,\alpha}$ with $\alpha > 0$; see also some generalization below to the case of less smoothness.

The article [5] provides some comparison of the Carnot–Carathéodory metric with the distance d_∞ as well as several other distance functions. It is worth noting that the quantity d_∞ is in general only a local quasimetric rather than a metric meaning that

$$d_\infty(x, z) \le Q(d_\infty(x, y) + d_\infty(y, z))$$

for all $x, y, z \in U$ and some constant $Q = Q(U) \ge 1$. Some authors state the Ball–Box Theorem as

$$\mathrm{Box}(x, C_1 r) \subset B_{cc}(x, r) \subset \mathrm{Box}(x, C_2 r),$$

where B_{cc} is a ball of d_{cc}, while Box is a ball of d_∞.

Gromov stated in [6] the Local Approximation Theorem for "sufficiently smooth vector fields" as

$$\left|d_{cc}(x,y)-\hat{d}^p_{cc}(x,y)\right|=o(\varepsilon)\quad \text{ as } \varepsilon\to 0 \text{ for } x,y\in B_{cc}(p,\varepsilon).$$

This statement is equivalent to item (3) because the metric $\hat{d}^p_{cc}$ is homogeneous under the dilation Δ^p_ε.

Generalizations of equiregular Carnot–Carathéodory spaces with C^1-smooth vector fields are introduced in [8, 9]. Acting as in these articles, we rely on the following definition:

DEFINITION 3. A connected C^∞-smooth manifold $\mathbb{M}$ of topological dimension N is called an *equiregular Carnot–Carathéodory space with $C^{r,\alpha}$-smooth vector fields*, where $r\in\mathbb{N}$ and $\alpha\in[0,1]$ provided that $C^{r,0}=C^r$ whenever the tangent bundle $T\mathbb{M}$ has a distinguished filtration by $C^{r,\alpha}$-smooth subbundles

$$H_1\subsetneq H_2\subsetneq\cdots\subsetneq H_m=T\mathbb{M} \tag{3}$$

such that $[H_i,H_j]\subset H_{i+j}$ for all $i,j=1,\dots,m$.

A Carnot–Carathéodory space $\mathbb{M}$ is called a *Carnot manifold* whenever $\mathbb{M}$ satisfies the stronger condition $H_k=\operatorname{span}\{H_{k-1},[H_i,H_j]:i+j=k\}$ for $k=2,\dots,m$. We call m the *depth* of $\mathbb{M}$.

Each classical equiregular sub-Riemannian space with $H\in C^{r+m-1,\alpha}$ is a $C^{r,\alpha}$-smooth Carnot manifold. Carnot manifolds are also equipped with the Carnot–Carathéodory metric; this is proved for the smoothness class $C^{1,\alpha}$ with $\alpha\in(0,1]$ in [9], and for C^1 in [10]. However, some points may exist in general that are unconnectable by a horizontal curve. In this case, the quasimetric d_∞ is used to study the properties of Carnot–Carathéodory spaces. The properties of C^1-smooth Carnot–Carathéodory spaces are described in the next theorem (cf. Theorem 1):

Theorem 2 [9, 11, 12]. *Suppose that C^1-smooth vector fields $X_1,\dots,X_N$ satisfy the commutator table*

$$[X_i,X_j](x)=\sum_{k:\sigma_k\le\sigma_i+\sigma_j}c_{ijk}(x)X_k(x)$$

and fix a point p. Then

(1) *There exists a tuple of vector fields $\widehat{X}'_1,\dots,\widehat{X}'_N$ in $\mathbb{R}^N$ such that*

$$\exp(u_1\widehat{X}'_1+\cdots+u_N\widehat{X}'_N)(0)=(u_1,\dots,u_N)$$

and

$$[\widehat{X}'_i,\widehat{X}'_j](u)=\sum_{k:\sigma_k=\sigma_i+\sigma_j}c_{ijk}(p)\widehat{X}'_k(u).$$

Furthermore, $\widehat{X}'_1,\dots,\widehat{X}'_N$ define the structure of a graded nilpotent Lie algebra (and the structure of a Carnot algebra for a Carnot manifold).

(2) *If $\widehat{X}^p_k=(\theta_p)_*\widehat{X}'_k$ then*

$$\widehat{X}^p_k(x)=\lim_{\varepsilon\to0}\left(\Delta^p_\varepsilon\right)_*^{-1}\varepsilon^{\sigma_k}X_k\left(\Delta^p_\varepsilon x\right),$$

where the limit is uniform on some neighborhood of p.

(3) *Using $\widehat{X}'_k$, construct the quasimetric $\hat{d}'_\infty$ by analogy with d_∞, and carry $\hat{d}'_\infty$ over to the manifold: $\hat{d}^p_\infty(x,y)=\hat{d}'_\infty\left(\theta_p^{-1}(x),\theta_p^{-1}(y)\right)$; we cannot immediately define $\hat{d}^p_\infty$ from $\widehat{X}_k$ because these $\widehat{X}_k$ are only continuous in general. Then*

$$\left|d_\infty(x,y)-\hat{d}^p_\infty(x,y)\right|=o(\varepsilon)$$

for $x,y\in\operatorname{Box}(p,\varepsilon)$ as $\varepsilon\to0$ and $o(\varepsilon)$ is uniform on some neighborhood of p.

The convergence of vector fields to homogeneous ones in coordinates of the first kind is obtained for $C^{1,\alpha}$-smooth vector fields in [9] and for C^1-smooth vector fields in [11]. The convergence of d_∞ to a homogeneous quasimetric of the local group for C^1-smooth vector fields is established in [12]. Observe that the coordinate system θ_p is just C^1-smooth for C^1-smooth vector fields.

The above results are obtained in the canonical coordinates of the first kind. However, it is convenient in some problems to use other coordinate systems, for instance, the canonical coordinate system of the second kind

$$(x_1, \dots, x_N) \mapsto \exp(x_N X_N) \circ \dots \circ \exp(x_1 X_1)(p)$$

is much used in [10].

In this regard, the following question arises: What conditions must the coordinate system satisfy for the items of Theorem 1 to hold? Such a condition is stated in [13] for smooth sub-Riemannian spaces. The class of privileged smooth coordinate systems for smooth spaces is described in [14]. Here we give the following simple geometric criterion for a smooth coordinate system to be privileged.

Theorem 3. *The analogs of items of Theorem 1 hold in the coordinates ϕ_p in a neighborhood of a point p if and only if*

$$\phi_p(\mathrm{Box}(0, C_1\varepsilon)) \subset B_{d_{cc}}(p, \varepsilon) \subset \phi_p(\mathrm{Box}(0, C_2\varepsilon))$$

for some $0 < C_1 \leq C_2 < \infty$ and all $0 < \varepsilon \leq \varepsilon_0$. Here $\mathrm{Box}(0, r) = \{x \in \mathbb{R}^N : |x_k|^{\sigma_k} \leq r\}$.

This article presents an independent proof of Theorem 3 in the equiregular C^m-smooth case (see Section 4); however, as Remark 7 implies, for vector fields of lower smoothness the claim is false. We describe the classes of coordinate systems ϕ_p in which some partial analogs of Theorem 1 are satisfied for Carnot–Carathéodory spaces with vector fields of lower smoothness.

Since convergence theorems are established in the canonical coordinates θ_p of the first kind, we describe these classes in terms of the *transition function* $\Phi_p = \phi_p^{-1} \circ \theta_p$; i. e., we obtain the conditions on the transition function in the new coordinates under which it preserves the claims of Theorem 1. In the smooth case, Theorem 3 implies the necessary and sufficient condition on the transition function Φ as follows:

$$\mathrm{Box}(0, C_1\varepsilon) \subset \Phi(\mathrm{Box}(0, \varepsilon)) \subset \mathrm{Box}(0, C_2\varepsilon) \tag{4}$$

for some $0 < C_1 \leq C_2 < \infty$ and all $\varepsilon \in (0, \varepsilon_0)$. Section 2 shows that in the case of insufficient smoothness of Φ condition (4) is still necessary (Theorem 6) but fails to be sufficient (Remark 7). Furthermore, in Section 2 we obtain a sufficient condition under which the homogeneous limit of quasimetrics is preserved in the new coordinates (Theorem 9). Namely: Assume that

(1) Φ is a homeomorphism;

(2) the limit

$$L(x) := \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x) \tag{5}$$

exists and is uniform on some neighborhood of the origin, where $\delta_\varepsilon(x_1, \dots, x_N) = (\varepsilon^{\sigma_1} x_1, \dots, \varepsilon^{\sigma_N} x_N)$;

(3) L is also a homeomorphism.

