ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 4, pp. 715–724.

EXPONENTIAL DECAY ESTIMATES FOR SOME COMPONENTS OF SOLUTIONS TO THE NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF THE LIVING SYSTEM MODELS

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Abstract: Studying the behavior of solutions to the Cauchy problem for a family of nonlinear functional differential equations with delay which arise in the living system models, we establish the conditions that provide some exponential decay estimates for components of solutions. We find the parameters of simultaneous exponential estimates as solutions to nonlinear inequalities built from the majorants of the mappings on the right-hand sides of the differential equations. We present the results of constructing the exponential estimates for the variables in an epidemic dynamics model.

DOI: 10.1134/S0037446620040126

Keywords: functional delay differential equation, Cauchy problem, global solvability, nonnegative solutions, exponential decay estimates, M-matrix, mathematical biology, living systems, epidemiology

Introduction. Many mathematical models describing the dynamics of living systems can be expressed as the Cauchy problem

$$\frac{dx(t)}{dt} = f(t, x_t) - (\mu + g(t, x_t))x(t), \quad t \geq 0, \tag{1}$$

$$x(t) = \psi(t), \quad t \in I_\omega = [-\omega, 0], \tag{2}$$

for a system of delay differential equations. In (1) and (2) we use the notation

$$x(t) = (x_1(t), \dots, x_m(t))^T, \quad \psi(t) = (\psi_1(t), \dots, \psi_m(t))^T,$$

$$f(t, x_t) = (f_1(t, x_t), \dots, f_m(t, x_t))^T, \quad g(t, x_t) = \operatorname{diag}(g_1(t, x_t), \dots, g_m(t, x_t)),$$

$$\mu = \operatorname{diag}\bigl(\mu_1, \dots, \mu_m\bigr), \quad m \geq 2,$$

where $x(t)$ is the sought function; the delayed variable $x_t : I_\omega \to R^m$ is defined as $x_t(\theta) = x(t + \theta)$ with $\theta \in I_\omega$ for each fixed $t \geq 0$; while $\psi(t)$ is the initial function, $f_i(t, x_t)$ and $g_i(t, x_t)$ are some mappings, and μ_i are constants for $1 \leq i \leq m$. By $dx(t)/dt$ we understand the componentwise right-hand derivative.

In applications the function $x(t)$ can represent the number of elements of some living system: cells, viruses, bacteria, microorganisms, individuals of various types or groups, and so on. Given $1 \leq i \leq m$, the mapping $f_i(t, x_t)$ describes the appearance rate of the elements of type i, the mapping $g_i(t, x_t)$ is such that $g_i(t, x_t)x_i(t)$ determines the extinction or transformation rate of elements of type i due to the interaction of the elements or the influence of the environment. The expression $\mu_i x_i(t)$ means the natural death or migration rate of type i elements, as well as the transition rate of type i elements.

Solutions $x(t)$ to (1), (2) were studied for linear and nonlinear systems (1) in [1–3]. These articles showed that $x(t)$ is nonnegative under a few conditions and proposed an approach to constructing componentwise upper estimates for $x(t)$. An important feature of some nonlinear models is the structure of equations in (1) which we can express as two blocks. In these systems the right-hand sides of the equations in the first and second blocks involve the components of $f(t, x_t)$ admitting upper estimates by linear mappings or constants. The presence of these estimates enables us to use the well-developed theory of

The author was supported by the Russian Foundation for Basic Research (Grant 18–29–10086).

Original article submitted November 27, 2019; revised May 4, 2020; accepted June 17, 2020.

monotone operators [4, Chapter 1, Section 3; 5; 6, Chapter 2, Subsection 2.3; Chapter 7, Subsection 7.1] and construct the estimates for the components of the solution $x(t)$ to (1), (2).

The goal of this article is to obtain the conditions that provide exponential decay estimates for some components of the solution $x(t)$ to (1), (2) on $[0,\infty)$.

1. The main assumptions. We use the notation and assumptions of [3]. Denote the norm of $v \in R^m$ by $\|v\|_{R^m} = \sum_{i=1}^m |v_i|$. Given $J=[a,b] \subset R$ and $A \subseteq R^m$, denote by $C(J,A)$ the set of all continuous functions $z : J \to A$ equipped with the norm

$$\|z\| = \max_{\theta \in J}(\|z(\theta)\|_{R^m}), \quad z \in C(J,R^m).$$

Consider the ball $B_d = \{z \in C(I_\omega, R^m) : \|z\| \le d\}$ in $C(I_\omega,R^m)$. If $u,w \in R^m$ then the inequalities $u<0$, $u>0$, $u \le w$, and $u \ge w$ are understood componentwise. If $x,y \in C(J,A)$ then we understand the relation $x(t) \le y(t)$ with $t \in J$ as inequality between the vectors.

