ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 4, pp. 743–754.

THE JUNCTION PROBLEM FOR TWO WEAKLY CURVED INCLUSIONS IN AN ELASTIC BODY

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Abstract: Under study are the boundary value problems that describe the equilibria of two-dimensional elastic bodies with thin weakly curved inclusions in the presence of delamination, which means that there is a crack between the inclusions and an elastic body. Some inequality-type nonlinear boundary conditions are imposed on the crack faces that exclude mutual penetration. This puts the problems into the class of those with unknown contact area. We assume that the inclusions have a contact point, find boundary conditions at the junction point, and justify passage to infinity with respect to the rigidity parameter of the thin inclusion. In particular, we obtain and analyze limit models.

DOI: 10.1134/S003744662004014X

Keywords: boundary value problem, elastic body, inclusion, crack, junction conditions

1. Introduction. This article studies junction boundary value problems for two weakly curved inclusions in an elastic body in the presence of crack delamination. On the crack faces we impose inequality-type nonlinear boundary conditions that prevent the mutual penetration of the opposite faces. The goal of this article is to find boundary conditions at the junction point and prove the solvability of the corresponding boundary value problems. We consider the junction of two elastic inclusions as well as of an elastic inclusion and a rigid inclusion.

The last few years have brought in the extensive studies of boundary value problems for the equilibria of elastic and inelastic bodies with cracks in the framework of models with the nonlinear boundary conditions on faces [1–7]. Similar results are obtained in the case of problems of equilibria for elastic bodies with thin inclusions in the presence of crack delamination [8–14]. Among the articles dealing with junction we note [15–22] which consider the junction problem of elastic objects. The junction problems of elastic, rigid, and semirigid inclusions in elastic bodies in the presence of delamination with nonlinear boundary conditions on the crack faces can be found in [23–28]. Thin elastic inclusions were described there using the Euler–Bernoulli and Timoshenko beam models.

In the case of weakly curved inclusions, treated in this article, a whole series of features appear due to the nonzero curvature of inclusions. In particular, the structure of the displacement field for rigid weakly curved inclusions is noticeably more complicated than for rectilinear inclusions. We consider the two possible variants of mutual contact of inclusions: The first variant corresponds to breaking composite inclusion at the junction point, and the second variant is characterized by the absence of a corner point, which corresponds to the conservation of the angle between the inclusions at this point during the deformation.

2. Statement of the equilibrium problem. The corner case. Let us state the equilibrium problem for an elastic body with two thin elastic weakly curved inclusions. Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary Γ; for $\gamma = \gamma_1 \cup \gamma_2 \cup \{(0,0)\}$ put $\Omega_\gamma = \Omega \setminus \bar{\gamma}$, where

$$\gamma_i = \{(x_1, x_2) \,|\, x_2 = \varphi(x_1), x_1 \in s_i\}, \quad s_1 = (-1, 0),\ s_2 = (0, 1),$$

and $\varphi : (-1, 1) \to \mathbb{R}$ is a given function with $\varphi(0) = 0$ of smoothness

$$\varphi \in H^1(s_i), \quad \varphi_{,11} \in L^\infty(s_i), \quad i = 1, 2.$$

The work is supported by the Mathematical Center in Akademgorodok under Agreement no. 075–15–2019–1613 with the Ministry of Science and Higher Education of the Russian Federation.

Original article submitted December 4, 2019; revised December 17, 2019; accepted December 25, 2019.

Assume that the median lines of the thin elastic inclusions coincide with γ_i. Therefore, the inclusions touch at $(0, 0)$. The elastic body occupies Ω_γ. Assume that γ has a corner point at $(0, 0)$. This term means that actually we have the two inclusions γ_1 and γ_2 unrelated to each other at $(0, 0)$. In other words, if we remove the surrounding elastic body then two inclusions γ_1 and γ_2 can move independently of each other.

To describe weakly curved inclusions, we use the Euler–Bernoulli model. For suitable models of elastic hollow shells, as well as the corresponding one-dimensional weakly curved beams; see [29, Chapter 1] for instance. Denote the unit normal to γ by $\nu = (\nu_1, \nu_2)$ and put $\tau = (\nu_2, -\nu_1)$. Denote by $B = \{b_{ijkl}\}$, for $i, j, k, l = 1, 2$, the available positive definite elasticity tensor:

$$b_{ijkl} = b_{jikl} = b_{klij},\ i, j, k, l = 1, 2; \quad b_{ijkl} \in L^\infty(\Omega),$$

$$b_{ijkl}\varphi_{ij}\varphi_{kl} \geq c_0|\varphi|^2 \quad \text{for all } \varphi_{ji} = \varphi_{ij},\ c_0 = \text{const} > 0.$$

There is summation implied over the repeating indices. Henceforth we assume all quantities with two lower indices symmetric with respect to these indices. Denote by $f = (f_1, f_2) \in L^2(\Omega)^2$ the given vector of exterior forces acting on the elastic body; by $k \in L^\infty(s_i)$, the known curvature of the median curves of the thin inclusions which is equal to $\varphi_{,11}(1 + (\varphi_{,1})^2)^{-3/2}$.

Suppose that the positive side, relative to the normal ν, of the inclusion γ delaminates, forming thus a crack between the elastic body and the inclusion. On the crack faces we impose inequality-type boundary conditions that ensure the mutual nonpenetration of the faces. To simplify the statement of junction boundary conditions at $(0, 0)$, assume that $\varphi_{,1}(0+) = 0$. This assumption preserves the generality of the result because we can always choose some coordinate system with $\varphi_{,1}(0+) = 0$.