Then in the new coordinate system the homogeneous limit of quasimetrics exists (similar to the limit of metrics in Theorem 1) and L is an isometry between the limit of quasimetrics in the original and new coordinate systems.

Section 3 shows that (5) is insufficient for the existence in the new coordinates of homogeneous limits of the basis vector fields (Remark 12). Then we obtain some sufficient condition (Theorem 14). Namely: Assume that

(1) Φ is a C^1-diffeomorphism;

(2) the uniform limit

$$\lambda(x) := \lim_{\varepsilon \to 0} D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon(x) \tag{6}$$

exists;

(3) $\det \lambda(0) \neq 0$.

Then in the new coordinates the homogeneous limit of vector fields exists, as in Theorem 1. Furthermore, λ is an isomorphism of homogeneous algebras between the limits in the original and new coordinate systems.

In Section 4 we prove that in the case of $\Phi \in C^m$, where m is the depth of the space, conditions (5) and (6) are equivalent to (4). The examples in Remarks 7 and 12 show that in the case of lower smoothness of the transition function all three conditions are distinct.

In Section 5 we prove that some coordinate systems, including the canonical coordinates of the second kind, satisfy (5) in the C^1-smooth case and (6) in the C^m-smooth case.

2. Homogeneous Approximation of Quasimetric Spaces

Definition 4. Given a neighborhood $U \subset \mathbb{R}^N$ of the origin, refer as a *quasimetric* on U to a function $d : U \times U \to \mathbb{R}$ such that

- d is continuous;
- $d(x, y) \geq 0$ for all $x, y \in U$ and $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) \leq Cd(y, x)$ for all $x, y \in U$ and some $C \geq 1$;
- $d(x, z) \leq Q(d(x, y) + d(y, z))$ for all $x, y, z \in U$ and some $Q \geq 1$.

The pair (U, d) is called a *quasimetric space*.

Definition 5. Given a tuple $(\sigma_1, \dots, \sigma_N)$ of positive reals, introduce in $\mathbb{R}^N$ the one-parameter group of *dilations*

$$\delta_\varepsilon(x_1, \dots, x_N) = (\varepsilon^{\sigma_1} x_1, \dots, \varepsilon^{\sigma_N} x_N), \quad \varepsilon > 0.$$

Definition 6. Define the *δ_ε-homogeneous quasinorm*

$$\|x\| = \|(x_1, \dots, x_N)\| = \max_{k=1,\dots,N} |x_k|^{\frac{1}{\sigma_k}}$$

on $\mathbb{R}^N$. Denote the set of $x \in \mathbb{R}^N$ with $\|x\| < r$ by $\operatorname{Box}(r)$. Observe that $\delta_\varepsilon \operatorname{Box}(r) = \operatorname{Box}(\varepsilon r)$.

Definition 7. A quasimetric $\hat{d}$ on $\mathbb{R}^N$ is called *δ_ε-homogeneous* whenever

$$\hat{d}(\delta_\varepsilon x, \delta_\varepsilon y) = \varepsilon\, d(x, y)$$

for all $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$. The triple $(\mathbb{R}^N, \delta_\varepsilon, \hat{d})$ is called a *δ_ε-homogeneous quasimetric space*.

Definition 8. Say that $(\mathbb{R}^N, \delta_\varepsilon, \hat{d})$ is a *δ_ε-homogeneous approximation* to (U, d) if the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(\delta_\varepsilon x, \delta_\varepsilon y) = \hat{d}(x, y) \tag{7}$$

exists and is uniform in x and y in some neighborhood of the origin. If the limit exists then we say that (U, d) *admits* a δ_ε-homogeneous approximation.

Remark 4. In the terminology of [15] the space $(\mathbb{R}^N, \hat{d})$ is a *local tangent cone* to the quasimetric space (U, d). This generalizes the concept of tangent cone to a metric space which is introduced by Gromov [16]. We avoid this terminology because the tangent cone is defined up to isometry. The approach here is a slightly more naive and distinguishes homogeneous approximations in distinct coordinate systems even if the resulting spaces are isometric.

Lemma 5. *If a quasimetric d admits a δ_ε-homogeneous approximation then there exist constants $0 < C_1 \leq C_2 < \infty$ and $r_0 > 0$ such that*

$$C_1 \|x\| \leq d(0, x) \leq C_2 \|x\|$$

for all $x \in \operatorname{Box}(r_0)$.

Proof. Suppose that a δ_ε-homogeneous quasimetric $\hat{d}$ is a δ_ε-homogeneous approximation to d. Then there exists $r_0 > 0$ such that

$$\frac{1}{\varepsilon} d(0, \delta_\varepsilon x) \to \hat{d}(0, x) \quad \text{as } \varepsilon \to 0 \tag{8}$$

uniformly in $x \in \operatorname{Box}(2r_0)$.

Put

$$m = \inf\{\hat{d}(0, v) : \|v\| = r_0\}, \quad M = \sup\{\hat{d}(0, v) : \|v\| = r_0\}.$$

There exists $\varepsilon_1 > 0$ such that

$$\frac{m}{2} < \frac{1}{\varepsilon} d(0, \delta_\varepsilon v) < 2M$$

for all $v \in \partial\,\mathrm{Box}(r_0)$ and $\varepsilon < \varepsilon_1$. Hence,

$$\frac{m}{2r_0}\|x\| = \frac{m}{2}\varepsilon < d(0, x) < 2M\varepsilon = \frac{2M}{r_0}\|x\|$$

for all $\varepsilon < \min\{\varepsilon_1, 2\}$ and $x \in \partial\,\mathrm{Box}(\varepsilon r_0)$. The proof of Lemma 5 is complete. $\square$

Theorem 6 (a necessary condition for homogeneous approximation in the new coordinates). *Consider a quasimetric space (U, d) admitting a δ_ε-homogeneous approximation and a homeomorphism $\Phi : U \to \Phi(U)$ with $\Phi(0) = 0$. Define the quasimetric ρ on $\Phi(U)$ by putting*

$$\rho(u, v) = d(\Phi^{-1}(u), \Phi^{-1}(v)).$$

If $(\Phi(U), \rho)$ admits a δ_ε-homogeneous approximation then there exist $\varepsilon_0 > 0$ and $0 < C_1 \le C_2 < \infty$ such that

$$\mathrm{Box}(C_1\varepsilon) \subset \Phi(\mathrm{Box}(\varepsilon)) \subset \mathrm{Box}(C_2\varepsilon) \tag{9}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. By Lemma 5, for the quasimetrics d and ρ there exist positive constants r_1, r_2, c_1, c_2, c_3, and c_4 such that

$$c_1\|x\| \le d(0, x) \le c_2\|x\|, \quad c_3\|y\| \le \rho(0, y) \le c_4\|y\|$$

for all $x \in \mathrm{Box}(r_1)$ and $y \in \mathrm{Box}(r_2)$. Since Φ is a homeomorphism of a neighborhood of the origin, there exist positive $r_3 \le r_2$ and $r_4 \le r_1$ such that $\mathrm{Box}(r_3) \subset \Phi^{-1}(\mathrm{Box}(r_2))$ and $\mathrm{Box}(r_4) \subset \Phi(\mathrm{Box}(r_3))$. Then

$$\frac{c_1}{c_4}\|x\| \le \frac{1}{c_4} d(0, x) = \frac{1}{c_4}\rho(0, \Phi(x)) \le \|\Phi(x)\| \le \frac{1}{c_3}\rho(0, \Phi(x)) = \frac{1}{c_3} d(0, x) \le \frac{c_2}{c_3}\|x\|$$

for all $x \in \mathrm{Box}(r_4)$. This implies the claim. $\square$

Remark 7. To show that the condition in Theorem 6 is insufficient in general, take the plane $\mathbb{R}^2$ with coordinates (x, y), the dilation $\delta_\varepsilon(x, y) = (\varepsilon x, \varepsilon^2 y)$, and the δ_ε-homogeneous metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + |y_1 - y_2|}.$$