Assume that the mappings, functions, and constants in (1) and (2) satisfy for each $1 \le i \le m$ the following that are denoted by **(H0)**:

(1) f_i, $g_i : R_+ \times C(I_\omega, A_\xi) \to R$, where $A_\xi = \{u \in R^m : u \ge \xi\}$ for some $\xi \in R^m$ with $\xi < 0$;

(2) f_i, $g_i : R_+ \times C(I_\omega, R_+^m) \to R_+$;

(3) $f_i(t,z)$ and $g_i(t,z)$ are continuous in $(t,z) \in R_+ \times C(I_\omega, A_\xi)$ and locally Lipschitz in z: For each $d \in R$ with $d>0$ there exist constants $L_f^{(i)} = L_f^{(i)}(\xi,d) > 0$ and $L_g^{(i)} = L_g^{(i)}(\xi,d) > 0$ such that

$$|f_i(t,z_1) - f_i(t,z_2)| \le L_f^{(i)}\,\|z_1 - z_2\|,$$

$$|g_i(t,z_1) - g_i(t,z_2)| \le L_g^{(i)}\,\|z_1 - z_2\|$$

for all $z_1, z_2 \in B_d \cap C(I_\omega, A_\xi)$ and $t \in [0,\infty)$;

(4) $\psi_i : I_\omega \to R_+$ is a continuous function;

(5) $\mu_i > 0$.

Following [7, Chapter 2, Subsection 2.1; 8, Chapter 1, Subsections 2.1 and 2.2], refer as a *solution to* (1), (2) on $[0,\infty)$ to a function $x(t)$ continuous on every finite interval $I_\omega \cup [0,\tau]$ with $\tau>0$ which possesses a continuous derivative on $[0,\tau)$ and satisfies (2) and (1) for all $t \in [0,\tau)$. Integrating (1) by the variation of constants formula with (2) taken into account, we arrive at the system of integral equations with initial data:

$$x(t) = e^{-\int_0^t (\mu + g(s,x_s))ds}\,\psi(0) + \int_0^t e^{-\int_a^t (\mu+g(s,x_s))ds} f(a,x_a)\,da, \quad t \ge 0, \tag{3}$$

$$x(t) = \psi(t), \quad t \in I_\omega. \tag{4}$$

In (3) by

$$e^{-\int_a^t (\mu+g(s,x_s))ds}, \quad 0 \le a \le t,$$

we understand the diagonal matrix that is constructed from the diagonal matrix $\mu + g(s,x_s)$. Refer as a *solution* to (3), (4) on $[0,\infty)$ to a function $x(t)$ continuous on every interval $I_\omega \cup [0,\tau]$ with $\tau>0$ and satisfying (4) and (3) for all $t \in [0,\tau]$. Using the standard approach [7, Chapter 2, Subsection 2.1], we find that problems (1), (2) and (3), (4) are equivalent. Consequently, we can apply (3), (4) to study the behavior of a solution $x(t)$ to (1), (2).

As shown in [3], Assumption (H0), together with some additional assumptions on $f(t,x_t)$ and the entries of $g(t,x_t)$, guarantees the existence, uniqueness, and nonnegativity of the solution $x(t)$ to (3), (4) on $[0,\infty)$. Moreover, some upper estimates for the components of $x(t)$ are constructed whose form depends on the assumptions.

The next section is devoted to constructing some upper estimates for the solution $x(t)$ to (3), (4) that involve exponential decay estimates for part of the components of $x(t)$.

2. The additional assumption and the main result. Introduce the notations: k is a fixed index, $1 \le k < m$; while $\eta = (\eta_{k+1}, \dots, \eta_m)^T$ is a vector with nonnegative components; and $D_\eta \subset R_+^m$ is as follows:

$$D_\eta = \{u \in R_+^m : 0 \le u_j \le \eta_j, k+1 \le j \le m\}.$$

Let us state Assumption **(H1)** on $f(t, x_t)$. Assume that for some $1 \le k < m$ the mapping $f(t, x_t)$ is expressed as

$$f(t, x_t) = (f_1(t, x_t), \dots, f_k(t, x_t), f_{k+1}(t, x_t), \dots, f_m(t, x_t))^T$$

and satisfies the following:

(1) for all $(t, x_t) \in R_+ \times C(I_\omega, R_+^m)$ we have

$$(f_{k+1}(t, x_t), \dots, f_m(t, x_t))^T \le p = (p_{k+1}, \dots, p_m)^T, \tag{5}$$

where $p_{k+1} > 0, \dots, p_m > 0$ are some constants;