The statement of the equilibrium problem for an elastic body with inclusions γ_1 and γ_2 is as follows: Find the displacement vector $u = (u_1, u_2)$ and the stress tensor $\sigma = \{\sigma_{ij}\}$ for $i, j = 1, 2$ defined on Ω_γ, as well as the displacements v and w of points of the thin inclusions defined on $s_1 \cup s_2$ such that

$$-\operatorname{div}\sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{2.1}$$

$$v_{,1111} + k(w_{,1} + kv) = [\sigma_\nu]p \quad \text{on } s_i, \quad i = 1, 2, \tag{2.2}$$

$$-w_{,11} - (kv)_{,1} = [\sigma_\tau]p \quad \text{on } s_i, \quad i = 1, 2, \tag{2.3}$$

$$u = 0 \text{ on } \Gamma; \quad v_{,11} = v_{,111} = w_{,1} + kv = 0, \quad x_1 = -1, 1, \tag{2.4}$$

$$[u_\nu] \geq 0, \quad v = u_\nu^-, \quad w = u_\tau^- \quad \text{on } \gamma, \tag{2.5}$$

$$\sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma, \tag{2.6}$$

$$[v(0)]_p = [w(0)]_p = \ [v_{,111}(0)]^p = [(w_{,1} + kv)(0)]^p = 0, \quad v_{,11}(0\pm) = 0. \tag{2.7}$$

Here $[h] = h^+ - h^-$ is the jump of h on γ, and $h^\pm$ are the values of h on the positive and negative sides of the cut γ in accordance with the chosen direction of the normal ν. Furthermore, $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the deformation tensor $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ for $i, j = 1, 2$; $\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j)$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_\tau = \sigma_{ij}\nu_j\tau_i$, $u_\nu = u\nu$, $u_\tau = u\tau$, and $p = \sqrt{1 + \varphi_{,1}^2}$.

The jumps $[\cdot]_p$ and $[\cdot]^p$ of the functions prescribed on the axis x_1 at $x_1 = 0$ are given as

$$[m(0)]_p = m(0+) - (p^{-1}m)(0-); \quad [m(0)]^p = m(0+) - (pm)(0-).$$

The second and third equalities in (2.5) should be understood as $v(x_1) = u_\nu^-(x_1, \varphi(x_1))$ for $x_1 \in s_i$, where $i = 1, 2$. Here (2.1) are the equilibrium equations of the elastic body and the state equation (the Hooke law), while (2.2) and (2.3) represent the equilibrium equations of the thin weakly curved inclusions; the right-hand sides of (2.2) and (2.3) describe the forces acting on the inclusions from the direction of the elastic body. The first boundary condition in (2.5) ensures the mutual nonpenetration of the crack faces, while the second and third conditions say that the displacements of points of the elastic body and the thin inclusions coincide on γ^-. The second group of (2.4) corresponds to the zero torque, zero shearing force, and zero dilation or contraction. As for boundary conditions (2.6), they are typical for the statements of boundary value problems of crack theory with unknown contact domain; see [1]. In particular, if contact is absent at the specified point x_0, i.e. $[u_\nu(x_0)] > 0$; we obtain the zero value of the surface force: $(\sigma\nu)^+(x_0) = 0$. On the other hand, if the surface force is nonzero, i.e., $\sigma_\nu^+(x_0) < 0$, then we have the contact condition $[u_\nu(x_0)] = 0$. Boundary conditions (2.7) describe the junction conditions for two inclusions at $(0,0)$. In particular, the torques vanish; the jumps $[\cdot]_p$ for the displacements of thin inclusions vanish; the jumps $[\cdot]^p$ of the shearing forces and tangential forces also vanish. Since at $(0,0)$ we have a corner point between the inclusions, the jump of $v_{,1}$ at 0 is nonzero in general.

As we will establish, (2.1)–(2.7) are precisely equivalent to the variational statement of the problem of minimizing the energy functional on a suitable set of functions. The energy functional includes the terms corresponding to the deformation energy of the elastic body, the work of exterior forces, the bending and dilation energy of the thin inclusion. Let us present the variational statement of (2.1)–(2.7). To this end, introduce the set of admissible displacements

$$K = \big\{(u,v,w) \mid u \in H^1_\Gamma(\Omega_\gamma)^2, v \in H^2(s_i),\ w \in H^1(s_i),\ i = 1,2,\\ [u_\nu] \ge 0,\ v = u_\nu^-,\ w = u_\tau^- \text{ on } \gamma\big\},$$

where the Sobolev space $H^1_\Gamma(\Omega_\gamma)$ is defined as

$$H^1_\Gamma(\Omega_\gamma) = \{\phi \in H^1(\Omega_\gamma) \mid \phi = 0 \text{ on } \Gamma\}.$$

Consider the energy functional

$$\pi(u,v,w) = \frac{1}{2}\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} fu + \frac{1}{2}\sum_{i=1}^{2}\int\limits_{s_i} v_{,11}^2 + \frac{1}{2}\sum_{i=1}^{2}\int\limits_{s_i} (w_{,1} + kv)^2.$$

For brevity, here and henceforth we denote $\sigma_{ij}(u)\varepsilon_{ij}(u)$ by $\sigma(u)\varepsilon(u)$. Then the minimization problem

$$\text{find } (u,v,w) \in K \text{ such that } \pi(u,v,w) = \inf_K \pi$$

has a solution satisfying the variational inequality

$$(u,v,w) \in K, \tag{2.8}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) + \sum_{i=1}^{2}\int\limits_{s_i} v_{,11}(\bar{v}_{,11} - v_{,11})$$
$$+ \sum_{i=1}^{2}\int\limits_{s_i} \{(w_{,1} + kv)(\bar{w}_{,1} + k\bar{v} - w_{,1} - kv)\} \ge 0 \quad \text{for all } (\bar{u},\bar{v},\bar{w}) \in K. \tag{2.9}$$

As the solvability of (2.8), (2.9) is established in [11], we omit the arguments. Note also that the solution is unique.

We can prove that problems (2.8), (2.9) and (2.1)–(2.7) are equivalent on the class of smooth solutions. This means that all relations in (2.1)–(2.7) follow from (2.8), (2.9); and, conversely, we can deduce (2.8) and (2.9) from (2.1)–(2.7).

Proposition 1. *Problems* (2.8), (2.9) *and* (2.1)–(2.7) *are equivalent on the class of sufficiently smooth solutions.*

Proof. Suppose that (2.8) and (2.9) hold. Firstly, using the corresponding substitutions into (2.9), we verify in the standard fashion that the equilibrium equation in (2.1) is satisfied. Boundary conditions (2.6) are typical for contact problems of this form, and so we omit their derivation; see [1, 11].