Consider the coordinate change $\Phi(x, y) = (x, y + f(x))$ with

$$f(x) = \begin{cases} \frac{x^2}{2}\sin\frac{1}{|x|^{1-\beta}}, & x \ne 0, \\ 0, & x = 0, \end{cases}$$

where $\beta \in (0, 1)$. Then $f \in C^{1,\beta} \setminus C^2$ and $f'(0) = 0$. Observe that $D\Phi(0) = \mathrm{Id}$; consequently, Φ is a $C^{1,\beta}$-diffeomorphism of neighborhoods of the origin. Moreover, Φ satisfies the estimate

$$\frac{1}{2}(|x|^2 + |y|) \le \frac{|x|^2}{2} + |y| \le |\Phi_1(x, y)|^2 + |\Phi_2(x, y)| \le \frac{3}{2}|x|^2 + |y| \le \frac{3}{2}(|x|^2 + |y|).$$

However, it is not difficult to verify that the metric $\rho(u, v) = d(\Phi(u), \Phi(v))$ admits no δ_ε-homogeneous approximation. Indeed,

$$\frac{1}{\varepsilon} d(\Phi(\varepsilon x_1, \varepsilon^2 y_1), \Phi(\varepsilon x_2, \varepsilon^2 y_2)) = \sqrt{(x_1 - x_2)^2 + |y_1 - y_2 - \tfrac{1}{\varepsilon^2}(f(\varepsilon x_1) - f(\varepsilon x_2))|},$$

where the expression

$$\frac{1}{\varepsilon^2}(f(\varepsilon x_1) - f(\varepsilon x_2)) = \frac{x_1}{2}\sin\frac{1}{|\varepsilon x_1|^{1-\beta}} - \frac{x_2}{2}\sin\frac{1}{|\varepsilon x_2|^{1-\beta}}$$

lacks any limit as $\varepsilon \to 0$ for $x_1 \ne x_2$.

It is possible to construct a similar example for the functions of class $C^{1,1}$. To this end we may consider, for instance, $f(x) = \int_0^x t \sin\frac{1}{t}\,dt$.

However, the hypotheses of Theorem 6 are sufficient for the mappings of class C^2 on this metric space, as we verify in Lemma 16.

Lemma 8. *Given a neighborhood $U \subset \mathbb{R}^N$ of the origin and a continuous mapping $\Phi : U \to \mathbb{R}^N$, for the uniform limit*

$$L(x) := \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x) \tag{10}$$

to exist on some neighborhood of the origin, it is necessary and sufficient that there exist a continuous δ_ε-homogeneous mapping $L : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\Phi_k(x) = L_k(x) + o(\varepsilon^{\sigma_k}), \quad k = 1, \dots, N, \tag{11}$$

as $\varepsilon \to 0$ and $x \in \mathrm{Box}(\varepsilon)$ for all coordinate functions L_k, and Φ_k, with $k = 1, \dots, N$. Here the coefficients σ_k for $k = 1, \dots, N$ are from Definition 5 of the dilation δ_ε.

Under each of these conditions, if Φ and L are homeomorphisms then the limit

$$\lim_{\varepsilon\to 0} \delta_\varepsilon^{-1} \circ \Phi^{-1} \circ \delta_\varepsilon(y) = L^{-1}(y)$$

also exists and is uniform on some neighborhood of the origin.

Proof. Suppose that (10) exists and is uniform in $x \in \mathrm{Box}(r_0)$. In this case the limit mapping L is continuous and

$$L_k(\delta_t x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\sigma_k}} \Phi_k(\delta_\varepsilon \delta_t x) = \lim_{\varepsilon\to 0} t^{\sigma_k} \frac{1}{(t\varepsilon)^{\sigma_k}} \Phi_k(\delta_{t\varepsilon} x) = t^{\sigma_k} L_k(x) \tag{12}$$

for all $t \in (0, 1]$; i. e., $\delta_t \circ L = L \circ \delta_t$. We can extend L to a continuous δ_ε-homogeneous mapping on the whole $\mathbb{R}^N$. Then

$$\Phi_k(\delta_\varepsilon x) = \varepsilon^{\sigma_k}(\varepsilon^{-\sigma_k}\Phi_k(\delta_\varepsilon(x))) = \varepsilon^{\sigma_k}(L_k(x) + o(1)) = L_k(\delta_\varepsilon x) + o(\varepsilon^{\sigma_k})$$

for all $x \in \mathrm{Box}(r_0)$. Conversely, suppose that (11) holds. Fix $r_0 > 0$ with $\overline{\mathrm{Box}}(r_0) \subset U$. Then

$$\frac{1}{\varepsilon^{\sigma_k}} \Phi_k(\delta_\varepsilon x) = \frac{1}{\varepsilon^{\sigma_k}} (L_k(\delta_\varepsilon x) + o(\varepsilon^{\sigma_k})) = L_k(x) + o(1) \tag{13}$$

as $\varepsilon \to 0$ uniformly in $x \in \mathrm{Box}(r_0)$. Thus, (10) and (11) are equivalent.

Furthermore, assume that Φ and L are homeomorphisms. Since L is continuous and δ_ε-homogeneous, it follows that

$$M = \sup_{x\neq 0} \frac{\|L(x)\|}{\|x\|} = \sup_{x\neq 0} \bigl\|\delta_{\|x\|}^{-1} L(x)\bigr\| = \sup_{x\neq 0} \bigl\|L\bigl(\delta_{\|x\|}^{-1} x\bigr)\bigr\| = \sup_{\|v\|=1} \|L(v)\| < \infty.$$

Since $L(x) \neq 0$ for $x \neq 0$, similarly we obtain

$$m = \sup_{x\neq 0} \frac{\|x\|}{\|L(x)\|} = \sup_{\|v\|=1} \frac{1}{\|L(v)\|} < \infty.$$

Therefore,

$$\frac{1}{m}\|x\| \le \|L(x)\| \le M\|x\|, \quad \frac{1}{M}\|y\| \le \|L^{-1}(y)\| \le m\|y\|$$

for all $x, y \in \mathbb{R}^N$. Consequently, there is a neighborhood V of the origin such that

$$\frac{1}{2m}\|x\| \le \|\Phi(x)\| \le 2M\|x\|, \quad \frac{1}{2M}\|y\| \le \|\Phi^{-1}(y)\| \le 2m\|y\|$$

for all $x \in V$ and $y \in \Phi(V)$. Put $\Phi^\varepsilon(x) = \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x)$ and take $r_1 \le r_0$ with $\mathrm{Box}(r_1) \subset V$ and $\mathrm{Box}\bigl(\frac{r_1}{2m}\bigr) \subset \Phi(V)$. Then

$$\|(\Phi^\varepsilon)^{-1}(y)\| = \frac{1}{\varepsilon}\|\Phi^{-1}(\delta_\varepsilon y)\| \le \frac{2m}{\varepsilon}\|\delta_\varepsilon y\| = 2m\|y\|$$

for all $y \in \mathrm{Box}\bigl(\frac{r_1}{2m}\bigr)$ and $\varepsilon > 0$; i. e., $(\Phi^\varepsilon)^{-1}\bigl(\mathrm{Box}\bigl(\frac{r_1}{2m}\bigr)\bigr) \subset \mathrm{Box}(r_1)$. From (13) we then infer that $\|\Phi^\varepsilon(x) - L(x)\| = o(1)$ as $\varepsilon \to 0$ uniformly in $x \in \mathrm{Box}(r_0)$. Hence,

$$y - L \circ (\Phi^\varepsilon)^{-1}(y) = \Phi^\varepsilon((\Phi^\varepsilon)^{-1}(y)) - L((\Phi^\varepsilon)^{-1}(y)) = o(1)$$

as $\varepsilon \to 0$ uniformly in $y \in \mathrm{Box}\bigl(\frac{r_1}{2m}\bigr)$. Since the continuous mapping L^{-1} is uniformly continuous on $\overline{\mathrm{Box}}\bigl(\frac{r_1}{2m}\bigr)$; therefore,

$$L^{-1}(y) - (\Phi^\varepsilon)^{-1}(y) = L^{-1}(y) - L^{-1}(L \circ (\Phi^\varepsilon)^{-1}(y)) = o(1)$$

as $\varepsilon \to 0$ uniformly on $\mathrm{Box}\bigl(\frac{r_1}{2m}\bigr)$. The proof of Lemma 8 is complete. $\square$

