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THE SUPERALGEBRAS OF JORDAN BRACKETS DEFINED BY THE n-DIMENSIONAL SPHERE

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Abstract: We study the generalized Leibniz brackets on the coordinate algebra of the n-dimensional sphere. In the case of the one-dimensional sphere, we show that each of these is a bracket of vector type. Each Jordan bracket on the coordinate algebra of the two-dimensional sphere is a generalized Poisson bracket. We equip the coordinate algebra of a sphere of odd dimension with a Jordan bracket whose Kantor double is a simple Jordan superalgebra. Using such superalgebras, we provide some examples of the simple abelian Jordan superalgebras whose odd part is a finitely generated projective module of rank 1 in an arbitrary number of generators. An analogous result holds for the Cartesian product of the sphere of even dimension and the affine line. In particular, in the case of the 2-dimensional sphere we obtain the exceptional Jordan superalgebra. The superalgebras we constructed give new examples of simple Jordan superalgebras.

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The superalgebras of Jordan brackets are of great importance in studying the structure of Jordan algebras and superalgebras. Some examples of superalgebras of Jordan brackets can be obtained by the Kantor doubling process starting from associative commutative superalgebras with a Jordan bracket (see [1–3]). The main properties of the superalgebras of Jordan brackets such as speciality, for example, were studied in [1–11]. In [12, 13], it was shown that the commutator with respect to the Novikov product defines a Jordan bracket on the associative commutative part of a Novikov–Poisson algebra. The Jordan superalgebra, constructed by this bracket, was shown to be special in [14]. The superalgebras of Jordan brackets of vector type play a significant role in the study of prime degenerate Jordan algebras (see [5, 6, 15]).

Note that if some Jordan bracket is given on an associative commutative algebra then the even part of the obtained Jordan superalgebra is associative, and the odd part is an associative module over the even part. Following [16], we call such superalgebras *abelian.*

The simple abelian Jordan superalgebras are close in their properties to the superalgebras of Jordan brackets. The simple Jordan superalgebras with associative even part were studied in [17–23]. In [17, 18], the unital simple special abelian Jordan superalgebras were described that are not isomorphic to the superalgebra of a nondegenerate bilinear form. As it is turned out, such superalgebras are the superalgebras of vector type with respect to some derivations, and their odd part is a finitely generated projective module of rank 1. Moreover, each of these superalgebras is embedded into a simple superalgebra of vector type which is constructed by a derivation. In [18–21], some examples were constructed of simple Jordan superalgebras of vector type with respect to two derivations. More precisely, the odd part of such a superalgebra is a projective module of rank 1, and it is generated as a module by at least two elements. Some examples of prime abelian Jordan superalgebras of vector type whose odd part is a finitely generated projective module of rank 1 with an arbitrary number of generators were constructed in [24].

It was shown in [23] that a simple (not necessarily special) abelian Jordan superalgebra is embedded

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into a simple superalgebra of a Jordan bracket. Moreover, the odd part of such a superalgebra as a module over the even part is a finitely generated projective module of rank 1.

It is worthy to mention the article [25] which stated the study of the simple Jordan superalgebras with associative even part. One of the methods of the present work is the study of action of a subalgebra of the Lie algebra of the vector fields of the n-dimensional sphere on the algebra of the regular functions of the n-dimensional sphere. In this connection, we distinguish the papers [26–28].

This article consists of three sections. In § 1, we study the simplicity conditions of the superalgebra constructed by the Kantor doubling process with the help of a generalized Leibniz bracket. We find some equivalent conditions of simplicity of a superalgebra and one of its subalgebras in terms of the conditions for the bracket given on a Z_2-graded associative commutative algebra. We also give a criterion of simplicity of a superalgebra in terms of the differential simplicity of an algebra with respect to some set of derivations. In § 2, we study some properties of the generalized Leibniz brackets that are defined on the algebra of regular functions of the n-dimensional sphere. In § 3, we provide some new examples of the unital simple abelian Jordan superalgebras not isomorphic to a superalgebra of a Jordan bracket.

§ 1. The Generalized Leibniz Brackets and the Kantor Double

Let F be a field of characteristic not 2. An algebra $A = A_0 + A_1$ is a Z_2-graded algebra if $A_0 \cap A_1 = 0$ and $A_i A_j = A_{i+j \bmod 2}$. The algebra A is a *superalgebra.* The space $A_0(A_1)$ is the *even* (*odd*) *part* of the Z_2-graded algebra A. The elements in $A_0 \cup A_1$ are *homogeneous.* The expression $|x|$, where $x \in A_0 \cup A_1$, denotes the parity index of a homogeneous element x:

$$|x| = \begin{cases} 0 & \text{if } x \in A_0 \quad (x \text{ is even}), \\ 1 & \text{if } x \in A_1 \quad (x \text{ is odd}). \end{cases}$$

Let G be the Grassmann algebra with unity 1 over F; i.e., G is an associative algebra given by the generators $1, e_1, e_2, \dots$ and the defining relations

$$e_i^2 = 0, \quad e_i e_j = -e_j e_i.$$

The products $1, e_{i_1} \dots e_{i_k}$, where $i_1 < i_2 < \dots < i_k$, form a basis for G. Let G_0 and G_1 be the vector subspaces spanned by the products of even and odd lengthes, respectively. Then $G = G_0 + G_1$ is a Z_2-graded algebra.

Let $A = A_0 + A_1$ be a Z_2-graded algebra. Then $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ is a subalgebra of $G \otimes A$ (the tensor product over a field F), and $G(A)$ is called the *Grassmann envelope* of A.