(2) the vector $\eta = (\eta_{k+1}, \dots, \eta_m)^T > 0$ with components

$$\eta_j = \frac{p_j}{\mu_j}, \quad k+1 \le j \le m, \tag{6}$$

is such that for all $(t, x_t) \in R_+ \times C(I_\omega, D_\eta)$ we have the estimate

$$\begin{aligned} (f_1(t, x_t), \dots, f_k(t, x_t))^T &\le L\big(x_t^{(k)}\big) \\ &= \sum_{i=0}^{n} L_{k,i} x^{(k)}(t - \omega_i) + \int_{-\omega}^{0} L_{k,n+1}(\theta) x^{(k)}(t + \theta)\, d\theta, \end{aligned} \tag{7}$$

where $x^{(k)}(t) = (x_1(t), \dots, x_k(t))^T$, the delays satisfy $0 < \omega_i \le \omega < \infty$ for $1 \le i \le n$, and $\omega_0 = 0$, while $L_{k,0}, \dots, L_{k,n}, L_{k,n+1}(\theta)$ are nonnegative matrices of size $k \times k$, the nonzero entries of $L_{k,n+1}(\theta)$ are Riemann integrable, and each row of the matrix

$$L_{[k]} = \sum_{i=0}^{n} L_{k,i} + \int_{-\omega}^{0} L_{k,n+1}(\theta)\, d\theta \tag{8}$$

is nonzero.

Resting on Assumption (H1), put

$$\mu_{[k]} = \operatorname{diag}\big(\mu_1, \dots, \mu_k\big), \quad I_{[k]} = \operatorname{diag}(1, \dots, 1), \quad \psi_{[k]}(t) = (\psi_1(t), \dots, \psi_k(t))^T.$$

Using the expression for $L(x_t^{(k)})$ in (7), consider the Cauchy problem for $y(t) = (y_1(t), \dots, y_k(t))^T$:

$$\begin{aligned} \frac{dy(t)}{dt} &= L(y_t) - \mu_{[k]} y(t) \\ &= \sum_{i=0}^{n} L_{k,i} y(t - \omega_i) + \int_{-\omega}^{0} L_{k,n+1}(\theta) y(t + \theta)\, d\theta - \mu_{[k]} y(t), \quad t \ge 0, \end{aligned} \tag{9}$$

$$y(t) = \psi_{[k]}(t), \quad t \in I_\omega. \tag{10}$$

Refer as a *solution* to (9), (10) on $[0,\infty)$ to a function $y(t)$ continuous on every finite interval $I_\omega \cup [0,\tau]$ with $\tau > 0$, which possesses a continuous derivative on $[0,\tau)$, and satisfies (10) and (9) for all $t \in [0,\tau)$; for $t = 0$ we understand by $dy(t)/dt$ the componentwise right-hand derivative. Integrating (9) by the variation of constants formula with (10) taken into account, we pass to the system of linear integral equations with the prescribed initial condition:

$$y(t) = e^{-\mu_{[k]}t}\psi(0) + \int_0^t e^{-\mu_{[k]}(t-a)} L(y_a)\, da, \quad t \geq 0, \tag{11}$$

$$y(t) = \psi_{[k]}(t), \quad t \in I_\omega. \tag{12}$$

Refer as a *solution* to (11), (12) on $[0,\infty)$ to a function $y(t)$ continuous on every finite interval $I_\omega \cup [0,\tau]$ with $\tau > 0$ which satisfies (12) and (11) for all $t \in [0,\tau]$. Finding a solution to (9), (10) is equivalent to solving (11), (12).

Observe that in the framework of Assumption (H1) the matrix $L_{[k]}$ given in (8) is nonnegative, while the offdiagonal entries of $\mu_{[k]} - L_{[k]}$ are nonpositive. This means that $\mu_{[k]} - L_{[k]}$ is a matrix of a special form. Say that $\mu_{[k]} - L_{[k]}$ is a *nonsingular M-matrix* whenever it is invertible and the inverse matrix $(\mu_{[k]} - L_{[k]})^{-1}$ is nonnegative. The series of criteria [9, Chapter 6; 10, Chapter 2, Section 36] enable us to test whether $\mu_{[k]} - L_{[k]}$ belongs to the family of nonsingular M-matrices.