Theorem 9 (a sufficient condition for homogeneous approximation in the new coordinates). *Given a neighborhood $U \subset \mathbb{R}^N$ of the origin, consider a continuous quasimetric d on U and a quasimetric $\hat{d}$ that is the δ_ε-homogeneous approximation of d. Let $\Phi : U \to \mathbb{R}^N$ be a homeomorphism onto a neighborhood of the origin such that there exists a δ_ε-homogeneous homeomorphism $L : \mathbb{R}^N \to \mathbb{R}^N$ satisfying the condition*

$$L(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x)$$

as $\varepsilon \to 0$ uniformly in $x \in \mathrm{Box}(r_0)$. Consider the quasimetric space $(\Phi^{-1}(U), \rho)$, where $\rho(u, v) = d(\Phi(u), \Phi(v))$. Then

(1) *the limit*

$$\hat{\rho}(u, v) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \rho(\delta_\varepsilon u, \delta_\varepsilon v)$$

exists for all $u, v \in \mathbb{R}^N$ and is uniform on some neighborhood $V \subset \Phi(U)$ of the origin;

(2) *$\hat{\rho}(u, v)$ is a continuous δ_ε-homogeneous quasimetric on $\mathbb{R}^N$;*

(3) *L is a δ_ε-homogeneous isometry between $(\mathbb{R}^N, \hat{d})$ and $(\mathbb{R}^N, \hat{\rho})$; i. e.,*

$$\delta_\varepsilon L(x) = L(\delta_\varepsilon x), \quad \hat{\rho}(x, y) = \hat{d}(L(x), L(y))$$

for all $x, y \in \mathbb{R}^N$.

Proof. Take $\Phi^\varepsilon = \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon$. Then

$$\rho(\delta_\varepsilon u, \delta_\varepsilon v) = d(\Phi(\delta_\varepsilon u), \Phi(\delta_\varepsilon v)) = d(\delta_\varepsilon \circ \Phi^\varepsilon(u), \delta_\varepsilon \circ \Phi^\varepsilon(v)).$$

Lemma 8 yields

$$\begin{aligned}\frac{1}{\varepsilon} \rho(\delta_\varepsilon u, \delta_\varepsilon v) &= \frac{1}{\varepsilon} d(\delta_\varepsilon \circ \Phi^\varepsilon(u), \delta_\varepsilon \circ \Phi^\varepsilon(v)) = \hat{d}(\Phi^\varepsilon(u), \Phi^\varepsilon(v)) + o(1) \\ &= \hat{d}(L(u) + o(1), L(v) + o(1)) + o(1)\end{aligned}$$

as $\varepsilon \to 0$, where all $o(1)$ are uniform in u and v in a neighborhood of the origin. Therefore,

$$\hat{\rho}(u, v) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \rho(\delta_\varepsilon u, \delta_\varepsilon v) = \hat{d}(L(u), L(v)).$$

Since L^{-1} is a homeomorphism, it follows that $\hat{\rho}$ is also a quasimetric on $\mathbb{R}^N$. Moreover,

$$\hat{\rho}(\delta_t u, \delta_t v) = \hat{d}(L(\delta_t u), L(\delta_t v)) = \hat{d}(\delta_t \circ L(u), \delta_t \circ L(v)) = t\hat{d}(L(u), L(v)) = t\hat{\rho}(u, v)$$

for all $t > 0$ and $u, v \in \mathbb{R}^N$. The proof of Theorem 9 is complete. □

Remark 10. In Theorem 9, the condition that the uniform limit exists is unnecessary. We can construct an example of Φ such that (10) lacks any limit, although the quasimetrics converge in the new coordinate system. Consider $\mathbb{C}$ with the Euclidean metric $d(z, w) = |z - w|$ and homothety as the dilation $\delta_\varepsilon(z) = \varepsilon z$ for $\varepsilon > 0$.

Define the mapping $\Phi : \mathbb{C} \to \mathbb{C}$ as

$$\Phi(re^{i\theta}) = r\, e^{i(\theta + \log r)}, \quad \Phi(0) = 0,$$

where $\theta \in [0, 2\pi]$. This mapping Φ is continuous because $\Phi(re^{0i}) = \Phi(re^{2\pi i})$ and $\Phi(re^{i\theta}) \to 0$ as $r \to 0$. Furthermore, the metric $d(z, w) = |\Phi(z) - \Phi(w)|$ is homogeneous:

$$\begin{aligned}\frac{1}{\varepsilon} |\Phi(\varepsilon r_1 e^{i\theta_1}) - \Phi(\varepsilon r_2 e^{i\theta_2})| &= \frac{1}{\varepsilon} |\varepsilon r_1 e^{i\theta_1 + i\log(\varepsilon r_1)} - \varepsilon r_2 e^{i\theta_2 + i\log(\varepsilon r_2)}| \\ &= |e^{i\log\varepsilon}||r_1 e^{i\theta_1 + i\log r_1} - r_2 e^{i\theta_2 + i\log r_2}| = |\Phi(r_1 e^{i\theta_1}) - \Phi(r_2 e^{i\theta_2})|.\end{aligned}$$

However, (10) for Φ is as follows:

$$\frac{1}{\varepsilon}\Phi(\varepsilon z)=\frac{1}{\varepsilon}\Phi(\varepsilon re^{i\theta})=re^{i\theta+i\log(\varepsilon r)}=re^{i\theta+i\log r}e^{i\log\varepsilon}=\Phi(z)e^{i\log\varepsilon}$$

and so there is no limit as $\varepsilon\to 0$. Here $|z-w|$ and $|\Phi(z)-\Phi(w)|$ are not isometric.

Let us apply the results of this section to Carnot–Carathéodory spaces. Consider an equiregular C^1-smooth space $\mathbb{M}$. In a neighborhood U of $p\in\mathbb{M}$ choose a basis $X_1,\dots,X_N$ subordinate to (3). Recall that, using the family

$$\theta_x(u_1,\dots,u_N)=\exp(u_1X_1+\dots+u_NX_N)(x),\quad x\in U,$$

of the canonical coordinates of the first kind, we define the quasimetric $d_\infty(x,y)=\max_{k=1,\dots,N}|u_k|^{\frac{1}{\sigma_k}}$ and the family of dilations

$$\Delta^p_\varepsilon:\theta_p(u_1,\dots,u_N)\mapsto\theta_p(\varepsilon^{\sigma_1}u_1,\dots,\varepsilon^{\sigma_N}u_N).$$

Theorems 2 and 9 imply the following statement:

Corollary 11. *Consider an equiregular Carnot–Carathéodory space $\mathbb{M}$ with C^1-smooth vector fields, a basis $X_1,\dots,X_N$ in a neighborhood of $p\in\mathbb{M}$ subordinate to* (3), *the canonical coordinate system θ_p of the first kind* (2), *and a homeomorphism $\phi_p:U\subset\mathbb{R}^N\to\mathbb{M}$ of a neighborhood of the origin onto a neighborhood of p. Define the family of dilations*

$$\widetilde{\Delta}^p_\varepsilon:\phi_p(x_1,\dots,x_N)\mapsto\phi_p(\varepsilon^{\sigma_1}x_1,\dots,\varepsilon^{\sigma_N}x_N).$$

If the limit

$$L^p(x):=\lim_{\varepsilon\to0}\delta_\varepsilon^{-1}\circ\phi_p^{-1}\circ\theta_p\circ\delta_\varepsilon(x)$$

exists and is uniform on some neighborhood of the origin and L^p is a homeomorphism then the limit

$$\tilde{d}^p_\infty(x,y)=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}d_\infty\bigl(\widetilde{\Delta}^p_\varepsilon x,\widetilde{\Delta}^p_\varepsilon y\bigr)$$

exists and is uniform on some neighborhood of the origin and $\tilde{d}^p_\infty$ is a $\widetilde{\Delta}^p_\varepsilon$-homogeneous quasimetric isometric to $\hat{d}^p_\infty$. The isometry is given by the mapping $\mathscr{L}^p=\phi_p\circ L^p\circ\theta_p^{-1}$: $\hat{d}^p_\infty(x,y)=\tilde{d}^p_\infty(\mathscr{L}^px,\mathscr{L}^py)$.