An associative superalgebra $A = A_0 + A_1$ is an *associative commutative superalgebra* provided that its Grassmann envelope $G(A)$ is an associative commutative algebra. Then

$$ab = (-1)^{|a||b|} ba$$

in A for homogeneous elements.

Let $\Gamma = \Gamma_0 + \Gamma_1$ be an associative commutative superalgebra over a field F, and let $\{\,,\} : \Gamma \times \Gamma \mapsto \Gamma$ be a super-skewsymmetric bilinear mapping on Γ which is called a *bracket.* Given Γ and $\{\,,\}$, we can construct the superalgebra $J(\Gamma, \{\,,\})$. Consider the direct sum of vector spaces $J(\Gamma, \{\,,\}) = \Gamma \oplus \Gamma\xi$, where $\Gamma\xi$ is an isomorphic copy of the vector space Γ. The product $(\cdot)$ on $J(\Gamma, \{\,,\})$ is defined as

$$a \cdot b = ab, \quad a \cdot b\xi = (ab)\xi, \quad a\xi \cdot b = (-1)^{|b|}(ab)\xi, \quad a\xi \cdot b\xi = (-1)^{|b|}\{a, b\},$$

where $a, b \in \Gamma_0 \cup \Gamma_1$, and ab is the product of a and b in Γ. Put $A_0 = \Gamma_0 + \Gamma_1\xi$ and $A_1 = \Gamma_0\xi + \Gamma_1$. Then $J(\Gamma, \{\,,\}) = A_0 + A_1$ is a superalgebra called the *Kantor double.*

Let D be an even derivation of a superalgebra Γ; i.e., $D(\Gamma_i) \subseteq \Gamma_i$, $i = 0, 1$. Then $\{\,,\}$ is a *generalized Leibniz bracket* or a *Leibniz D-bracket* provided that

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|} b\{a, c\} - D(a)bc \tag{1}$$

for $a, b, c \in \Gamma_0 \cup \Gamma_1$. If $D = 0$ then $\{\,,\}$ is a *Leibniz bracket.*

Note that if $\{\,,\}$ is a generalized Leibniz bracket then

$$\langle a,b\rangle=\{a,b\}-D(a)b+aD(b)$$

is a Leibniz bracket.

Let Γ be a unital superalgebra. Putting $b=c=1$ in (1), we get $D(a)=\{a,1\}$.

Given a generalized Leibniz bracket $\{\,,\}$, we define the Jacobian

$$J(a,b,c)=\{a,\{b,c\}\}+(-1)^{|a||b|+|a||c|}\{b,\{c,a\}\}+(-1)^{|a||c|+|b||c|}\{c,\{a,b\}\}$$

of a,b,c and put

$$S(a,b,c)=\{a,b\}D(c)+(-1)^{|a||b|+|a||c|}\{b,c\}D(a)+(-1)^{|a||c|+|b||c|}\{c,a\}D(b). \tag{2}$$

If Γ is a unital superalgebra then $S(1,a,b)=0$ for all $a,b\in\Gamma$.

Lemma 1. *Let $\Gamma=\Gamma_0+\Gamma_1$ be an associative commutative superalgebra, and let $\{\,,\}$ be a generalized Leibniz bracket on Γ. Then*

$$S(ab,u,v)=aS(b,u,v)+(-1)^{|a||b|}bS(a,u,v), \tag{3}$$

$$J(ab,u,v)=aJ(b,u,v)+(-1)^{|a||b|}bJ(a,u,v)+ab(D(\{u,v\})-\{D(u),v\}-\{u,D(v)\}) \tag{4}$$

for homogeneous elements. If Γ is a unital superalgebra then

$$J(ab,u,v)=aJ(b,u,v)+(-1)^{|a||b|}bJ(a,u,v)-abJ(1,u,v). \tag{5}$$

Proof. It suffices to prove these identities for the associative commutative algebra Γ. Show (3). Take $a,b,u,v\in\Gamma$. Then

$$S(ab,u,v)=\{ab,u\}D(v)+\{u,v\}D(ab)+\{v,ab\}D(u)$$
$$=(a\{b,u\}+\{a,u\}b+abD(u))D(v)+\{u,v\}D(ab)+(\{v,a\}b+a\{v,b\}-D(v)ab)D(u)$$
$$=a(\{b,u\}D(v)+\{u,v\}D(b)+\{v,b\}D(u))$$
$$+b(\{a,u\}D(v)+\{u,v\}D(a)+\{v,a\}D(u))=aS(b,u,v)+bS(a,u,v).$$

Identity (4) is proved analogously.

Let Γ be a unital superalgebra. Then

$$J(1,u,v)=-(D(\{u,v\})-\{D(u),v\}-\{u,D(v)\}),$$

whence (5) follows. $\square$

Lemma 2. *Let $\Gamma=\Gamma_0+\Gamma_1$ be a unital associative commutative superalgebra, and let $\{\,,\}$ be a generalized Leibniz bracket on Γ. Define the bracket*

$$\langle a,b\rangle=\{a,b\}-D(a)b+D(b)a.$$

Then

$$J(a,b,c)_{\langle,\rangle}=J(a,b,c)+S(a,b,c)-aJ(1,b,c)$$
$$-(-1)^{|a|(|b|+|c|)}bJ(1,c,a)-(-1)^{|c|(|a|+|b|)}cJ(1,a,b),$$

where $J(a,b,c)_{\langle,\rangle}$ is the Jacobian of $\langle\,,\rangle$.