Furthermore, choose in (2.9) test functions of the form $(\bar{u}, \bar{v}, \overline{w}) = (u, v, w) \pm (\tilde{u}, \tilde{v}, \widetilde{w})$, where $(\tilde{u}, \tilde{v}, \widetilde{w}) \in K$, with $[\tilde{u}_\nu] = 0$ on γ. This yields

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int\limits_{\Omega_\gamma} f\tilde{u} + \sum_{i=1}^{2} \int\limits_{s_i} v_{,11}\tilde{v}_{,11} + \sum_{i=1}^{2} \int\limits_{s_i} (w_{,1} + kv)(\widetilde{w}_{,1} + k\tilde{v}) = 0;$$

consequently, integrating by parts, we obtain

$$\begin{aligned} &- \int\limits_{\gamma} [\sigma_\nu]\tilde{u}_\nu - \int\limits_{\gamma} [\sigma_\tau \tilde{u}_\tau] + \sum_{i=1}^{2} \int\limits_{s_i} v_{,1111}\tilde{v} \\ &- \sum_{i=1}^{2} \int\limits_{s_i} (w_{,1} + kv)_{,1}\tilde{w} + \sum_{i=1}^{2} \int\limits_{s_i} (w_{,1} + kv)k\tilde{v} + v_{,11}\tilde{v}_{,1}|_{-1}^{0} \\ &+ v_{,11}\tilde{v}_{,1}|_{0}^{1} - v_{,111}\tilde{v}|_{-1}^{0} - v_{,111}\tilde{v}|_{0}^{1} + (w_{,1} + kv)\tilde{w}|_{-1}^{0} + (w_{,1} + kv)\widetilde{w}|_{0}^{1} = 0. \end{aligned} \tag{2.10}$$

Assume for the time being that the test functions in (2.10) satisfy the conditions $\tilde{v} = \tilde{v}_{,1} = \widetilde{w} = 0$ for $x_1 = -1, 0, 1$. The vanishing of the derivative $\tilde{v}_{,1}$ at $x_1 = 0$ means that $\tilde{v}_{,1}(0\pm) = 0$. We arrive at the identity

$$- \int\limits_{\gamma} [\sigma_\nu]\tilde{u}_\nu - \int\limits_{\gamma} [\sigma_\tau \tilde{u}_\tau] + \sum_{i=1}^{2} \int\limits_{s_i} v_{,1111}\tilde{v} - \sum_{i=1}^{2} \int\limits_{s_i} (w_{,1} + kv)_{,1}\tilde{w} + \sum_{i=1}^{2} \int\limits_{s_i} (w_{,1} + kv)k\tilde{v} = 0,$$

valid for all chosen functions. This implies equilibrium equations (2.2) and (2.3) because $d\gamma = pdx_1$. Return to (2.10). The available smoothness of the solution and the second and third boundary conditions in (2.5) yield $[v(0)]_p = [w(0)]_p = 0$. Indeed, γ is a Lipschitz curve. Since $u \in H^1(\Omega_\gamma)^2$, it follows that $u|_{\gamma^-} \in H^{1/2}(\gamma)$. Therefore, by the boundary conditions $v = u_\nu^-$ and $w = u_\tau^-$ on γ and the available smoothness of v and w, the displacements of inclusions at $(0, 0)$ must coincide. The functions v and w are the displacements of the thin inclusions in the normal and tangent directions. Thus, the gluing of displacements at $(0, 0)$ leads to the boundary conditions $[v(0)]_p = [w(0)]_p = 0$. Here we account for the assumption $\varphi_{,1}(0+) = 0$. We should also keep in mind that the jumps $[\cdot]_p$ of the test functions $\tilde{v}$ and $\widetilde{w}$ at $x_1 = 0$ vanish. Therefore, from (2.10) we obtain the second group of boundary conditions (2.4) and, moreover,

$$(v_{,11}\tilde{v}_{,1})(0-) - (v_{,11}\tilde{v}_{,1})(0+) + [v_{,111}(0)]^p\tilde{v}(0+) - [(w_{,1} + kv)(0)]^p\widetilde{w}(0+) = 0.$$

This implies the remaining boundary conditions in (2.7). Thus, from (2.8) and (2.9) we obtain all relations in (2.1)–(2.7).

Conversely, suppose that (2.1)–(2.7) hold. Multiply the first equation in (2.1) by $\bar{u} - u$, while (2.2) and (2.3) by $\bar{v} - v$ and $\overline{w} - w$ respectively, then integrate over Ω_γ and s_i. Integrating by parts yields

$$\begin{aligned} &\int\limits_{\gamma} [\sigma\nu(\bar{u} - u)] + \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) + \sum_{i=1}^{2} \int\limits_{s_i} v_{,11}(\bar{v}_{,11} - v_{,11}) \\ &+ \sum_{i=1}^{2} \int\limits_{s_i} (w_{,1} + kv)(\bar{w}_{,1} + k\bar{v} - w_{,1} - kv) - \sum_{i=1}^{2} \int\limits_{s_i} [\sigma_\tau]p(\overline{w} - w) \\ &- \sum_{i=1}^{2} \int\limits_{s_i} [\sigma_\nu]p(\bar{v} - v) - v_{,11}(\bar{v}_{,1} - v_{,1})|_{-1}^{0} - v_{,11}(\bar{v}_{,1} - v_{,1})|_{0}^{1} + v_{,111}(\bar{v} - v)|_{-1}^{0} \\ &+ v_{,111}(\bar{v} - v)|_{0}^{1} - (w_{,1} + kv)(\bar{w} - w)|_{-1}^{0} - (w_{,1} + kv)(\overline{w} - w)|_{0}^{1} = 0. \end{aligned} \tag{2.11}$$

In order to obtain (2.9), it suffices to establish that the sum of boundary terms in (2.11), including the integrals over γ and s_i, is not positive. In view of (2.7) and the second group of conditions in (2.4), it suffices to show that

$$\int_\gamma [\sigma_\nu(\bar{u}_\nu - u_\nu)] + \int_\gamma [\sigma_\tau(\bar{u}_\tau - u_\tau)] - \sum_{i=1}^2 \int_{s_i} [\sigma_\tau] p(\overline{w} - w) - \sum_{i=1}^2 \int_{s_i} [\sigma_\nu] p(\bar{v} - v) \le 0;$$

i.e., taking the boundary conditions (2.5) and (2.6) into account,

$$\int_\gamma [\sigma_\nu(\bar{u}_\nu - u_\nu)] - \sum_{i=1}^2 \int_{s_i} [\sigma_\nu] p(\bar{v} - v) \le 0.$$

The last inequality is easy to verify using (2.6).