Lemma 1. *Suppose that the matrix $L_{[k]}$ in* (H1) *is such that $\mu_{[k]} - L_{[k]}$ is a nonsingular M-matrix. Then the solution $y(t)$ to* (9), (10) *satisfies the estimate*

$$0 \leq y(t) \leq ce^{-rt}, \quad t \in I_\omega \cup [0,\infty), \tag{13}$$

where $c \in R^k$ and $r \in R$ are such that

$$c > 0, \quad \left(\mu_{[k]} - rI_{[k]} - \sum_{i=0}^{n} e^{r\omega_i} L_{k,i} - \int_{-\omega}^{0} e^{-r\theta} L_{k,n+1}(\theta)\, d\theta\right) c \geq 0, \tag{14}$$

$$c \geq \max_{t\in I_\omega}(e^{rt}\psi_{[k]}(t)), \quad 0 < r < \min(\mu_1, \dots, \mu_k). \tag{15}$$

Proof. Take some $c \in R^k$ and $r \in R$. Introduce the function

$$v(t) = ce^{-rt}, \quad t \in I_\omega \cup [0,\infty). \tag{16}$$

Resting on the hypotheses, apply Theorem 3 of [1] to (9), (10) or (11), (12). We infer that there are $c \in R^k$ and $r \in R$ satisfying (14), (15) and such that $v(t)$ in (16) satisfies

$$e^{-\mu_{[k]}t}\psi_{[k]}(0) + \int_0^t e^{-\mu_{[k]}(t-a)} L(v_a)\, da \leq v(t), \quad 0 \leq t < \infty, \tag{17}$$

$$\psi_{[k]}(t) \leq v(t), \quad t \in I_\omega. \tag{18}$$

Moreover, (11), (12), (17), and (18) imply that for all $t \in I_\omega \cup [0,\infty)$ the solution $y(t)$ to (9), (10) satisfies $0 \leq y(t) \leq v(t) = c\exp(-rt)$; i.e., (13) holds. $\square$

The methods for finding $c \in R^k$ and $r \in R$ in (13) appear in Section 3 of [1] and Section 2 of [2].

Proceed to constructing the exponential decay estimates for the solution $x(t)$ to (1), (2) for the first k components. We use the approach of [3]. Fix $\tau > 0$. By $C_\psi \subset C([-\omega,\tau], R^m)$ we understand the set of all functions $x \in C([-\omega,\tau], R^m)$ with $x(t) = \psi(t)$ for $t \in I_\omega$. Refer as a *solution* to (3), (4) on $[0,\tau]$ to $x \in C_\psi$ such that $x(t)$ satisfies (3) for all $t \in [0,\tau]$.

Denote by $C_{\psi,0} \subset C_\psi$ the set of all functions $x \in C_\psi$ with $x(t) \geq 0$ for $t \in [-\omega, \tau]$. Take some function $\upsilon = \upsilon(t) = (\upsilon_1(t), \dots, \upsilon_m(t))^T$ continuous on $[-\omega, \tau]$ and having nonnegative components. Denote the set of functions $x \in C_\psi$ with $0 \leq x(t) \leq \upsilon(t)$ for $t \in [-\omega, \tau]$ by $C_{\psi,0,\upsilon}$.

Basing on (3) and (4), introduce the operator F that associates to each $x \in C_{\psi,0}$ the function $F(x) \in C_{\psi,0}$ defined as

$$F(x)(t) = \psi(t), \quad t \in I_\omega,$$

$$F(x)(t) = e^{-\int_0^t (\mu + g(s,x_s))\,ds}\psi(0) + \int_0^t e^{-\int_a^t (\mu+g(s,x_s))\,ds} f(a, x_a)\,da, \quad t \in [0,\tau].$$

Lemma 2. *Suppose that Assumptions* (H0) *and* (H1) *are satisfied, while $\mu_{[k]} - L_{[k]}$ is a nonsingular M-matrix and the components of the initial function ψ satisfy*

$$\max_{t \in I_\omega} \psi_j(t) \leq \eta_j, \quad k+1 \leq j \leq m.$$

Then there are $\beta \in R^m$, $r \in R$, $\beta > 0$, and $r > 0$ such that for

$$\upsilon(t) = (\beta_1 e^{-rt}, \dots, \beta_k e^{-rt}, \beta_{k+1}, \dots, \beta_m)^T, \quad t \in R,$$

and each $\tau > 0$ the set $C_{\psi,0,\upsilon}$ is invariant under F.

Proof. Using the hypotheses, assume that the components of the vector

$$\upsilon(t) = (\upsilon_1(t), \dots, \upsilon_k(t), \upsilon_{k+1}(t), \dots, \upsilon_m(t))^T,$$

appearing in the definition of $C_{\psi,0,\upsilon}$, are of the form

$$\upsilon^{(k)}(t) = (\upsilon_1(t), \dots, \upsilon_k(t)) = (c_1 e^{-rt}, \dots, c_k e^{-rt}),$$

$$(\upsilon_{k+1}(t), \dots, \upsilon_m(t)) = (\eta_{k+1}, \dots, \eta_m), \quad t \in R,$$

where $c = (c_1, \dots, c_k)^T$ and r are the parameter of (16), while the constants $\eta_{k+1}, \dots, \eta_m$ are given by (6). Put

$$\mu_* = \operatorname{diag}(\mu_{k+1}, \dots, \mu_m), \quad \psi_*(t) = (\psi_{k+1}(t), \dots, \psi_m(t))^T, \quad t \in I_\omega.$$

Fix $\tau > 0$. Take $x \in C_{\psi,0,\upsilon}$. By assumption, $\psi_*(t) \leq \eta$ for all $t \in I_\omega$. Invoking (15), we infer that

$$0 \leq F(x)(t) = \psi(t) \leq \upsilon(t)$$

for $t \in I_\omega$.