Proof. By the definition of $\langle\,,\rangle$ we have

$$\langle a,\langle b,c\rangle\rangle=\{a,\langle b,c\rangle\}-D(a)\langle b,c\rangle+aD(\langle b,c\rangle)=\{a,\{b,c\}\}+\{a,b\}D(c)+aD(\{b,c\})$$
$$-\{a,D(b)\}c+(-1)^{|a||b|}b\{a,D(c)\}-D(a)\{b,c\}-(-1)^{|a||b|}D(b)\{a,c\}$$
$$-2D(a)bD(c)+2D(a)D(b)c-aD^2(b)c+abD^2(c),$$

whence the required result follows by direct computation. $\square$

Let $\Lambda=\Lambda_0+\Lambda_1$ be a Z_2-graded associative commutative algebra. Put $\Gamma_0=\Lambda$, $\Gamma_1=0$, and $\Gamma=\Gamma_0+\Gamma_1=\Lambda$. Assume that a generalized Leibniz bracket is given on Λ which is consistent with the Z_2-grading of Λ. Then $J(\Lambda_0,\Lambda_1,\{\,,\})=\Lambda_0+\Lambda_1\xi$ is a subsuperalgebra of $J(\Lambda,\{\,,\})=J(\Gamma,\{\,,\})$.

Theorem 1. *Let Λ be a unital Z_2-graded associative commutative algebra without zero divisors, and let $\Lambda_0 = \Lambda_1\Lambda_1$. Then the superalgebra $J(\Lambda, \{\,,\})$ is simple if and only if the subsuperalgebra $J(\Lambda_0, \Lambda_1, \{\,,\})$ is simple.*

PROOF. Let $J(\Lambda_0, \Lambda_1, \{\,,\})$ be simple. Assume that I is an ideal of $J(\Lambda, \{\,,\})$. Then $I = K + L\xi$. Since I is an ideal, K and L are some ideals of Λ. Analogously,

$$K\xi \subseteq (K\Lambda)\xi \subseteq K \cdot \Lambda\xi \subseteq L\xi, \quad \{K, \Lambda\} \subseteq K\xi \cdot \Lambda\xi \subseteq L\xi \cdot \Lambda\xi \subseteq K.$$

Put $K_0 = K \cap \Lambda_0$. Then K_0 is an ideal of Λ_0, and $\{K_0\Lambda_1, \Lambda_1\} \subseteq K \cap \Lambda_0 = K_0$. Put $R = K_0 + (K_0\Lambda_1)\xi$. Then $R \cdot \Lambda_0 \subseteq R$, and

$$R \cdot \Lambda_1\xi \subseteq K_0 \cdot \Lambda_1\xi + \{K_0\Lambda_1, \Lambda_1\} \subseteq R.$$

Hence, R is an ideal of $J(\Lambda_0, \Lambda_1, \{\,,\})$. Then either $1 \in K_0$ or $K_0 = 0$. If $1 \in K_0$ then $1 \in K$, and $I = J(\Lambda, \{\,,\})$.

Let $K_0 = 0$ and $a_0 + a_1 \in K$, where $a_0 \in \Lambda_0$ and $a_1 \in \Lambda_1$. Then

$$(a_0 + a_1)(a_0 - a_1) = a_0^2 - a_1^2 \in K \cap \Lambda_0 = K_0 = 0.$$

Since Λ does not contain zero divisors, $a_0 + a_1 = 0$ and $K = 0$.

Thus, $J(\Lambda, \{\,,\})$ is a simple superalgebra.

Let $J(\Lambda, \{\,,\})$ be a simple superalgebra. Assume that I is an ideal of $J(\Lambda_0, \Lambda_1, \{\,,\})$. Then $I = K_0 + K_1\xi$, where K_0 is an ideal of Λ_0, and K_1 is a submodule of the Λ_0-module Λ_1. Also we get

$$\{K_0\Lambda_1, \Lambda_1\} \subseteq (K_0 \cdot \Lambda_1\xi) \cdot \Lambda_1\xi \subseteq K_1\xi \cdot \Lambda_1\xi \subseteq K_0.$$

Since $\{\,,\}$ is a generalized Leibniz bracket and $\Lambda_0 = \Lambda_1\Lambda_1$; therefore,

$$\{K_0, \Lambda_1\} \subseteq \{\Lambda_1, K_0\} \subseteq \{\Lambda_1, K_0\Lambda_1\Lambda_1\}$$
$$\subseteq \{K_0\Lambda_1, \Lambda_1\}\Lambda_1 + K_0\Lambda_1\{\Lambda_1, \Lambda_1\} + D(\Lambda_1)K_0\Lambda_1\Lambda_1 \subseteq K_0\Lambda_1.$$

Since $1 \in \Lambda_1\Lambda_1$, we have $1 = \sum_i a_i b_i$, where $a_i, b_i \in \Lambda_1$. Take $r \in K_0$. Then

$$D(r) = \{r, 1\} = \Big\{r, \sum_i a_i b_i\Big\} = \sum_i a_i\{r, b_i\} + \sum_i \{r, a_i\}b_i - \sum_i D(r)a_i b_i.$$

Since

$$\sum_i a_i\{r, b_i\} + \sum_i \{r, a_i\}b_i \in \{K_0, \Lambda_1\}\Lambda_1 \subseteq K_0\Lambda_1\Lambda_1 \subseteq K_0\Lambda_0 \subseteq K_0;$$

therefore, $2D(r) \in K_0$, whence

$$\{K_0, \Lambda_0\} \subseteq \{K_0, \Lambda_1\Lambda_1\} \subseteq \{K_0, \Lambda_1\}\Lambda_1 + D(K_0)\Lambda_1\Lambda_1 \subset K_0.$$

Analogously, $\{K_0\Lambda_1, \Lambda_0\} \subseteq K_0\Lambda_1$.