The proof of Proposition 1 is complete. □

3. Passage to the limit at infinity with respect to the rigidity parameter. In this section we study passage to the limit with respect to the rigidity parameter in (2.8), (2.9). Assume that the rigidity of one inclusion depends on a parameter $\delta > 0$. In (2.8), (2.9) this parameter equals 1. Introduce now a parameter into the model and study the behavior of solution as $\delta \to \infty$.

Therefore, consider the boundary value problem of the form (2.1)–(2.7) with the parameter δ. In this case we decorate all unknown functions with δ. We need to find the displacement vector $u^\delta = (u_1^\delta, u_2^\delta)$ and the stress tensor $\sigma^\delta = \{\sigma_{ij}^\delta\}$, for $i, j = 1, 2$, defined on Ω_γ, as well as the displacements v^δ and w^δ of points of the thin inclusion defined for $x_1 \in s_i$, with $i = 1, 2$, such that

$$-\operatorname{div} \sigma^\delta = f, \quad \sigma^\delta - B\varepsilon(u^\delta) = 0 \quad \text{in } \Omega_\gamma,$$

$$\delta^{i-1}\big(v^\delta_{,1111} + k\big(w^\delta_{,1} + kv^\delta\big)\big) = \big[\sigma^\delta_\nu\big] p \quad \text{on } s_i,\ i = 1, 2,$$

$$\delta^{i-1}\big(-w^\delta_{,11} - (kv^\delta)_{,1}\big) = \big[\sigma^\delta_\tau\big] p \quad \text{on } s_i,\ i = 1, 2,$$

$$u^\delta = 0 \text{ on } \Gamma, \quad v^\delta_{,11} = v^\delta_{,111} = w^\delta_{,1} + kv^\delta = 0,\ x_1 = -1, 1,$$

$$\big[u^\delta_\nu\big] \ge 0, \quad v^\delta = u^{\delta-}_\nu, \quad w^\delta = u^{\delta-}_\tau \quad \text{on } \gamma,$$

$$\sigma^{\delta+}_\nu \le 0, \quad \sigma^{\delta+}_\tau = 0, \quad \sigma^{\delta+}_\nu\big[u^\delta_\nu\big] = 0 \quad \text{on } \gamma,$$

$$[v^\delta(0)]_p = [w^\delta(0)]_p = \big[v^\delta_{,111}(0)\big]^p = \big[\big(w^\delta_{,1} + kv^\delta\big)(0)\big]^p = 0, \quad v^\delta_{,11}(0\pm) = 0.$$

It is clear that the rigidity parameter equals 1 for the inclusion γ_1 and δ for the inclusion γ_2. The stated problem can be expressed as the variational inequality

$$(u^\delta, v^\delta, w^\delta) \in K, \tag{3.1}$$

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\bar{u} - u^\delta) + \sum_{i=1}^2 \int_{s_i} \delta^{i-1}\big(w^\delta_{,1} + kv^\delta\big)\big(\overline{w}^\delta_{,1} - w^\delta_{,1}\big)$$

$$+ \sum_{i=1}^2 \int_{s_i} \delta^{i-1}\big\{v^\delta_{,11}\big(\bar{v}_{,11} - v^\delta_{,11}\big) + k\big(w^\delta_{,1} + kv^\delta\big)(\bar{v} - v^\delta)\big\} - \int_{\Omega_\gamma} f(\bar{u} - u^\delta) \ge 0 \text{ for all } (\bar{u}, \bar{v}, \overline{w}) \in K. \tag{3.2}$$

Firstly, let us obtain a priori estimates in (3.1), (3.2). From (3.1), (3.2) for $\alpha > 0$ we infer that

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) - \int_{\Omega_\gamma} f u^\delta + \sum_{i=1}^2 \int_{s_i} \delta^{i-1}\big(w^\delta_{,1} + kv^\delta\big)^2$$

$$+ \sum_{i=1}^2 \int_{s_i} \delta^{i-1}\big(v^\delta_{,11}\big)^2 \pm \alpha \sum_{i=1}^2 \int_{s_i} \{(v^\delta)^2 + (w^\delta)^2\} = 0. \tag{3.3}$$

Choosing a small value of $\alpha > 0$ and using the Korn inequality, the equalities $v^\delta = u_\nu^{\delta-}$ and $w^\delta = u_\tau^{\delta-}$ on γ, and Embedding Theorems, we have

$$\frac{1}{2}\int\limits_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) - \sum_{i=1}^{2} \alpha \int\limits_{s_i} \{(v^\delta)^2 + (w^\delta)^2\} \geq c_1 \|u^\delta\|^2_{H^1_\Gamma(\Omega_\gamma)^2}, \quad c_1 > 0. \tag{3.4}$$

Hence, we see that

$$\|u^\delta\|^2_{H^1_\Gamma(\Omega_\gamma)^2} \leq c \tag{3.5}$$

uniformly in δ. Moreover, for all $\beta > 0$, $\delta \geq \delta_0 > 0$, and $i = 1, 2$ we obtain

$$\begin{aligned}\alpha(v^\delta)^2 + \alpha(w^\delta)^2 + \delta^{i-1}\big\{\big(v^\delta_{,11}\big)^2 + \big(w^\delta_{,1}\big)^2 + k^2\big(v^\delta\big)^2 + 2kw^\delta_{,1}v^\delta\big\} \\ \geq \alpha(v^\delta)^2 + \alpha(w^\delta)^2 + \delta_0^{i-1}\left\{\big(v^\delta_{,11}\big)^2 + \beta\big(w^\delta_{,1}\big)^2 - \frac{\beta k^2(v^\delta)^2}{1-\beta}\right\}.\end{aligned}$$