Express $F(x)(t)$ componentwise:

$$F(x)(t) = (F_1(x)(t), \dots, F_k(x)(t), F_{k+1}(x)(t), \dots, F_m(x)(t))^T, \quad t \in I_\omega \cup [0,\tau].$$

Verify that $0 \leq F(x)(t) \leq \upsilon(t)$ for all $t \in [0,\tau]$. Constructing the estimates presented below, we appreciate the inequalities $0 \leq x(t) \leq \upsilon(t)$ for $t \in [-\omega, \tau]$.

Resting on (5) and (6), we find that

$$\begin{aligned}
&(F_{k+1}(x)(t), \dots, F_m(x)(t))^T \\
&\leq e^{-\mu_* t}\psi_*(0) + \int_0^t e^{-\mu_*(t-a)}(f_{k+1}(t, x_a), \dots, f_m(t, x_a))^T da \\
&\leq e^{-\mu_* t}\eta + \int_0^t e^{-\mu_*(t-a)} p\,da = \eta = (\upsilon_{k+1}(t), \dots, \upsilon_m(t))^T, \quad t \in [0,\tau].
\end{aligned}$$

Using (6), (7), and Lemma 1, we obtain

$$
\begin{aligned}
(F_1(x)(t), \dots, F_k(x)(t))^T &\le e^{-\mu_{[k]} t}\psi_{[k]}(0) \\
&+ \int_0^t e^{-\mu_{[k]}(t-a)}(f_1(t, x_a), \dots, f_k(t, x_a))^T \, da \\
&\le e^{-\mu_{[k]} t}\psi_{[k]}(0) + \int_0^t e^{-\mu_{[k]}(t-a)} L\big(x_a^{(k)}\big)\, da \\
&\le e^{-\mu_{[k]} t}\psi_{[k]}(0) + \int_0^t e^{-\mu_{[k]}(t-a)} L\big(\upsilon_a^{(k)}\big)\, da \\
&\le (\upsilon_1(t), \dots, \upsilon_k(t))^T, \quad t \in [0, \tau].
\end{aligned}
$$

Consequently, $F(x) \in C_{\psi,0,\upsilon}$ holds for all $x \in C_{\psi,0,\upsilon}$. The parameters of the function $\upsilon(t)$ are independent of τ. Since the choice of τ is arbitrary, the proof of Lemma 2 is complete. □

Theorem. *In the hypothesis of Lemma 2 problem* (1), (2) *on* $[0, \infty)$ *admits a unique solution* $x(t)$, *and for all* $t \in I_\omega \cup [0, \infty)$ *we have the componentwise estimates*

$$0 \le x_i(t) \le c_i e^{-rt}, \quad 1 \le i \le k, \tag{19}$$

$$0 \le x_j(t) \le \eta_j, \quad k+1 \le j \le m, \tag{20}$$

where $c = (c_1, \dots, c_k)^T$ *and* r *satisfy* (14), (15), *while* $\eta_{k+1}, \dots, \eta_m$ *are given in* (6).

PROOF. Using the hypotheses together with Lemmas 1 and 2, we infer that $C_{\psi,0,\upsilon}$ is invariant under F for every $\tau > 0$. Applying Lemma 2 of [3], we establish that problem (3), (4) on $[0, \tau]$ admits a unique solution $x = x(t)$; furthermore, $x \in C_{\psi,0,\upsilon}$. Since the choice of τ is arbitrary and the components of $\upsilon(t)$ are independent of τ, we conclude that (3), (4) admits a unique solution $x = x(t)$ on $[0, \infty)$ and this solution satisfies (19) and (20). □

3. Example. Consider the mathematical model describing the spread of infection in the population of some domain. Its variables are the number $x_1(t)$ of latently infected (not infectious) individuals and the number $x_2(t)$ of sick (infectious) individuals, and the number $x_3(t)$ of individuals susceptible to the infection. The model equations are of the form

$$\frac{dx_1(t)}{dt} = \sigma_1 \beta x_2(t-\omega_1)x_3(t-\omega_1) - (\lambda_1 + \gamma)x_1(t) + \sigma_3 \eta x_2(t-\omega_3), \tag{21}$$

$$\frac{dx_2(t)}{dt} = \sigma_2 \beta x_2(t-\omega_2)x_3(t-\omega_2) + \gamma x_1(t) - (\lambda_2 + \eta)x_2(t), \quad t \ge 0, \tag{22}$$

$$\frac{dx_3(t)}{dt} = \rho(t) - \lambda_3 x_3(t) - \beta x_2(t)x_3(t), \tag{23}$$

$$x_1(0) = x_1^{(0)}, \ x_2(t) = x_2^{(0)}(t), \ x_3(t) = x_3^{(0)}(t), \ t \in I_\omega = [-\max\{\omega_1, \omega_2, \omega_3\}, 0]. \tag{24}$$