Let $K = K_0 + K_0\Lambda_1$. Then K is an ideal of Λ, and

$$\{K, \Lambda\} \subseteq \{K_0, \Lambda\} + \{K_0\Lambda_1, \Lambda_0\} + \{K_0\Lambda_1, \Lambda_1\} \subseteq K_0 + K_0\Lambda_1 = K.$$

So, $K + K\xi$ is an ideal of $J(\Lambda, \{\,,\})$. Hence, either $1 \in K_0$ or $K_0 = 0$. If $1 \in K_0$ then $I = J(\Lambda_0, \Lambda_1, \{\,,\})$.

Let $K_0 = 0$. Then $I = K_1\xi$. Thus,

$$\{K_1, \Lambda_1\} \subseteq K_1\xi \cdot \Lambda_1\xi \subseteq K_0 = 0.$$

Hence, $K_1\Lambda_1 + K_1\xi$ is an ideal of $J(\Lambda_0, \Lambda_1, \{\,,\})$. By the above, either $1 \in K_1\Lambda_1$ or $K_1\Lambda_1 = 0$.

If $1 \in K_1\Lambda_1$ then $\Lambda_1 = K_1$, and $\{\Lambda_1, \Lambda_1\} = 0$. Let $a \in \Lambda_1$ and $1 = \sum_i a_i b_i$, where $a_i, b_i \in \Lambda_1$. Then

$$D(a) = \{a, 1\} = \Big\{a, \sum_i a_i b_i\Big\} = \sum_i a_i\{a, b_i\} + \sum_i \{a, a_i\}b_i - \sum_i D(a)a_i b_i = -D(a).$$

Consequently, $2D(a) = 0$, whence

$$\{\Lambda_1, \Lambda_0\} \subseteq \{\Lambda_1, \Lambda_1 \cdot \Lambda_1\} \subseteq \{\Lambda_1, \Lambda_1\}\Lambda_1 + D(\Lambda_1)\Lambda_1 \cdot \Lambda_1 = 0.$$

Analogously, $\{\Lambda_0, \Lambda_0\} = 0$. Thus, $\{\Lambda, \Lambda\} = 0$, and $\Lambda\xi$ is an ideal of $J(\Lambda, \{\,,\})$. Hence, $1 \notin K_1\Lambda_1$. Then $K_1\Lambda_1 = 0$ and $K_1 = 0$, since $\Lambda_1\Lambda_1 = \Lambda_0$. Therefore, $I = 0$.

So, $J(\Lambda_0, \Lambda_1, \{\,,\})$ is a simple superalgebra. $\square$

Let Λ be a unital associative commutative algebra with a generalized Leibniz bracket $\{\,,\}$. Given $u, v \in \Lambda$, we define the mapping $D_{u,v} : \Lambda \mapsto \Lambda$ by putting $D_{u,v}(a) = \{au, v\} - a\{u, v\}$.

Lemma 3. *$D_{u,v}$ is a derivation of Λ. If the superalgebra $J(\Lambda, \{\, ,\})$ is simple then Λ is a simple differential algebra with respect to the derivations in $\mathscr{D} = \{D_{u,v} \mid u, v \in \Lambda\}$. Conversely, if Λ is a simple differential algebra with respect to the derivations in $\mathscr{D}$ then either $\{\Lambda, \Lambda\} = 0$ or $J(\Lambda, \{\, ,\})$ is a simple superalgebra.*

Proof. Take $a, b \in \Lambda$ and $u, v \in \Lambda$. By (1)

$$
\begin{aligned}
D_{u,v}(ab) &= \{abu, v\} - ab\{u, v\} = a\{bu, v\} + \{a, v\}bu + D(v)abu - ab\{u, v\} \\
&= aD_{u,v}(b) + b\{au, v\} - ba\{u, v\} = D_{u,v}(a)b + aD_{u,v}(b).
\end{aligned}
$$

Hence, $D_{u,v}$ is a derivation of Λ.

Let $J(\Lambda, \{\, ,\})$ be simple. Assume that I is an ideal of Λ invariant under the derivations in $\mathscr{D}$. Then

$$\{Iu, v\} \subseteq D_{u,v}(I) + I\{u, v\} \subseteq I$$

for all $u, v \in \Lambda$. Therefore, $\{I\Lambda, \Lambda\} \subseteq I$. Hence, $\{I, \Lambda\} \subseteq I$.

Let $K = I + I\xi$. Then $K \cdot \Lambda \subseteq K$ and $K \cdot \Lambda\xi \subseteq K + \{I, \Lambda\} \subseteq K$. Hence, K is an ideal of $J(\Lambda, \{\, ,\})$. Since $J(\Lambda, \{\, ,\})$ is simple; therefore, either $K = 0$ or $K = J(\Lambda, \{\, ,\})$. From here we infer that either $I = 0$ or $I = \Lambda$.