Since $\alpha - \frac{\delta_0^{i-1}\beta k^2}{1-\beta} \geq \frac{\alpha}{2}$ for β small, where $i = 1, 2$, using (3.5), (3.4), and (3.3) for $\delta \geq \delta_0$, we infer that

$$\|v^\delta\|^2_{H^2(s_i)} \leq c, \quad \|w^\delta\|^2_{H^1(s_i)} \leq c, \quad i = 1, 2, \tag{3.6}$$

$$\delta \int\limits_{s_2} \big(v^\delta_{,11}\big)^2 + \delta \int\limits_{s_2} \big(w^\delta_{,1} + kv^\delta\big)^2 \leq c. \tag{3.7}$$

By (3.5)–(3.7), we may assume that, as $\delta \to \infty$, we have

$$(u^\delta, v^\delta, w^\delta) \to (u, v, w) \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2 \times H^2(s_i) \times H^1(s_i), \quad i = 1, 2, \tag{3.8}$$

$$v(x_1) = a_0 + a_1x_1; \quad w_{,1}(x_1) + k(x_1)v(x_1) = 0, \quad x_1 \in s_2,\ a_0, a_1 \in \mathbb{R}. \tag{3.9}$$

Introduce the set

$$\begin{aligned}K_r = \big\{(u, v, w) \mid u \in H^1_\Gamma(\Omega_\gamma)^2,\ v \in H^2(s_1),\ w \in H^1(s_1), \\ [u_\nu] \geq 0,\ v = u_\nu^-,\ w = u_\tau^- \text{ on } \gamma,\ (u_\nu^-, u_\tau^-)|_{\gamma_2} \in L(0,1)\big\}\end{aligned}$$

of admissible displacements for the limit problem, where

$$L(0,1) = \{(v, w) \mid v(x_1) = b_0 + b_1x_1,\ w_{,1}(x_1) + k(x_1)v(x_1) = 0,\ x_1 \in s_2;\ b_0, b_1 \in \mathbb{R}\}.$$

We should note that the solution lying in K_r means that the displacement u^- on γ_2 has prescribed structure; namely, the displacement $u_\nu^- = v$ must be a continuous affine function, while the displacement u_τ^- must be determined from the solution to the ordinary differential equation on the interval s_2. This equation on w contains as its right-hand side both the curvature k of the inclusion and the displacement v.

It is clear that the limit functions u, v, and w of (3.8) and (3.9) satisfy the condition $(u, v, w) \in K_r$. Basing on (3.8) and (3.9), we can pass to the limit in (3.1) and (3.2). To this end, take an arbitrary test function $(\bar{u}, \bar{v}, \overline{w}) \in K_r$. Then $(\bar{u}, \bar{v}, \overline{w}) \in K$. Upon passing to the limit as $\delta \to \infty$ in (3.1) and (3.2), we find that the limit element (u, v, w) satisfies the variational inequality

$$(u, v, w) \in K_r, \tag{3.10}$$

$$\begin{aligned}\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) + \int\limits_{s_1} v_{,11}(\bar{v}_{,11} - v_{,11}) \\ + \int\limits_{s_1} (w_{,1} + kv)(\overline{w}_{,1} + k\bar{v} - w_{,1} - kv) \geq 0 \ \text{ for all } \ (\bar{u}, \bar{v}, \overline{w}) \in K_r.\end{aligned} \tag{3.11}$$

Thus, we have justified the following theorem.

Theorem 1. *As $\delta \to \infty$, the solutions to problems* (3.1), (3.2) *converge in the sense of* (3.8), (3.9) *to the solution to* (3.10), (3.11).

It is interesting to write down the differential statement of (3.10), (3.11). We need to find the displacement vector $u = (u_1, u_2)$ and the stress tensor $\sigma = \{\sigma_{ij}\}$, for $i, j = 1, 2$, defined on Ω_γ, as well as the displacements v and w of points of the thin inclusions defined on $s_1 \cup s_2$ such that

$$-\operatorname{div}\sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{3.12}$$

$$v_{,1111} + k(w_{,1} + kv) = [\sigma_\nu]p, \quad -w_{,11} - (kv)_{,1} = [\sigma_\tau]p \quad \text{on } s_1, \tag{3.13}$$

$$u = 0 \text{ on } \Gamma; \quad v_{,11} = v_{,111} = w_{,1} + kv = 0, \quad x_1 = -1, \tag{3.14}$$

$$[u_\nu] \geq 0, \quad \sigma_\nu^+ \leq 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma, \tag{3.15}$$

$$v = u_\nu^-,\ w = u_\tau^- \text{ on } \gamma; \quad (u_\nu^-, u_\tau^-)|_{\gamma_2} \in L(0,1), \tag{3.16}$$

$$[v(0)]_p = [w(0)]_p = 0, \quad v_{,11}(0-) = 0, \tag{3.17}$$

$$\int\limits_{s_2} [\sigma_\nu]p\bar{v} + \int\limits_{s_2} [\sigma_\tau]p\overline{w} + v_{,111}(0-)\bar{v}(0) - (w_{,1} + kv)(0-)\overline{w}(0) = 0 \text{ for all } (\bar{v}, \overline{w}) \in L(0,1). \tag{3.18}$$

Proposition 2. *Problems* (3.10), (3.11) *and* (3.12)–(3.18) *are equivalent on the class of sufficiently smooth solutions.*

Proof. Suppose that (3.10), (3.11) hold. Firstly, let us establish in the usual fashion the validity of (3.12). Moreover, we can use the standard arguments to verify boundary conditions (3.15) on γ; see [1, 11]. As we have already noted, these boundary conditions are typical for problems of this form. Furthermore, insert into (3.11) the test functions $(\bar{u}, \bar{v}, \overline{w}) = (u, v, w) \pm (\tilde{u}, \tilde{v}, \widetilde{w})$, where $(\tilde{u}, \tilde{v}, \widetilde{w}) \in K_r$, with $[\tilde{u}_\nu] = 0$ on γ. This yields