Remark 12. In Corollary 11, if $\mathscr{L}^p$ is a C^1-diffeomorphism then we can define the vector fields $\widetilde{X}^p_j=\mathscr{L}^p_*\widehat{X}^p_j$. These are homogeneous under dilation in the new coordinates; however, in general we cannot assert that they are homogeneous limits of the vector fields Φ_*X_j. Consider, for instance, $\mathbb{R}^2_{x,y}$ with the collection of vector fields $\bigl\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\bigr\}$, the dilation $\delta_\varepsilon(x,y)=(\varepsilon x,\varepsilon^2y)$, and the transition mapping $\Phi(x,y)=(x,y+f(x))$, where

$$f(x)=\begin{cases}x^3\sin\frac{1}{x}, & x\neq0,\\ 0, & x=0.\end{cases}$$

Then Φ is a $C^{1,1}$-diffeomorphism of neighborhoods of the origin and

$$\delta_\varepsilon^{-1}\circ\Phi\circ\delta_\varepsilon(x,y)=\begin{pmatrix}x\\ y+\varepsilon x^3\sin\frac{1}{\varepsilon x}\end{pmatrix}\to\begin{pmatrix}x\\ y\end{pmatrix}$$

as $\varepsilon\to0$. However, $\Phi_*\frac{\partial}{\partial x}=\frac{\partial}{\partial x}+\bigl(3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}\bigr)\frac{\partial}{\partial y}$, and the expression

$$(\delta_\varepsilon)_*^{-1}\varepsilon\Phi_*\frac{\partial}{\partial x}(\varepsilon x,\varepsilon^2y)=\frac{\partial}{\partial x}+\left(3\varepsilon x^2\sin\frac{1}{\varepsilon x}-x\cos\frac{1}{\varepsilon x}\right)\frac{\partial}{\partial y}$$

lacks any limit as $\varepsilon\to0$ for $x\neq0$.

3. Homogeneous Approximation to Vector Fields

In this section we present a sufficient condition on the transition function Φ for which there exist homogeneous approximations to basis vector fields of a Carnot–Carathéodory space in the new coordinates.

Definition 9. Assume that a dilation δ_ε is prescribed in a neighborhood $U \subset \mathbb{R}^N$ of the origin. Say that a continuous vector field X on U *admits a δ_ε-homogeneous approximation of degree* r whenever the limit

$$\widehat{X}(x) := \lim_{\varepsilon\to 0}(\delta_\varepsilon^{-1})_*\varepsilon^r X(\delta_\varepsilon x) \tag{14}$$

exists and is uniform on some neighborhood of the origin. Observe that in this case the vector field $\widehat{X}$ is δ_ε-homogeneous of degree r.

Lemma 13. *Given $\Phi \in C^1(U, \mathbb{R}^N)$, for the uniform limit*

$$\lambda(x) := \lim_{\varepsilon\to 0} D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon(x) \tag{15}$$

to exist on some neighborhood of the origin, it is necessary and sufficient that for all $k, l \in \{1, \dots, N\}$ with $\sigma_k > \sigma_l$ there are some continuous functions $\lambda_{kl} : \mathbb{R}^N \to \mathbb{R}$ δ_ε-homogeneous of degree $\sigma_k - \sigma_l$ and satisfying the condition

$$\frac{\partial\Phi_k}{\partial x_l}(x) = \lambda_{kl}(x) + o(\varepsilon^{\sigma_k-\sigma_l}) \tag{16}$$

as $\varepsilon \to 0$, where all $o(\cdot)$ are uniform in $x \in \mathrm{Box}(\varepsilon)$.

Under these hypotheses, if $\Phi(0) = 0$ then the uniform limit

$$L(x) := \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x) \tag{17}$$

exists and $\lambda = DL$.

Proof. Since the matrix $D\delta_\varepsilon$ is diagonal with $\varepsilon^{\sigma_1}, \dots, \varepsilon^{\sigma_N}$ at the diagonal, we have

$$\left[D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon\right]_{kl}(x) = \varepsilon^{\sigma_l-\sigma_k}\frac{\partial\Phi_k}{\partial x_l}(\delta_\varepsilon x).$$

Necessity. Take $V = \mathrm{Box}(r_0)$ and suppose that the limit in (15) exists and is uniform on V. This is equivalent to the property that for all $k, l \in \{1, \dots, N\}$ the uniform limits of the coordinate functions

$$\lambda_{kl}(x) := \lim_{\varepsilon\to 0} \varepsilon^{\sigma_l-\sigma_k}\frac{\partial\Phi_k}{\partial x_l}(\delta_\varepsilon x), \quad x \in V,$$

exist. Then the functions λ_{kl} are continuous as the uniform limits of continuous functions and $\lambda_{kl}(\delta_t x) = t^{\sigma_k-\sigma_l}\lambda_{kl}(x)$ for $x \in V$ and $t \in (0, 1]$. We can extend λ_{kl} by homogeneity to functions on $\mathbb{R}^N$.

Sufficiency. Assume (16). Then in the case $\sigma_l \geq \sigma_k$ we have

$$\varepsilon^{\sigma_l-\sigma_k}\frac{\partial\Phi_k}{\partial x_l}(\delta_\varepsilon x) \to \begin{cases} \frac{\partial\Phi_k}{\partial x_l}(0), & \sigma_l = \sigma_k, \\ 0, & \sigma_l > \sigma_k, \end{cases}$$

as $\varepsilon \to 0$ uniformly on some compact neighborhood of the origin. For $\sigma_l < \sigma_k$ we obtain

$$\varepsilon^{\sigma_l-\sigma_k}\frac{\partial\Phi_k}{\partial x_l}(\delta_\varepsilon x) = \varepsilon^{\sigma_l-\sigma_k}\big(\lambda_{kl}(\delta_\varepsilon x) + o(\varepsilon^{\sigma_k-\sigma_l})\big) = \lambda_{kl}(x) + o(1),$$

where $o(\cdot)$ is uniform in x.

Existence of the limit in (17). Put $\Phi^\varepsilon = \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon$. If $\Phi(0) = 0$ then $\Phi^\varepsilon(0) = 0$ for all $\varepsilon > 0$. Suppose that $D\Phi^\varepsilon(x) \to \lambda(x)$ as $\varepsilon \to 0$ uniformly in $x \in \overline{\mathrm{Box}}(r_0)$. Given $\varepsilon_1, \varepsilon_2 > 0$ and $x \in \mathrm{Box}(r_0)$, we have

$$|\Phi^{\varepsilon_1}(x) - \Phi^{\varepsilon_2}(x)| = |(\Phi^{\varepsilon_1} - \Phi^{\varepsilon_2})(x) - (\Phi^{\varepsilon_1} - \Phi^{\varepsilon_2})(0)| \leq r_0 \sup_{y\in\overline{\mathrm{Box}}(r_0)} \|D(\Phi^{\varepsilon_1} - \Phi^{\varepsilon_2})(y)\|.$$

Since the family of $D\Phi^\varepsilon$ is fundamental in the uniform norm, we conclude that so is the family of Φ^ε. Hence, the uniform limit $\Phi^\varepsilon(x) \to L(x)$ as $\varepsilon \to 0$ exists for $x \in \mathrm{Box}(r_0)$. Furthermore, since $D\Phi^\varepsilon$ converge uniformly, $DL(x) = \lim_{\varepsilon\to 0} D\Phi^\varepsilon(x) = \lambda(x)$ for $x \in \mathrm{Box}(r_0)$. The proof of Lemma 13 is complete. $\square$

Theorem 14 (a sufficient condition for approximation of vector fields in the new coordinates). *Given a continuous vector field X in a neighborhood U of the origin, assume that the uniform limit*

$$\widehat{X}(x) = \lim_{\varepsilon\to 0}\bigl(\delta_\varepsilon^{-1}\bigr)_*\varepsilon^r X(\delta_\varepsilon x)$$

exists for some $r > 0$.