Let Λ be a simple differential algebra under the derivations in $\mathscr{D}$. Assume that I is an ideal of $J(\Lambda, \{\, ,\})$. Then $I = K + L\xi$, where K and L are some ideals of Λ. Furthermore,

$$\{K\Lambda, \Lambda\} \subseteq (K \cdot \Lambda\xi) \cdot \Lambda\xi \subseteq L\xi \cdot \Lambda\xi \subseteq K.$$

Hence, K is an ideal of Λ invariant under the derivations in $\mathscr{D}$. Then either $1 \in K$ or $K = 0$. We may assume that $K = 0$. Therefore, $I = L\xi$, and

$$\{L, \Lambda\} \subseteq L\xi \cdot \Lambda\xi \subseteq K = 0.$$

It follows from here that L is an ideal of Λ, which is invariant under the derivations in $\mathscr{D}$. Then either $1 \in L$ or $L = 0$. If $1 \in L$ then $L = \Lambda$. Hence, $\{\Lambda, \Lambda\} = 0$. Therefore, if $\{\Lambda, \Lambda\} \neq 0$ then $1 \notin L$, whence $L = 0$ and $I = 0$.

The lemma is proved. $\square$

Lemma 4. *Let $\Lambda = \Lambda_0 + \Lambda_1$ be a unital Z_2-graded associative commutative algebra, and let $\mathscr{D}$ be a set of even derivations of Λ. Assume that Λ is a simple differential algebra as a Z_2-graded algebra under the derivations in $\mathscr{D}$. Then either $\Lambda_1 = 0$ or $\Lambda_0 = \Lambda_1\Lambda_1$. Furthermore, either Λ is a simple differential algebra under the derivations in $\mathscr{D}$ or $\Lambda = \Lambda_0 + s\Lambda_0$, where $s \in \Lambda_1$, $s^2 = 1$, and $D(s) = 0$ for every $D \in \mathscr{D}$.*

Proof. Without loss of generality we may assume that $\mathscr{D} = \{D\}$. Note that $\Lambda_1\Lambda_1 + \Lambda_1$ is a Z_2-graded ideal of Λ, which is invariant under D. If $\Lambda_1\Lambda_1 = 0$ then Λ_1 is a Z_2-graded ideal of Λ, which is invariant under D. Therefore, either $\Lambda_1 = 0$ or $\Lambda_0 = \Lambda_1\Lambda_1$.

Let I be a proper ideal of Λ invariant under D. We may assume that $\Lambda_1 \neq 0$. Put $I_0 = I \cap \Lambda_0$. Then $I_0 + I_0\Lambda_1$ is a Z_2-graded ideal of Λ invariant under D. Therefore, either $1 \in I_0$ or $I_0 = 0$. Hence, we may assume that $I_0 = I \cap \Lambda_0 = 0$. Put $I_1 = I \cap \Lambda_1$. Then $I_1\Lambda_1 + I_1$ is a Z_2-graded ideal of Λ invariant under D. Thus, either $I_1 = \Lambda_1$ or $I_1 = 0$. Since $I_1\Lambda_1 + I_1 \subseteq I$; therefore, we may assume that $I_1 = I \cap \Lambda_1 = 0$.

Let

$$I_0 = \{a_0 \in \Lambda_0 \mid \exists a_1 \in \Lambda_1,\ a_0 + a_1 \in I\}.$$

Then I_0 is an ideal of Λ_0, and $D(I_0) \subseteq I_0$. Hence, $I_0 + I_0\Lambda_1$ is a Z_2-graded ideal of Λ invariant under D. Therefore, either $I_0 = 0$ or $I_0 = \Lambda_0$. If $I_0 = 0$ then $I \subseteq I \cap \Lambda_1 = 0$.

Hence, $I_0 = \Lambda_0$. Then $1 + s \in I$ for some $s \in \Lambda_1$, whence

$$(1 + s)(1 - s) = 1 - s^2 \in I \cap \Lambda_0 = 0.$$

Consequently, $s^2 = 1$ and $\Lambda_1 = s\Lambda_0$. Furthermore,

$$D(s) = D(1 + s) \in I \cap \Lambda_1 = 0.$$

The lemma is proved. $\square$

Lemma 5. *Let $\Lambda = \Lambda_0 + \Lambda_1$ be a unital Z_2-graded associative commutative algebra and $\Lambda_0 = \Lambda_1\Lambda_1$. Then Λ_1 is a finitely generated projective Λ_0-module of rank 1.*

Proof. Let $\Lambda_0 = \Lambda_1\Lambda_1$. Then $1 = \sum_i = x_i y_i$, where $x_i, y_i \in \Lambda_1$. Hence,

$$\Lambda_1 = \Lambda_0 x_1 + \dots + \Lambda_0 x_n;$$

i.e., Λ_1 is a finitely generated Λ_0-module. Consider the Λ_0-module $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1$. Then $x_1 \otimes y_1 + \dots + x_n \otimes y_n$ generates the Λ_0-module $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1$. Indeed,

$$u \otimes v = ((x_1 y_1 + \dots + x_n y_n) u \otimes v) = x_1 y_1 u \otimes v + \dots + x_n y_n u \otimes v$$
$$= x_0 \otimes y_1 uv + \dots + x_n \otimes y_n uv = uv(x_1 \otimes y_1 + \dots + x_n \otimes y_n)$$

for all $u, v \in \Lambda_1$. Then the mapping

$$\sum_i u_i \otimes v_i \mapsto \sum_i u_i v_i$$

is an isomorphism of Λ_0-modules $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1$ and Λ_0. Therefore, Λ_1 is a projective Λ_0-module of rank 1.