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int\limits_{\Omega_\gamma} f\tilde{u} + \int\limits_{s_1} v_{,11}\tilde{v}_{,11} + \int\limits_{s_1} (w_{,1} + kv)(\widetilde{w}_{,1} + k\tilde{v}) = 0.$$

Integrating by parts, we obtain

$$\begin{aligned} -\int\limits_{\gamma} [\sigma_\nu]\tilde{u}_\nu - \int\limits_{\gamma} [\sigma_\tau\tilde{u}_\tau] + \int\limits_{s_1} v_{,1111}\tilde{v} - \int\limits_{s_1} (w_{,1} + kv)_{,1}\tilde{w} \\ + \int\limits_{s_1} (w_{,1} + kv)k\tilde{v} + v_{,11}\tilde{v}_{,1}|_{-1}^{0} - v_{,111}\tilde{v}|_{-1}^{0} + (w_{,1} + kv)\tilde{w}|_{-1}^{0} = 0. \end{aligned} \tag{3.19}$$

Suppose firstly that in (3.19) the test functions satisfy $\tilde{v} = \tilde{v}_{,1} = \widetilde{w} = 0$ for $x_1 = -1, 0-$. This yields the following identity, valid for all chosen functions:

$$\begin{aligned} -\int\limits_{\gamma_1} [\sigma_\nu]\tilde{v} - \int\limits_{\gamma_1} [\sigma_\tau]\widetilde{w} - \int\limits_{\gamma_2} [\sigma_\nu]\tilde{v} - \int\limits_{\gamma_2} [\sigma_\tau]\widetilde{w} + \int\limits_{s_1} v_{,1111}\tilde{v} \\ - \int\limits_{s_1} (w_{,1} + kv)_{,1}\tilde{w} + \int\limits_{s_1} (w_{,1} + kv)k\tilde{v} = 0. \end{aligned} \tag{3.20}$$

(3.20) implies (3.13) and, moreover,

$$\int\limits_{s_2} [\sigma_\nu]p\tilde{v} + \int\limits_{s_2} [\sigma_\tau]p\tilde{w} = 0 \quad \text{for all } (\tilde{v}, \widetilde{w}) \in L(0,1),\ \tilde{v}(0) = 0,\ \widetilde{w}(0) = 0.$$

Return to (3.19). Since the solution is smooth, $[v(0)]_p = [w(0)]_p = 0$. Hence, we obtain the second group of boundary conditions of (3.14) and also $v_{,11}(0-) = 0$. Therefore, (3.19) implies the identity

$$\int\limits_{\gamma_2} [\sigma_\nu]\tilde{v} + \int\limits_{\gamma_2} [\sigma_\tau]\widetilde{w} + v_{,111}(0-)\tilde{v}(0) - (w_{,1} + kv)(0-)\widetilde{w}(0) = 0, \tag{3.21}$$

valid for all $(\tilde{v}, \widetilde{w})$, which coincides with (3.18).

Observe that (3.21) is a corollary of (3.18). Thus, all relations in (3.12)–(3.18) are justified on using (3.10) and (3.11).

Let us show that (3.12)–(3.18) imply (3.10) and (3.11). From (3.12) and (3.13) for $(\bar{u}, \bar{v}, \overline{w}) \in K_r$, upon integration we obtain

$$\begin{gathered}\int\limits_{\gamma} [\sigma\nu(\bar{u} - u)] + \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) \\ + \int\limits_{s_1} v_{,11}(\tilde{v}_{,11} - v_{,11}) + \int\limits_{s_1} (w_{,1} + kv)(\overline{w}_{,1} + k\bar{v} - w_{,1} - kv) \\ - \int\limits_{s_1} [\sigma_\tau]p(\overline{w} - w) - \int\limits_{s_1} [\sigma_\nu]p(\bar{v} - v) \\ -v_{,11}(\bar{v}_{,1} - v_{,1})|_{-1}^{0} + v_{,111}(\bar{v} - v)|_{-1}^{0} - (w_{,1} + kv)(\overline{w} - w)|_{-1}^{0} = 0.\end{gathered} \tag{3.22}$$

In order to deduce (3.11) from (3.22), it suffices to demonstrate that

$$\begin{gathered}\int\limits_{\gamma_1} [\sigma_\nu(\bar{u}_\nu - u_\nu)] + \int\limits_{\gamma_1} [\sigma_\tau(\bar{u}_\tau - u_\tau)] \\ + \int\limits_{\gamma_2} [\sigma_\nu(\bar{u}_\nu - u_\nu)] + \int\limits_{\gamma_2} [\sigma_\tau(\bar{u}_\tau - u_\tau)] - \int\limits_{s_1} [\sigma_\tau]p(\overline{w} - w) - \int\limits_{s_1} [\sigma_\nu]p(\bar{v} - v) \\ +(v_{,111}(\bar{v} - v))(0-) - ((w_{,1} + kv)(\overline{w} - w))(0-) \le 0.\end{gathered}$$

The validity of the last inequality is not difficult to verify using (3.15), (3.16), and (3.18).

The proof of Proposition 2 is complete. □

Thus, the differential statement of (3.10), (3.11) as (3.12)–(3.18), along with junction condition (3.17), includes (3.18). This identity involves both a nonlocal term (the integral over s_2), and the values of the required functions at $0-$.