In the initial data (24) the functions $x_2^{(0)}(t)$ and $x_3^{(0)}(t)$ are nonnegative and continuous, while $x_1^{(0)} \ge 0$. The function $\rho(t)$ is nonnegative, continuous, and bounded above by some constant $\rho^* > 0$, with $t \in [0, \infty)$. All parameters in (21)–(23) are positive. The meaning of $\rho(t)$ is the growth rate of the group of susceptible individuals. The parameters λ_1, λ_2, and λ_3 determine the death rate of individuals and the rate of migration to other domains. The parameter β reflects the contact rate of susceptible and sick individuals. The parameters $\sigma_1 > 0$ and $\sigma_2 > 0$, with $\sigma_1 + \sigma_2 < 1$, count the share of susceptible individuals, who upon getting infected develop either latent or active stage of the disease. The durations of this passage from the infection time are described by the delay constants ω_1 and ω_2. The parameter γ stands for the rate of spontaneous development of the disease in latently infected individuals. The parameter η determines the rate of passage of sick individuals into the noninfectious stage of the disease as a consequence of beginning self-treatment or treatment in medical institutions. The duration of this stage is described by the delay constant ω_3. After being in the noninfectious stage of the disease, the share $0 < \sigma_3 < 1$ of the previous individuals joins the group of latently infected individuals.

Observe that the parameter η can account for the rate of diagnostic process of sick individuals with various methods. The function $\rho(t)$ and the parameter β can vary depending on the various measures taken by the healthcare system of the domain.

We may regard (21)–(24) as a version of the mathematical model of the spread of tuberculosis. In contrast to the available models [11, 12], we can complement (21)–(23) with the variables $w_1(t)$ and $w_2(t)$ reflecting the individuals in the intermediate (latent) stages of infection, as well as the variable $w_3(t)$ describing the number of sick individuals in the noninfectious stage of the disease:

$$w_1(t) = \int\limits_{t-\omega_1}^{t} e^{-\lambda_3(t-a)}\nu_1\beta x_2(a)x_3(a)\,da,$$

$$w_2(t) = \int\limits_{t-\omega_2}^{t} e^{-\lambda_3(t-a)}\nu_2\beta x_2(a)x_3(a)\,da,$$

$$w_3(t) = \int\limits_{t-\omega_3}^{t} e^{-\lambda_3(t-a)}\eta x_2(a)\,da, \quad t \geq 0,$$

where $\nu_1 > 0$ and $\nu_2 > 0$ with $\nu_1 + \nu_2 = 1$ are some constants. To derive these relations, we used [13, 14].

We can reduce the integral relations for $w_1(t)$, $w_2(t)$, and $w_3(t)$ to the Cauchy problem of the form

$$\begin{gathered}
\frac{dw_1(t)}{dt} = \nu_1\beta x_2(t)x_3(t) - \lambda_3 w_1(t) - e^{-\lambda_3\omega_1}\nu_1\beta x_2(t-\omega_1)x_3(t-\omega_1),\\
\frac{dw_2(t)}{dt} = \nu_2\beta x_2(t)x_3(t) - \lambda_3 w_2(t) - e^{-\lambda_3\omega_2}\nu_2\beta x_2(t-\omega_2)x_3(t-\omega_2),\\
\frac{dw_3(t)}{dt} = \eta x_2(t) - \lambda_3 w_3(t) - e^{-\lambda_3\omega_3}\eta x_2(t-\omega_3), \quad t \geq 0,\\
w_1(0) = \int\limits_{-\omega_1}^{0} e^{\lambda_3 a}\nu_1\beta x_2^{(0)}(a)x_3^{(0)}(a)\,da, \quad w_2(0) = \int\limits_{-\omega_2}^{0} e^{\lambda_3 a}\nu_2\beta x_2^{(0)}(a)x_3^{(0)}(a)\,da,\\
w_3(0) = \int\limits_{-\omega_3}^{0} e^{\lambda_3 a}\eta x_2^{(0)}(a)\,da.
\end{gathered}$$