The lemma is proved. □

A generalized Leibniz bracket $\{\,,\}$ on a superalgebra $\Gamma = \Gamma_0 + \Gamma_1$ is a *generalized Poisson bracket* (see [29]) provided that

$$\{a, \{b, c\}\} + (-1)^{|a|(|b|+|c|)}\{b, \{c, a\}\} + (-1)^{|c|(|a|+|b|)}\{c, \{a, b\}\} = 0$$

for all $a, b, c \in \Gamma_0 \cup \Gamma_1$; i.e., $(\Gamma, \{\,,\})$ is a Lie superalgebra. Furthermore, if $D = 0$ then $\{\,,\}$ is a *Poisson bracket.*

A superalgebra $J = J_0 + J_1$ is a *Jordan superalgebra* provided that its Grassmann envelope $G(J)$ is a Jordan algebra; i.e.,

$$xy = yx, \quad (x^2 y)x = x^2(yx)$$

in $G(J)$.

If the Kantor double $J(\Gamma, \{\,,\})$, constructed by a bracket $\{\,,\}$, is a Jordan superalgebra then $\{\,,\}$ is a *Jordan bracket.*

A bracket $\{\,,\}$ on a unital superalgebra $\Gamma = \Gamma_0 + \Gamma_1$ is Jordan (see [1, 2]) if and only if the following hold:

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|} b\{a, c\} - \{a, 1\}bc. \tag{6}$$

$$\{a, b\}\{c, 1\} + (-1)^{|a||b|+|a||c|}\{b, c\}\{a, 1\} + (-1)^{|a||c|+|b||c|}\{c, a\}\{b, 1\}$$
$$= \{a, \{b, c\}\} + (-1)^{|a||b|+|a||c|}\{b, \{c, a\}\} + (-1)^{|a||c|+|b||c|}\{c, \{a, b\}\}, \tag{7}$$
$$\{\{d, d\}, d\} = -\{d, d\}\{d, 1\}, \tag{8}$$

where $a, b, c \in \Gamma_0 \cup \Gamma_1$, $d \in \Gamma_1$.

Thus, the bracket is Jordan if $J(a, b, c) = S(a, b, c)$ for all $a, b, c \in \Gamma_0 \cup \Gamma_1$.

The mapping $D : a \mapsto \{a, 1\}$ is a derivation of Γ by (6). Then (6) is equivalent to

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|} b\{a, c\} - D(a)bc; \tag{9}$$

i.e., $\{\,,\}$ is a generalized Leibniz bracket.

If $\{\,,\}$ is a Jordan bracket then $J(1, a, b) = S(1, a, b) = 0$ for all $a, b \in \Gamma$. Hence, D is a derivation of $(\Gamma, \{\,,\})$.

A Jordan bracket $\{\,,\}$ is a *bracket of vector type* provided that $\{a, b\} = D(a)b - aD(b)$ for all $a, b \in \Gamma$. Denote a Jordan superalgebra of a bracket of vector type by $J(\Gamma, D)$.

A Jordan bracket is a *bracket of Poisson type* provided that $D(a) = 0$ for every $a \in \Gamma$.

Let $J = A + M$ be a Jordan superalgebra with even part A and odd part M. Let $B = B_0 + B_1$ be an associative superalgebra with a product $*$. Defining on B the supersymmetric product

$$a \circ_s b = \frac{1}{2}(a * b + (-1)^{|a||b|} b * a), \quad a, b \in B_0 \cup B_1,$$

we obtain a Jordan superalgebra B^+. A Jordan superalgebra J is *special* provided that J is embedded (as a Z_2-graded algebra) into a superalgebra B^+ for a suitable associative superalgebra B. A superalgebra J is *exceptional* if J is not special.

§ 2. The Generalized Leibniz Brackets on the Coordinate Algebra of the n-Dimensional Sphere

Let $F[x_0, \dots, x_n]$ be the polynomial algebra in the variables $x_0, x_1, \dots, x_n$. Consider the polynomial $S^n(x_0, \dots, x_n) = x_0^2 + \dots + x_n^2 - 1$. Let

$$\Lambda(n) = F[x_0, \dots, x_n]/(S^n(x_0, \dots, x_n))$$

be the quotient algebra of $F[x_0, \dots, x_n]$ by the ideal $(S^n(x_0, \dots, x_n))$ generated by $S^n(x_0, \dots, x_n)$. Identify the images of $x_0, x_1, \dots, x_n$ under the canonical homomorphism $F[x_0, \dots, x_n] \mapsto \Lambda(n)$ with $x_0, x_1, \dots, x_n$, respectively. Let $\Lambda(n)_0$ be a subalgebra of $\Lambda(n)$ generated by the monomials of even degree, and let $\Lambda(n)_1 = \Lambda(n)_0 x_0 + \dots + \Lambda(n)_0 x_n$. Then $\Lambda(n) = \Lambda(n)_0 + \Lambda(n)_1$ is a Z_2-graded algebra. Denote the product of a and b in $\Lambda(n)$ by ab. The algebra $\Lambda(n)$ lacks zero divisors (for example, see [30]). By Lemma 5, $\Lambda(n)_1$ is a projective $\Lambda(n)_0$-module of rank 1.