Take a C^1-diffeomorphism $\Phi : U \to \mathbb{R}^N$ of neighborhoods of the origin with $\Phi(0) = 0$. Suppose that the uniform limit

$$\lambda(x) := \lim_{\varepsilon\to 0} D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon(x)$$

exists in a neighborhood of the origin and $\det\lambda(0) \neq 0$*. Put $Y(y) = \Phi_* X(\Phi^{-1}(y))$. Then the limit*

$$\widehat{Y}(y) = \lim_{\varepsilon\to 0}\bigl(\delta_\varepsilon^{-1}\bigr)_*\varepsilon^r Y(\delta_\varepsilon y)$$

exists, is uniform on some neighborhood of the origin, and $\widehat{Y}(y) = L_*\widehat{X}(L^{-1}(y))$, where the mapping L is defined by (17)*.*

Proof. Indeed, by Lemma 13 there is a neighborhood of the origin in which the uniform limits

$$L(x) = \lim_{\varepsilon\to 0}\delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x), \quad L_*(x) = DL(x) = \lim_{\varepsilon\to 0}\bigl(\delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon\bigr)_*(x)$$

exist. Since $\det DL(0) = \det\lambda(0) \neq 0$, it follows that $L(x)$ is a diffeomorphism of neighborhoods of the origin. By Lemma 8, the uniform limit

$$L^{-1}(y) = \lim_{\varepsilon\to 0}\delta_\varepsilon^{-1} \circ \Phi^{-1} \circ \delta_\varepsilon(y)$$

exists in a neighborhood of the origin. Consequently, in a sufficiently small neighborhood

$$\begin{aligned}\bigl(\delta_\varepsilon^{-1}\bigr)_*\varepsilon^r Y(\delta_\varepsilon y) &= \bigl(\delta_\varepsilon^{-1}\bigr)_*\varepsilon^r\Phi_* X(\Phi^{-1}(\delta_\varepsilon y))\\ &= \bigl(\delta_\varepsilon^{-1}\circ\Phi\circ\delta_\varepsilon\bigr)_*\bigl(\delta_\varepsilon^{-1}\bigr)_*\varepsilon^r X\circ\delta_\varepsilon\bigl(\delta_\varepsilon^{-1}\circ\Phi^{-1}\circ\delta_\varepsilon(y)\bigr) \to L_*\widehat{X}(L^{-1}(y))\end{aligned}$$

as $\varepsilon \to 0$ uniformly in y. $\square$

Theorems 2 and 14 immediately yield the following statement:

Corollary 15. *Consider an equiregular Carnot–Carathéodory space $\mathbb{M}$ with C^1-smooth vector fields, a basis $X_1, \dots, X_N$ for $T\mathbb{M}$ in a neighborhood of $p \in \mathbb{M}$ which is subordinate to* (3)*, the canonical coordinate system θ_p of the first kind* (2)*, and a C^1-diffeomorphism $\phi_p : U \subset \mathbb{R}^N \to \mathbb{M}$ from a neighborhood of the origin onto a neighborhood of p. Define the family of dilations*

$$\widetilde{\Delta}_\varepsilon^p : \phi_p(x_1, \dots, x_N) \mapsto \phi_p(\varepsilon^{\sigma_1}x_1, \dots, \varepsilon^{\sigma_N}x_N).$$

If the uniform limit

$$\lambda^p(x) := \lim_{\varepsilon\to 0} D\delta_\varepsilon^{-1} \circ D\phi_p^{-1} \circ D\theta_p \circ D\delta_\varepsilon(x)$$

exists in a neighborhood of the origin and $\det\lambda^p(0) \neq 0$ *then*

(1) *the uniform limit*

$$\widetilde{X}_k^p(x) = \lim_{\varepsilon\to 0}\bigl(\widetilde{\Delta}_\varepsilon^p\bigr)_*^{-1}\varepsilon^{d_k}X_k\bigl(\widetilde{\Delta}_\varepsilon^p x\bigr)$$

exists in a neighborhood of the origin;

(2) *the hypotheses of Corollary 11 hold, while the mappings L^p and $\mathscr{L}^p$ defined in Corollary 11 are continuously differentiable and $\widetilde{X}_k^p = \mathscr{L}_*^p\widehat{X}_k^p$.*

4. Transition and Smoothness

In this section we point out conditions on the smoothness of the transition function Φ under which the necessary condition (9) is also sufficient.

We use the standard multi-index notation. If $\alpha = (\alpha_1, \dots, \alpha_N)$, where α_k are nonnegative integers for $k = 1, \dots, N$, then

$$|\alpha| = \alpha_1 + \dots + \alpha_N,\ \alpha! = \alpha_1! \cdot \ldots \cdot \alpha_N!,\ x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_N^{\alpha_N},\ D^\alpha \Phi = \frac{\partial \Phi^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.$$

Introduce also the *weight* of multi-indices as $\sigma(\alpha) = \sigma_1 \alpha_1 + \dots + \sigma_N \alpha_N$.

Lemma 16. *Consider $\Phi : U \to \mathbb{R}^N$ and for the coordinate functions of this mapping assume that $\Phi_k \in C^{\sigma_k}(U)$ for $k = 1, \dots, N$. The following are equivalent:*

(1) *There exist constants $C > 0$ and $\varepsilon_0 > 0$ such that $\Phi(\mathrm{Box}(\varepsilon)) \subset \mathrm{Box}(C\varepsilon)$ for all $0 < \varepsilon \le \varepsilon_0$.*
(2) *$\Phi_k(x) = O(\varepsilon^{\sigma_k})$ as $\varepsilon \to 0$ for $x \in \mathrm{Box}(\varepsilon)$.*
(3) *$D^\alpha \Phi_k(0) = 0$ for all multi-indices α with $\sigma(\alpha) < \sigma_k$.*
(4) *The limits*

$$L(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x), \tag{18}$$

$$DL(x) = \lim_{\varepsilon \to 0} D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon(x) \tag{19}$$

exist and are uniform on some neighborhood of the origin.

If any of these conditions hold then coordinate functions of L are polynomials.

Proof. Equivalence of items (1) and (2) is obvious: Item (2) is a coordinate expression of item (1). To show that item (3) is equivalent to (2), expand the coordinate functions of Φ into Taylor polynomials to the corresponding orders:

$$\Phi_k(x) = P_k(x) + o(|x|^{\sigma_k}) = \sum_{\alpha : |\alpha| \le \sigma_k} \frac{D^\alpha \Phi_k(0)}{\alpha!} x^\alpha + o(|x|^{\sigma_k}). \tag{20}$$

Observe that $x^\alpha = O(\varepsilon^{\sigma(\alpha)})$ for $x \in \mathrm{Box}(\varepsilon)$. Hence, $\Phi_k(x) = O(\varepsilon^{\sigma_k})$ if and only if $D^\alpha \Phi_k(0) = 0$ for all α with $\sigma(\alpha) < \sigma_k$.

Let us verify that item (3) is necessary and sufficient for the limit of (18) to exist. Indeed, using the expansion in (20), for the coordinate function Φ_k we obtain

$$\frac{1}{\varepsilon^{\sigma_k}} \Phi_k(\delta_\varepsilon x) = \sum_{\alpha : \sigma(\alpha) < \sigma_k} \frac{D^\alpha \Phi_k(0)}{\alpha! \varepsilon^{\sigma_k - \sigma(\alpha)}} x^\alpha + \sum_{\alpha : \sigma(\alpha) = \sigma_k} \frac{D^\alpha \Phi_k(0)}{\alpha!} x^\alpha + o(1).$$

This expression has a limit as $\varepsilon \to 0$ if and only if its first term vanishes; i. e., if item (3) is satisfied. In this case, the second term yields an expression for the coordinate function of the limit $L_k(x)$.