Accounting for the structure of equations for $w_i(t)$, where $i = 1, 2, 3$, we can express the parameters σ_1, σ_2, and σ_3 in (21) and (22) as

$$\sigma_1 = e^{-\lambda_3\omega_1}\nu_1, \quad \sigma_2 = e^{-\lambda_3\omega_2}\nu_2, \quad \sigma_3 = e^{-\lambda_3\omega_3}.$$

System (21)–(23) with initial data (24) corresponds to (1), (2). The right-hand sides of (21)–(23) satisfy Assumption (H0). Assume moreover that

$$x_3^{(0)}(t) \leq x_3^{(*)} = \rho^*/\lambda_3, \quad t \in I_\omega. \tag{25}$$

From the structure of the right-hand sides of this system, we find that the nonnegative continuous functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ satisfy the estimates $0 \le x_3(t) \le x_3^*$ for $t \in I_\omega \cup [0, \infty)$, and

$$\frac{dx_1(t)}{dt} \le \sigma_1 \beta x_3^* x_2(t-\omega_1) - (\lambda_1 + \gamma) x_1(t) + \sigma_3 \eta x_2(t - \omega_3),$$

$$\frac{dx_2(t)}{dt} \le \sigma_2 \beta x_3^* x_2(t-\omega_2) + \gamma x_1(t) - (\lambda_2 + \eta) x_2(t),$$

$$\frac{dx_3(t)}{dt} \le \rho^* - \lambda_3 x_3(t), \quad t \ge 0.$$

Consequently, we can express the vector p and the matrices $L_{k,i}$, L_k, and $\mu_{[k]} - L_{[k]}$ in Assumption (H1) for $m = 3$, $k = 2$, and $k+1 = 3$, namely:

$$p = p_3 = \rho^*, \quad f_3(t, x_t) \le p_3 = \rho^*,$$

$$\mu_{[2]} = \begin{pmatrix} \lambda_1 + \gamma & 0 \\ 0 & \lambda_2 + \eta \end{pmatrix}, \quad L_{2,0} = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad L_{2,1} = \begin{pmatrix} 0 & \sigma_1 \beta x_3^* \\ 0 & 0 \end{pmatrix},$$

$$L_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \beta x_3^* \end{pmatrix}, \quad L_{2,3} = \begin{pmatrix} 0 & \sigma_3 \eta \\ 0 & 0 \end{pmatrix}, \quad L_{2,4}(\theta) \equiv 0, \ \theta \in I_\omega,$$

$$L_{[2]} = \begin{pmatrix} 0 & \sigma_1 \beta x_3^* + \sigma_3 \eta \\ \gamma & \sigma_2 \beta x_3^* \end{pmatrix}, \quad \mu_{[2]} - L_{[2]} = \begin{pmatrix} \lambda_1 + \gamma & -\sigma_1 \beta x_3^* - \sigma_3 \eta \\ -\gamma & \lambda_2 + \eta - \sigma_2 \beta x_3^* \end{pmatrix}.$$

Turning to the hypotheses of the theorem, we require that $\mu_{[2]} - L_{[2]}$ be a nonsingular M-matrix. Apply the criterion that all principal minors of $\mu_{[2]} - L_{[2]}$ be positive [9, Chapter 6, Section 2]. The first minor equals $\lambda_1 + \gamma$ and is obviously positive. The second principal minor coincides with the determinant of the matrix in question and is of the form

$$\det(\mu_{[2]} - L_{[2]}) = (\lambda_1 + \gamma)(\lambda_2 + \eta)(1 - R_0),$$

where

$$R_0 = \frac{\sigma_3 \gamma \eta}{(\lambda_1 + \gamma)(\lambda_2 + \eta)} + \frac{\beta(\lambda_1 \sigma_2 + \gamma(\sigma_1 + \sigma_2))}{(\lambda_1 + \gamma)(\lambda_2 + \eta)} x_3^*.$$

Putting

$$R_0 < 1, \tag{26}$$

we infer that $\mu_{[2]} - L_{[2]}$ is a nonsingular M-matrix.

Consequently, if (25) and (26) hold then the components $x_1(t)$ and $x_2(t)$ of the solution to (21)–(24) satisfy the estimates

$$0 \le x_1(t) \le c_1 e^{-rt}, \quad 0 \le x_2(t) \le c_2 e^{-rt}, \quad t \in I_\omega \cup [0, \infty). \tag{27}$$

Put $c_{[2]} = (c_1, c_2)^T$ and $\psi_{[2]}(t) = \big(x_1^{(0)}, x_2^{(0)}(t)\big)^T$ for $t \in I_\omega$, as well as

$$H(r) = (h_{ij}(r)) = \mu_{[2]} - rI_{[2]} - L_{2,0} - e^{r\omega_1} L_{2,1} - e^{r\omega_2} L_{2,2} - e^{r\omega_3} L_{2,3}, \quad r \in R.$$