Lemma 6. *Let $\{\,,\}$ be a generalized Leibniz bracket on $\Lambda(1)$. Then $\{\,,\}$ is a Jordan bracket of vector type with respect to the derivation $D : a \mapsto \{a, 1\}$. If the characteristic of F is zero then the Jordan superalgebra $J(\Lambda(1)_0, \Lambda(1)_1, \{\,,\})$ is simple with $D = x_1 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_1}$. If the equation $t^2 + 1 = 0$ is unsolvable in F then the $\Lambda(1)_0$-module $\Lambda(1)_1$ is not one-generated, i.e., $\Lambda(1)_1$ is not isomorphic to $\Lambda(n)_0$.*

Proof. Let $\{\,,\}$ be a generalized Leibniz bracket on $\Lambda(1)$. Put $x = x_0$ and $y = x_1$. Then

$$D(a) = \{a, 1\} = \{a, x^2 + y^2\} = 2\{a, x\}x + 2\{a, y\}y - D(a)$$

for every $a \in \Lambda(1)$. Hence,

$$D(a) = \{a, x\}x + \{a, y\}y.$$

Thus,

$$\begin{cases} D(x) = \{x, y\}y, \\ D(y) = \{y, x\}x, \end{cases}$$

whence

$$\{x, y\} = \{x, y\}x^2 + \{x, y\}y^2 = D(x)y - xD(y).$$

Then $\{\,,\}$ is a bracket of vector type by (1).

Let $D = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Then D is an even derivation of the Z_2-graded algebra $\Lambda(1)$. Therefore, the bracket

$$\{a, b\} = D(a)b - aD(b)$$

is consistent with the Z_2-grading of $\Lambda(1) = \Lambda(1)_0 + \Lambda(1)_1$. Put $D_{11} = x^2 D$, $D_{22} = y^2 D$, and $D_{12} = xyD$. The equalities

$$\{ax, bx\} = D_{11}(a)b - aD_{11}(b), \quad \{ay, by\} = D_{22}(a)b - aD_{22}(b),$$
$$\{ax, by\} = ab + D_{12}(a)b - aD_{12}(b)$$

hold, where $a, b \in \Lambda(1)_0$. If the characteristic of F is zero then $J(\Lambda(1)_0, \Lambda(1)_1, \{\,,\})$ is simple as it was shown in [19, 20]. It was also shown there that the $\Lambda(1)_0$-module $\Lambda(1)_1$ is not generated by one element if $t^2 + 1 = 0$ is unsolvable in F. $\square$

Lemma 7. *Each Jordan bracket on $\Lambda(2)$ is a generalized Poisson bracket. Let $\{\,,\}$ be a generalized Leibniz bracket on $\Lambda(2)$. Then the following are equivalent:*

(i) *$\{\,,\}$ is a Jordan bracket;*

(ii) *$D : a \mapsto \{a, 1\}$ is a derivation of $(\Lambda(2), \{\,,\})$;*

(iii) *$J(1, x_i, x_j) = 0$ for some $i \neq j$.*

Proof. Let $\{\,,\}$ be a generalized Leibniz bracket on $\Lambda(2)$. Put $x = x_0$, $y = x_1$, and $z = x_2$. By Lemma 1

$$S(1, x, y) = S(x^2 + y^2 + z^2, x, y)$$
$$= 2S(x, x, y)x + 2S(y, x, y)y + 2S(z, x, y)z = 2S(x, y, z)x.$$

Since $S(1,x,y)=0$, we have $S(x,y,z)x=0$. Analogously, $S(x,y,z)y=S(x,y,z)z=0$. Thus,

$$S(x,y,z)=S(x,y,z)x^2+S(x,y,z)y^2+S(x,y,z)z^2=0.$$

Hence, $S(a,b,c)=0$ for all $a,b,c\in\Lambda(2)$ by Lemma 1.

It follows from here that each Jordan bracket on $\Lambda(2)$ is a generalized Poisson bracket, i.e., $J(a,b,c)=0$ for all $a,b,c\in\Lambda(2)$.

Assume (i). Then $J(1,a,b)=S(1,a,b)$ for all $a,b\in\Lambda(2)$. Since $S(1,a,b)=0$, we have $J(1,a,b)=0$. So, (ii) holds.

Clearly, (ii) implies (iii).

Show that

$$J(x,y,z)x=J(1,y,z)=-D(\{y,z\})+\{y,D(z)\}+\{D(y),z\}.$$

By Lemma 1 we get

$$J(1,y,z)=J(x^2+y^2+z^2,y,z)=2xJ(x,y,z)-J(1,y,z)(x^2+y^2+z^2).$$

Hence, $J(1,y,z)=xJ(x,y,z)$. Analogously, $J(1,x,y)=zJ(x,y,z)$, and $J(1,x,z)=-yJ(x,y,z)$.

Assume (iii). Consider the case $J(1,y,z)=0$. Then $J(x,y,z)x=0$. Since $\Lambda(2)$ lacks zero divisors, $J(x,y,z)=0$. Thus, $J(a,b,c)=0$ for all $a,b,c\in\Lambda(2)$ by Lemma 1. Thus, $\{\,,\}$ is a Jordan bracket on $\Lambda(2)$; i.e., (i) holds. $\square$

Corollary 1. *Each solution u_1, u_2, u_3 to the system*

$$\begin{cases} u_1y+u_2z=0,\\ u_1x-u_3z=0,\\ u_2x+u_3y=0 \end{cases}$$

in $\Lambda(2)$ defines a Jordan bracket of Poisson type on $\Lambda(2)$. In particular, each Leibniz bracket on $\Lambda(2)$ is a Jordan bracket of Poisson type. Each Jordan bracket on $\Lambda(2)$ is a sum of Jordan brackets of vector and Poisson types.