Consider the limit in (19). Since $D\delta_\varepsilon$ is the diagonal matrix with entries $\varepsilon^{\sigma_1}, \dots, \varepsilon^{\sigma_N}$ at the diagonal, we have

$$\left[D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon\right]_{kl}(x) = \varepsilon^{\sigma_l - \sigma_k} \frac{\partial \Phi_k}{\partial x_l}(\delta_\varepsilon x).$$

Since $\frac{\partial \Phi_k}{\partial x_l} \in C^{\sigma_k - 1}(U)$, it follows that

$$\frac{\partial \Phi_k}{\partial x_l}(x) = \frac{\partial P_k}{\partial x_l}(x) + o(|x|^{\sigma_k - 1}),$$

where P_k is the Taylor polynomial for Φ_k in (20). Furthermore, $\frac{\partial (x^\alpha)}{\partial x_l}$ is a δ_ε-homogeneous monomial of degree $\sigma(\alpha) - \sigma_l$. Thus,

$$\varepsilon^{\sigma_l - \sigma_k} \frac{\partial \Phi_k}{\partial x_l}(\delta_\varepsilon x) = \sum_{\alpha : \sigma(\alpha) < \sigma_k} \frac{D^\alpha \Phi_k(0)}{\alpha! \varepsilon^{\sigma_k - \sigma(\alpha)}} \frac{\partial (x^\alpha)}{\partial x_l} + \sum_{\alpha : \sigma(\alpha) = \sigma_k} \frac{D^\alpha \Phi_k(0)}{\alpha!} \frac{\partial (x^\alpha)}{\partial x_l} + o(1).$$

We infer again that this expression has a limit as $\varepsilon \to 0$ if and only if its first term vanishes; i. e., item (3) holds. In this case, the second term is $\frac{\partial L_k}{\partial x_l}(x)$. The proof of Lemma 16 is complete. $\square$

Corollary 17. *Consider an equiregular Carnot–Carathéodory space $\mathbb{M}$ of depth m with C^m-smooth vector fields. Take $p \in \mathbb{M}$ and the canonical system of coordinates θ_p of the first kind in a neighborhood of p. A C^m-smooth coordinate system ϕ_p in a neighborhood of p satisfies the hypotheses of Corollaries 11 and 15 if and only if there exist constants $0 < C_1 \le C_2 < \infty$ and $\varepsilon_0 > 0$ such that*

$$\phi_p(\mathrm{Box}(C_1\varepsilon)) \subset \theta_p(\mathrm{Box}(\varepsilon)) \subset \phi_p(\mathrm{Box}(C_2\varepsilon)) \tag{21}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Put $\Phi = \theta_p^{-1} \circ \phi_p$.

Necessity. Under the hypotheses of Corollary 11 the uniform limits

$$L(x) = \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x)$$

exist on some neighborhood of the origin. By Theorem 6, there are constants $0 < C_1 \le C_2 < \infty$ and $\varepsilon_0 > 0$ such that $\mathrm{Box}(C_1\varepsilon) \subset \Phi(\mathrm{Box}(\varepsilon)) \subset \mathrm{Box}(C_2\varepsilon)$. This directly yields (21).

Sufficiency. Suppose that (21) holds. By Lemma 16, the uniform limits

$$L(x) = \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x), \quad \lambda(x) = DL(x) = \lim_{\varepsilon\to 0} D\delta_\varepsilon^{-1} \circ D\Phi \circ D\delta_\varepsilon(x),$$

$$L^{-1}(x) = \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1} \circ \Phi^{-1} \circ \delta_\varepsilon(x), \quad \lambda^{-1}(x) = DL^{-1}(x) = \lim_{\varepsilon\to 0} D\delta_\varepsilon^{-1} \circ D\Phi^{-1} \circ D\delta_\varepsilon(x)$$

exist on some neighborhood of the origin. Furthermore, the coordinate functions of L, L^{-1}, λ, and λ^{-1} are polynomials. Thus, the requirements of Corollary 11 are met: L exists and is a homeomorphism, while the conditions of Corollary 15 are met as well: the mapping λ exists and $\det \lambda \neq 0$. $\square$

Remark 18. Examples in Remarks 7 and 12 show that the smoothness assumptions on Φ cannot be improved in general; i. e., the hypotheses of Theorems 6, 9, and 14 are equivalent for C^m-smooth mappings and differ substantially in the case of lower smoothness, for instance, for $C^{m-1,1}$.

Remark 19 (an alternative proof of Theorem 3). For Carnot manifolds with C^1-smooth vector fields we have the Ball–Box Theorem (claim 2 of Theorem 1; for a proof see [17, Theorem 8] for instance). This theorem and Corollary 17 imply Theorem 3 of the Introduction for Carnot manifolds with C^m-smooth vector fields.

5. Canonical Coordinate Systems

Consider an equiregular Carnot–Carathéodory space $\mathbb{M}$ with C^r-smooth vector fields with $r \ge 1$ and a basis $X_1, \dots, X_N$ for $T\mathbb{M}$ in a neighborhood of $p \in \mathbb{M}$ which is subordinate to (3). Split the tuple of vector fields $\{X_i\}_{i=1}^N$ into L disjoint tuples $\{X_{j,1}, \dots, X_{j,k_j}\}$, for $j = 1, \dots, L$, and consider the mapping

$$\begin{aligned}\phi_p(u_1, \dots, u_N) = \exp(u_{L,1}X_{L,1} + \dots + u_{L,k_L}X_{L,k_L}) \circ \cdots \\ \circ \exp(u_{2,1}X_{2,1} + \dots + u_{2,k_2}X_{2,k_2}) \circ \exp(u_{1,1}X_{1,1} + \dots + u_{1,k_1}X_{1,k_1})(p).\end{aligned} \tag{22}$$

Then $\phi_p \in C^k$ and $\frac{\partial \phi_p}{\partial u_i}(0) = X_i(p)$. Consequently, ϕ_p is a C^r-diffeomorphism from a neighborhood of the origin onto a neighborhood of p. The *canonical coordinate system of the second kind*

$$\theta_p^2(u_1, \dots, u_N) = \exp(u_N X_N) \circ \exp(u_{N-1}X_{N-1}) \circ \dots \circ \exp(u_1 X_1)(p)$$

is a particular case of this mapping.

Theorem 20. *Consider an equiregular Carnot–Carathéodory space $\mathbb{M}$ with C^1-smooth vector fields and $p \in \mathbb{M}$. Using* (22)*, define the one-parameter family of dilations*

$$\Delta_\varepsilon^p : \phi_p(x_1, \dots, x_N) \mapsto \phi_p(\varepsilon^{\sigma_1} x_1, \dots, \varepsilon^{\sigma_N} x_N).$$

Then the limit

$$\tilde{d}_\infty^p(x, y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d_\infty\big(\Delta_\varepsilon^p x, \Delta_\varepsilon^p y\big)$$

exists, is uniform on some neighborhood of p, and $\tilde{d}_\infty^p$ is a Δ_ε^p-homogeneous quasimetric isometric to the quasimetric $\hat{d}_\infty^p$ of Theorem 2. If $\mathbb{M}$ is a depth m space with C^m-smooth vector fields then the limit

$$\widetilde{X}_k^p(x) = \lim_{\varepsilon \to 0} \big(\Delta_\varepsilon^p\big)_*^{-1} \varepsilon^{d_k} X_k\big(\Delta_\varepsilon^p x\big)$$

exists and is uniform on some neighborhood of p, while the vector fields $\widetilde{X}_k^p$ determine the structure of a graded nilpotent Lie algebra isomorphic to the algebra of Theorem 2.

We prove this theorem using the following result:

Theorem 21 [12]. *Consider an equiregular Carnot–Carathéodory space $\mathbb{M}$ with C^1-smooth vector fields, $p \in \mathbb{M}$, and a basis $X_1, \dots, X_N$ in a neighborhood of p which is subordinate to* (3)*. If $\widehat{X}_k^p$, for $k = 1, \dots, N$, are nilpotent approximations to these vector fields constructed using the canonical coordinates of the first kind as in Theorem 2 then there exists a neighborhood U of p such that, given two absolutely continuous curves $\gamma, \hat{\gamma} : [0, 1] \to \mathbb{M}$ with $\gamma(0) = \hat{\gamma}(0) \in U$ and*

$$\dot{\gamma}(t) = \sum_{i=1}^N b_i(t) X_i(\gamma(t)), \quad \dot{\hat{\gamma}}(t) = \sum_{i=1}^N b_i(t) \widehat{X}_i^p(\hat{\gamma}(t)),$$

where the measurable functions $b_i(t)$ satisfy the condition

$$\int_0^1 |b_i(t)|\, dt < S\varepsilon^{\sigma_i}, \quad S < \infty,\ i = 1, \dots, N, \tag{23}$$

we have

$$\max\{d_\infty(\gamma(1), \hat{\gamma}(1)), \hat{d}_\infty(\gamma(1), \hat{\gamma}(1))\} \le o(1) \cdot \varepsilon,$$

where $o(1)$ is uniform on U and in all tuples $\{b_i(t)\}_{i=1}^N$ of functions satisfying (23).