To find the constants c_1, c_2, and r in (27), use (14) and (15) to obtain

$$c_{[2]} > 0, \quad H(r) c_{[2]} \ge 0, \tag{28}$$

$$c_{[2]} \ge \max_{t \in I_\omega} (e^{rt} \psi_{[2]}(t)), \quad 0 < r < \min(\lambda_1 + \gamma, \lambda_2 + \eta). \tag{29}$$

Following [2], seek the constant r as the positive root of the equation

$$\det H(r) = (\lambda_1 + \gamma - r)(\lambda_2 + \eta - r - e^{r\omega_2}\sigma_2\beta x_3^*) - \gamma(e^{r\omega_1}\sigma_1\beta x_3^* + e^{r\omega_3}\sigma_3\eta) = 0. \tag{30}$$

Observe that $H(0) = \mu_{[2]} - L_{[2]}$ and $\det H(0) > 0$ by (26). Using (30), pass to the equation for the required constant r:

$$\lambda_2 + \eta - r - e^{r\omega_2}\sigma_2\beta x_3^* = \gamma \frac{e^{r\omega_1}\sigma_1\beta x_3^* + e^{r\omega_3}\sigma_3\eta}{\lambda_1 + \gamma - r}, \quad 0 \le r < \min(\lambda_1 + \gamma, \lambda_2 + \eta). \tag{31}$$

It is clear that the left-hand side of (31) is a decreasing function $\varphi_1(r)$, while the right-hand side is an increasing function $\varphi_2(r)$. Since $\det H(0) > 0$, it follows that $\varphi_1(0) > \varphi_2(0)$. Consequently, there exists a unique root $0 < r_* < \min(\lambda_1 + \gamma, \lambda_2 + \eta)$ of (31) and accordingly (30).

To find the vector $c_{[2]}$, use (28) and consider the equation $H(r_*)u = 0$, where $u = (u_1, u_2)^T \in R^2$:

$$h_{11}(r_*)u_1 + h_{12}(r_*)u_2 = 0, \quad h_{21}(r_*)u_1 + h_{22}(r_*)u_2 = 0.$$

Since $\det H(r_*) = 0$ and $h_{11}(r_*) = \lambda_1 + \gamma - r_* \neq 0$, the required u is of the form

$$u = (u_1, u_2)^T = \alpha u^* = \alpha(u_1^*, 1)^T,$$

where $\alpha \in R$ is an arbitrary constant, while

$$u_1^* = (e^{r_*\omega_1}\sigma_1\beta x_3^* + e^{r_*\omega_3}\sigma_3\eta)/(\lambda_1 + \gamma - r_*).$$

Putting $\alpha > 0$, we establish that r_* and $c_{[2]} = \alpha u^*$ satisfy (28). We can choose the vector $c_{[2]}$ as $c_{[2]} = \alpha_* u^*$, where the constant $\alpha_* > 0$ is determined by the condition that (29) holds, and it obviously depends on the components of the initial function $\psi_{[2]}(t)$.

Observe that the additional variables $w_1(t)$, $w_2(t)$, and $w_3(t)$ satisfy the estimates similar to (27), namely:

$$0 \le w_i(t) \le d_i e^{-rt}, \quad i = 1, 2, 3, \ t \in [0, \infty), \tag{32}$$

where the constants $d_1 > 0$, $d_2 > 0$, and $d_3 > 0$ can be expressed in terms of c_2, $x_3^{(*)}$, and the parameters appearing in the integral expressions for these variables.

In the framework of this model, we may interpret (25) and (26) as conditions for the exponential decay of the epidemic process in the selected population. Following the accepted terminology [15, 16], we refer to the constant R_0 appearing in (26) as the *basic reproduction number*. This constant reflects the average number of sick (infectious) individuals reproduced in the population relative to one latently infected individuum, one sick individuum, and x_3^* susceptible individuals.

Using the expression for R_0 and the parameters σ_1, σ_2, and σ_3, we obtain the estimate

$$R_0 < \widehat{R}_0 = \frac{\gamma e^{-\lambda_3\omega_3}}{\lambda_1 + \gamma} + \frac{\beta(\gamma\nu_1 e^{-\lambda_3\omega_1} + (\lambda_1 + \gamma)\nu_2 e^{-\lambda_3\omega_2})x_3^*}{(\lambda_1 + \gamma)(\lambda_2 + \eta)}. \tag{33}$$

It is clear from (33) that for each collection of parameters of the model there are sufficiently large ω_3 and η and sufficiently small γ, β, and x_3^* for which, in some combination, the inequality $\widehat{R}_0 < 1$ holds. The values of these parameters and estimates (27) and (32) can be used for planning the work of the healthcare system of the domain to diagnose and treat sick individuals, as well as to eradicate the infection in the course of some span of time.

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