The results remain valid in the case of linear boundary conditions on the crack faces. Let us state the corresponding boundary value problems. Problem (2.1)–(2.7) becomes the following. Find the displacement vector $u = (u_1, u_2)$ and the stress tensor $\sigma = \{\sigma_{ij}\}$, for $i, j = 1, 2$, defined on Ω_γ, as well as the displacements v and w of points of the thin inclusions defined on $s_1 \cup s_2$ such that

$$-\mathrm{div}\ \sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma,$$

$$v_{,1111} + k(w_{,1} + kv) = [\sigma_\nu]p \quad \text{on } s_i, \quad i = 1, 2,$$

$$-w_{,11} - (kv)_{,1} = [\sigma_\tau]p \quad \text{on } s_i, \quad i = 1, 2,$$

$$u = 0 \quad \text{on } \Gamma; \quad v_{,11} = v_{,111} = w_{,1} + kv = 0, \quad x_1 = -1, 1,$$

$$\sigma^+\nu = 0, \quad v = u_\nu^-, \quad w = u_\tau^- \quad \text{on } \gamma,$$

$$[v(0)]_p = [w(0)]_p = [v_{,111}(0)]^p = [(w_{,1} + kv)(0)]^p = 0, \quad v_{,11}(0\pm) = 0.$$

Limit junction boundary value problem (3.12)–(3.18) for elastic and rigid thin inclusions also changes in the case of linear boundary conditions on the crack faces. The statement takes the following form: Find the displacement vector $u = (u_1, u_2)$ and the stress tensor $\sigma = \{\sigma_{ij}\}$, for $i, j = 1, 2$, defined on Ω_γ, as well as the displacements v and w of points of the thin inclusions defined on $s_1 \cup s_2$ such that

$$-\operatorname{div}\sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma,$$

$$v_{,1111} + k(w_{,1} + kv) = [\sigma_\nu]p, \; -w_{,11} - (kv)_{,1} = [\sigma_\tau]p \quad \text{on } s_1,$$

$$u = 0 \text{ on } \Gamma; \quad v_{,11} = v_{,111} = w_{,1} + kv = 0, \quad x_1 = -1,$$

$$\sigma^+\nu = 0, \quad v = u_\nu^-, \quad w = u_\tau^- \quad \text{on } \gamma; \quad (u_\nu^-, u_\tau^-)|_{\gamma_2} \in L(0,1),$$

$$[v(0)]_p = [w(0)]_p = 0, \quad v_{,11}(0-) = 0,$$

$$\int\limits_{s_2} [\sigma_\nu]p\bar{v} + \int\limits_{s_2} [\sigma_\tau]p\overline{w} + v_{,111}(0-)\bar{v}(0) - (w_{,1} + kv)(0-)\overline{w}(0) = 0 \text{ for all } (\bar{v}, \overline{w}) \in L(0,1).$$

4. The equilibrium problem. No corner case. This section presents the results of analysis of the equilibrium problem for an elastic body with two weakly curved inclusions γ_1 and γ_2 in the case lacking a corner at the point $(0,0)$. The absence of a corner at $(0,0)$ means that the inclusions γ_1 and γ_2 are connected to each other. Actually we have one inclusion γ with a smoothness violation of the median line possible at $(0,0)$. Since the smoothness of the function φ can be violated at 0, as in Section 2, we should write down equilibrium equations for γ_1 and γ_2. Therefore, we still consider two inclusions γ_1 and γ_2, while at $(0,0)$ we must impose junction conditions. One of these conditions is the conservation of the angle between the inclusions during deformation. As above, assume for simplicity that $\varphi_{,1}(0+) = 0$.

The statement of the corresponding equilibrium problem is as follows: Find the displacement vector $u = (u_1, u_2)$ and the stress tensor $\sigma = \{\sigma_{ij}\}$, for $i, j = 1, 2$, defined on Ω_γ, as well as the displacements v and w of points of the thin inclusions defined on $s_1 \cup s_2$ such that

$$-\operatorname{div}\sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{4.1}$$

$$v_{,1111} + k(w_{,1} + kv) = [\sigma_\nu]p, \; -w_{,11} - (kv)_{,1} = [\sigma_\tau]p \quad \text{on } s_i, \quad i = 1, 2, \tag{4.2}$$

$$u = 0 \quad \text{on } \Gamma; \quad v_{,11} = v_{,111} = w_{,1} + kv = 0, \quad x_1 = -1, 1, \tag{4.3}$$

$$[u_\nu] \geq 0, \quad v = u_\nu^-, \quad w = u_\tau^-, \quad \sigma_\nu^+ \leq 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma, \tag{4.4}$$

$$[v(0)]_p = [w(0)]_p = [v_{,1}(0)] = 0, \tag{4.5}$$

$$[v_{,11}(0)] = [v_{,111}(0)]^p = [(w_{,1} + kv)(0)]^p = 0, \tag{4.6}$$

where $[m(0)] = m(0+) - m(0-)$. Clearly, along with the conservation of the angle between the inclusions at the junction, the jump of the momentum at $(0,0)$ also vanishes. Problem (4.1)–(4.6) admits a variational statement. Introduce the set of admissible displacements

$$K^0 = \big\{(u, v, w) \mid u \in H^1_\Gamma(\Omega_\gamma)^2, v \in H^2(s_i), \; w \in H^1(s_i), \; i = 1, 2,$$
$$[u_\nu] \geq 0, \; v = u_\nu^-, \; w = u_\tau^- \text{ on } \gamma, \; [v_{,1}(0)] = 0\big\}.$$

As above, the energy functional is

$$\pi(u, v, w) = \frac{1}{2}\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} fu + \frac{1}{2}\sum_{i=1}^{2}\int\limits_{s_i} v_{,11}^2 + \frac{1}{2}\sum_{i=1}^{2}\int\limits_{s_i} (w_{,1} + kv)^2,$$

while the minimization problem

$$\text{find } (u,v,w) \in K^0 \text{ such that } \pi(u,v,w) = \inf_{K^0} \pi$$

has a unique solution satisfying the variational inequality

$$(u,v,w) \in K^0, \tag{4.7}$$

$$\begin{aligned} &\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}-u) - \int_{\Omega_\gamma} f(\bar{u}-u) + \sum_{i=1}^{2} \int_{s_i} v_{,11}(\bar{v}_{,11} - v_{,11}) \\ &+ \sum_{i=1}^{2} \int_{s_i} \{(w_{,1} + kv)(\overline{w}_{,1} + k\bar{v} - w_{,1} - kv)\} \geq 0 \quad \text{for all } (\bar{u},\bar{v},\overline{w}) \in K^0. \end{aligned} \tag{4.8}$$

We can verify that problems (4.7), (4.8) and (4.1)–(4.6) are equivalent on the class of smooth solutions in the sense that we can obtain (4.7), (4.8) from (4.1)–(4.6) and, conversely, all relations in (4.1)–(4.6) follow from (4.7) and (4.8).