Proof of Theorem 20. Take δ_ε-homogeneous vector fields $\widehat{X}_1', \dots, \widehat{X}_N'$ of Theorem 2. Define $\hat{\phi}_p' : \mathbb{R}^N \to \mathbb{R}^N$ as

$$\hat{\phi}_p'(u_1, \dots, u_N) = \exp(u_{L,1}\widehat{X}_{L,1}' + \dots + u_{L,k_L}\widehat{X}_{L,k_L}') \circ \cdots$$
$$\circ \exp(u_{2,1}\widehat{X}_{2,1}' + \dots + u_{2,k_2}\widehat{X}_{2,k_2}') \circ \exp(u_{1,1}\widehat{X}_{1,1}' + \dots + u_{1,k_1}\widehat{X}_{1,k_1}')(0).$$

Observe that $\hat{\phi}_p'$ is a C^∞-diffeomorphism onto $\mathbb{R}^N$. Since $\widehat{X}_j'$ is homogeneous of degree σ_j, for all $u, v \in \mathbb{R}^N$ we have

$$\delta_\varepsilon \circ \exp(u_1\widehat{X}_1' + \dots + u_N\widehat{X}_N')(v) = \exp(\varepsilon^{\sigma_1}u_1\widehat{X}_1' + \dots + \varepsilon^{\sigma_N}u_N\widehat{X}_N')(\delta_\varepsilon v).$$

Consequently, $\delta_\varepsilon \circ \hat{\phi}_p' = \hat{\phi}_p' \circ \delta_\varepsilon$.

Theorem 21 shows that

$$\hat{d}_\infty^p\left(\exp\left(\sum_{k=1}^N \varepsilon^{\sigma_k} u_k X_k \right)(x), \theta_p \circ \exp\left(\sum_{k=1}^N \varepsilon^{\sigma_k} u_k \widehat{X}_k' \right) \circ \theta_p^{-1}(x) \right) = o(\varepsilon)$$

for each tuple $(u_1, \dots, u_N)$ of constants, where $o(\varepsilon)$ is uniform in x in a neighborhood of p; and in $(u_1, \dots, u_N)$, in a neighborhood of the origin. Thus,

$$\hat{d}_\infty^p(\phi_p(\delta_\varepsilon u), \theta_p \circ \hat{\phi}_p'(\delta_\varepsilon u)) = o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

uniformly in u. The quasimetric $\hat{d}^p_\infty$ is homogeneous in θ_p; consequently,

$$\frac{1}{\varepsilon}\hat{d}^p_\infty\big(\phi_p(\delta_\varepsilon u), \theta_p \circ \hat{\phi}'_p(\delta_\varepsilon u)\big) = \frac{1}{\varepsilon}\hat{d}^p_\infty\big(\theta_p \circ \theta_p^{-1} \circ \phi_p(\delta_\varepsilon u), \theta_p \circ \delta_\varepsilon \circ \hat{\phi}'_p(u)\big)$$
$$= \hat{d}^p_\infty\big(\theta_p \circ \delta_\varepsilon^{-1} \circ \theta_p^{-1} \circ \phi_p(\delta_\varepsilon u), \theta_p \circ \hat{\phi}'_p(u)\big) = o(1).$$

Hence, we conclude that the limit

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \circ \theta_p^{-1} \circ \phi_p \circ \delta_\varepsilon = \hat{\phi}'_p$$

exists and is uniform on some neighborhood of the origin, and the hypotheses of Corollary 11 hold. The mapping $\mathscr{L}^p = \theta_p \circ \hat{\phi}'_p \circ \phi_p^{-1}$: $\tilde{d}^p_\infty(x,y) = \hat{d}^p_\infty(\mathscr{L}^p x, \mathscr{L}^p y)$ provides an isometry between the quasimetrics.

In the case of Carnot–Carathéodory spaces with C^m-smooth vector fields both coordinate systems θ_p and ϕ_p are also C^m-smooth; consequently, by Theorem 6 and Corollary 17 the hypotheses of Corollary 15 hold and $\mathscr{L}^p$ determines a Lie algebra isomorphism: $\widetilde{X}^p_k(x) = (\mathscr{L}^p_*)^{-1}\widehat{X}^p_k(\mathscr{L}^p x)$ for $k = 1, \dots, N$. The proof of Theorem 21 is complete. $\square$

References

1. Rashevskii P. K., "On the connectivity of two arbitrary points of a totally nonholonomic space by an admissible curve," Uchen. Zap. Mosk. Ped. Inst. Ser. Fiz.-Mat. Nauk, vol. 3, no. 2, 83–94 (1938).
2. Chow W. L., "Über systeme von linearen partiallen differentialgleichungen erster ordnung," Math. Ann., Bd 117, 98–105 (1939).
3. Rothschild L. P. and Stein E. M., "Hypoelliptic differential operators and nilpotent groups," Acta Math., vol. 137, no. 3–4, 247–320 (1976).
4. Metivier G., "Fonction spectrale et valeurs propres d'une classe d'opérateurs non elliptiques," Comm. Partial Differ. Equ., vol. 1, 467–519 (1976).
5. Nagel A., Stein E. M., and Wainger S., "Balls and metrics defined by vector fields. I: Basic properties," Acta Math., vol. 155, 103–147 (1985).
6. Gromov M., "Carnot–Carathéodory spaces seen from within," in: *Progress in Mathematics*, Birkhäuser, Basel (1996), 79–323 (Sub-Riemannian Geometry; Vol. 144).
7. Bramanti M., Brandolini L., and Pedroni M., "On the lifting and approximation theorem for nonsmooth vector fields," Indiana Univ. Math. J., vol. 59, no. 6, 2093–2138 (2010).
8. Vodopyanov S., "Geometry of Carnot–Carathéodory spaces and differentiability of mappings," in: *The Interaction of Analysis and Geometry. Contemporary Mathematics*, vol. 424, Amer. Math. Soc., Providence (2007), 247–301.
9. Karmanova M. and Vodopyanov S., "Geometry of Carnot–Carathéodory spaces, differentiability, coarea and area formulas," in: *Analysis and Mathematical Physics. Trends in Mathematics*, Birkhäuser, Basel (2009), 233–335.
10. Basalaev S. and Vodopyanov S., "Approximate differentiability of mappings of Carnot–Carathéodory spaces," Eurasian Math. J., vol. 4, no. 2, 10–48 (2013).
11. Greshnov A. V., "Proof of Gromov's theorem on homogeneous nilpotent approximation for vector fields of class C^1," Siberian Adv. Math., vol. 23, no. 3, 180–191 (2013).
12. Karmanova M. B., "Fine properties of basis vector fields on Carnot–Carathéodory spaces under minimal assumptions on smoothness," Sib. Math. J., vol. 55, no. 1, 87–99 (2014).
13. Bellaïche A., "The tangent space in sub-Riemannian geometry," in: *Progress in Mathematics*, Birkhäuser, Basel (1996), 4–84 (Sub-Riemannian Geometry; Vol. 144).
14. Choi W. and Ponge R., *Privileged Coordinates and Tangent Groupoid for Carnot Manifolds* (2020) (https://arXiv1510.05851v1[math.DG]).
15. Selivanova S. V., "The tangent cone to a quasimetric space with dilations," Sib. Math. J., vol. 51, no. 2, 313–324 (2010).
16. Gromov M., "Groups of polynomial growth and expanding maps," Inst. Hautes Etudes Sci. Publ. Math., vol. 53, no. 18, 53–73 (1981).
17. Karmanova M. and Vodopyanov S., "On local approximation theorem on equiregular Carnot–Carathéodory spaces," in: *Geometric Control Theory and Sub-Riemannian Geometry*, Springer, Cham (2014), 241–262 (Springer INdAM Ser.; Vol. 5).

S. G. Basalaev
Sobolev Institute of Mathematics
Novosibirsk State University, Novosibirsk, Russia
E-mail address: sbasalaev@gmail.com