As in Section 3, we can introduce a parameter $\delta > 0$ into the model (4.1)–(4.6) and study passage to the limit as the parameter tends to infinity, while fixing the rigidity of the inclusion γ_1 at the value 1, and the rigidity of the inclusion γ_2 equal to δ. Consider the variational inequality

$$(u^\delta, v^\delta, w^\delta) \in K^0, \tag{4.9}$$

$$\begin{aligned} &\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\bar{u}-u^\delta) + \sum_{i=1}^{2} \int_{s_i} \delta^{i-1} v^\delta_{,11}(\bar{v}_{,11} - v^\delta_{,11}) \\ &+ \sum_{i=1}^{2} \int_{s_i} \delta^{i-1} \big\{ \big(w^\delta_{,1} + kv^\delta\big)\big(\overline{w}_{,1} + k\bar{v} - w^\delta_{,1} - kv^\delta\big)\big\} - \int_{\Omega_\gamma} f(\bar{u}-u^\delta) \geq 0 \text{ for all } (\bar{u},\bar{v},\overline{w}) \in K^0. \end{aligned} \tag{4.10}$$

It turns out that, as $\delta \to \infty$, the inclusion γ_2 becomes rigid; i.e., the displacement field for this inclusion has prescribed structure.

In order to state the limit problem corresponding to $\delta \to \infty$, introduce the set of admissible displacements

$$\begin{aligned} K_r^0 = \big\{ &(u,v,w) \mid u \in H^1_\Gamma(\Omega_\gamma)^2,\ v \in H^2(s_1),\ w \in H^1(s_1), \\ &[u_\nu] \geq 0,\ v = u^-_\nu,\ w = u^-_\tau \text{ on } \gamma,\ (u^-_\nu, u^-_\tau)|_{\gamma_2} \in L(0,1),\ [v_{,1}(0)] = 0 \big\}. \end{aligned}$$

The set $L(0,1)$ is defined in the same way as in Section 3. Skipping details, note that from the sequence of solutions to problems (4.9), (4.10) we can extract a subsequence so that, as $\delta \to \infty$, we have

$$(u^\delta, v^\delta, w^\delta) \to (u,v,w) \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2 \times H^2(s_i) \times H^1(s_i), \quad i = 1,2, \tag{4.11}$$

$$v(x_1) = a_0 + a_1 x_1; \quad w_{,1}(x_1) + k(x_1)v(x_1) = 0, \quad x_1 \in s_2;\ a_0, a_1 \in \mathbb{R}. \tag{4.12}$$

Furthermore, the limit functions u, v, and w satisfy the variational inequality

$$(u,v,w) \in K_r^0, \tag{4.13}$$

$$\begin{aligned} &\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}-u) - \int_{\Omega_\gamma} f(\bar{u}-u) + \int_{s_1} v_{,11}(\bar{v}_{,11} - v_{,11}) \\ &+ \int_{s_1} (w_{,1} + kv)(\overline{w}_{,1} + k\bar{v} - w_{,1} - kv) \geq 0 \quad \text{for all } (\bar{u},\bar{v},\overline{w}) \in K_r^0. \end{aligned} \tag{4.14}$$

We state the result as the next theorem:

Theorem 2. *As $\delta \to \infty$, the solutions to problems* (4.9), (4.10) *converge in the sense of* (4.11), (4.12) *to the solution to* (4.13), (4.14).

Finally, let us present the differential statement of (4.13), (4.14). We need to find the displacement vector $u = (u_1, u_2)$ and the stress tensor $\sigma = \{\sigma_{ij}\}$, for $i, j = 1, 2$, defined on Ω_γ, as well as the displacements v and w of points of the thin inclusions defined on $s_1 \cup s_2$ such that

$$-\operatorname{div} \sigma = f, \quad \sigma - B\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{4.15}$$

$$v_{,1111} + k(w_{,1} + kv) = [\sigma_\nu]p, \quad -w_{,11} - (kv)_{,1} = [\sigma_\tau]p \quad \text{on } s_1, \tag{4.16}$$

$$u = 0 \text{ on } \Gamma; \quad v_{,11} = v_{,111} = w_{,1} + kv = 0, \quad x_1 = -1, \tag{4.17}$$

$$[u_\nu] \ge 0, \quad \sigma_\nu^+ \le 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma, \tag{4.18}$$

$$v = u_\nu^-, \; w = u_\tau^- \text{ on } \gamma; \quad (u_\nu^-, u_\tau^-)|_{\gamma_2} \in L(0,1), \tag{4.19}$$

$$[v(0)]_p = [w(0)]_p = [v_{,1}(0)] = 0, \tag{4.20}$$

$$\int\limits_{s_2} [\sigma_\nu]p\bar{v} + \int\limits_{s_2} [\sigma_\tau]p\overline{w} + v_{,111}(0-)\bar{v}(0)$$
$$-v_{,11}(0-)\bar{v}_{,1}(0) - (w_{,1} + kv)(0-)\overline{w}(0) = 0 \quad \text{for all } (\bar{v}, \overline{w}) \in L(0,1). \tag{4.21}$$

Clearly, in comparison with the corresponding boundary condition (3.18), in the statement of the equilibrium problem with a corner between inclusions nonlocal boundary condition (4.21) involves the second derivative of the function v at 0 from the left.

On the class of smooth solutions, problems (4.13), (4.14) and (4.15)–(4.21) are equivalent. We omit the details of the proof because they are generally similar to those in Section 3.

As in the end of Section 3, in the no-corner case considered in Section 4 we can state the junction problem for two thin elastic weakly curved inclusions, as well as the junction problem for elastic and rigid inclusions in the case of linear boundary conditions on the crack faces. But we will omit the details.

We emphasize that all results of this article are established for nonlinear models excluding mutual penetration of the opposite crack faces. At the same time, similar results hold in the simpler case of classical linear boundary conditions on the crack faces.

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