Proof. Let $\{\,,\}$ be a Jordan bracket on $\Lambda(2)$. Then (6) holds, and $D:a\mapsto\{a,1\}$ is a derivation of $\Lambda(2)$. Define the new bracket on $\Lambda(2)$ by putting

$$\langle a,b\rangle=\{a,b\}-D(a)b+aD(b).$$

Then $\langle\,,\rangle$ is a Leibniz bracket, and $\langle\,,\rangle$ is a Jordan bracket of Poisson type by Lemma 7. It follows from here that $\{\,,\}$ is a sum of a bracket of vector type and a Jordan bracket of Poisson type. $\square$

Theorem 2. *The algebra $\Lambda(n)$ may be equipped with a Jordan bracket $\{\,,\}_n$ for every $n\geq 1$, which is consistent with the Z_2-grading of $\Lambda(n)=\Lambda(n)_0+\Lambda(n)_1$ and such that the following hold:*

1. *If $n=1$ then $\{a,b\}_1=D(a)b-aD(b)$ for $a,b\in\Lambda(1)$, where $D=x_1\frac{\partial}{\partial x_0}-x_0\frac{\partial}{\partial x_1}$.*
2. *Let k be even, $k\leq n$, and $z(k)=\sum_{i=0}^k(-1)^ix_i$. Then $\{z(k),x_i\}_n=0$ for $i\leq k$.*
3. *The element $z(2)=x_0-x_1+x_2$ belongs to the center of the Lie algebra $(\Lambda(2),\{\,,\}_2)$.*
4. *$I=z(2)\Lambda(2)_1+z(2)\Lambda(2)_0\xi$ is an ideal of $J(\Lambda(2)_0,\Lambda(2)_1,\{\,,\}_2)$.*

Proof. Using the definition of Jordan bracket, put $\{a,b\}_n=-\{b,a\}_n$ for all $a,b\in\Lambda(n)$. Define the bracket $\{\,,\}_n$ and a derivation D on the generators $1,x_0,\dots,x_n$ as follows:

$$\{x_i,x_j\}_n=1,\quad 0\leq i<j\leq n,$$

$$D(x_i)_n=\{x_i,1\}_n=-\sum_{k=0}^{i-1}x_k+\sum_{k=i+1}^{n}x_k,\quad i=0,\dots,n.$$

Using (1), $\{\,,\}_n$ may be extended to $\Lambda(n)$. The bracket $\{\,,\}_n$ is correctly defined.

Show that (7) holds. By Lemma 1, it suffices to verify (7) for the generators of $\Lambda(n)$. Let $0 \le i < j < k \le n$. Then

$$J(x_i, x_j, x_k) = \{x_i, 1\}_n - \{x_j, 1\}_n + \{x_k, 1\}_n = D(x_i) - D(x_j) + D(x_k).$$

On the other hand,

$$S(x_i, x_j, x_k) = D(x_k) + D(x_i) - D(x_j).$$

Hence, $J(x_i, x_j, x_k) = S(x_i, x_j, x_k)$. If $i = j$ then

$$J(x_i, x_i, x_k) = S(x_i, x_i, x_k) = 0.$$

By Lemma 1

$$J(1, x_j, x_k) = J\left(\sum_{i=0}^{n} x_i^2, x_j, x_k\right) = \sum_{i=0}^{n} 2x_i J(x_i, x_j, x_k) - J(1, x_j, x_k).$$

Consequently, $J(1, x_j, x_k) = \sum_{i=0}^{n} x_i J(x_i, x_j, x_k)$. By the above,

$$J(x_i, x_j, x_k) = S(x_i, x_j, x_k).$$

Hence,

$$J(1, x_j, x_k) = \sum_{i=0}^{n} x_i J(x_i, x_j, x_k) = \sum_{i=0}^{n} x_i S(x_i, x_j, x_k) = \frac{1}{2} S(1, x_j, x_k)$$

by Lemma 1. Since $S(1, x_j, x_k) = 0$, we have $J(1, x_j, x_k) = 0$.

Thus, (7) holds, i.e., $\{\,,\}_n$ is a Jordan bracket. Clearly, $\{\,,\}_n$ is consistent with the Z_2-grading of $\Lambda(n)$.

Item 1 holds by Lemma 6.

Prove Item 2: Let $i \le k$. Then

$$\{z(k), x_i\}_n = \sum_{j=0}^{i-1} (-1)^j + \sum_{j=i+1}^{k} -(-1)^j.$$

If i is even then $\sum_{j=0}^{i-1} (-1)^j = 0$, and $\sum_{j=i+1}^{k} -(-1)^j = 0$. Then $\{z(k), x_i\}_n = 0$. If i is odd then $\sum_{j=0}^{i-1} (-1)^j = 1$, and $\sum_{j=i+1}^{k} -(-1)^j = -1$. Then $\{z(k), x_i\}_n = 0$.

Prove Item 3: By Lemma 7, $(\Lambda(2), \{\,,\}_2)$ is a Lie algebra. By Item 2, $D(z(2)) = \{z(2), 1\}_2 = 0$ and $\{z(2), x_i\}_2 = 0$, where $i = 0, 1, 2$. Thus, $\varphi : a \in \Lambda(2) \mapsto \{z(2), a\}_2$ is a derivation of $\Lambda(2)$ by (1). Since $\varphi(1) = \varphi(x_i) = 0$ for $i = 0, 1, 2$; therefore, $\varphi = 0$. Hence, $\{z(2), \Lambda(2)\}_2 = 0$, i.e., $z(2)$ belongs to the center of the Lie algebra $(\Lambda(2), \{\,,\}_2)$.

Prove Item 4: Let $I = z(2)\Lambda(2)_1 + z(2)\Lambda(2)_0\xi$. By the definition of product in $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$ we get

$$I \cdot \Lambda(2)_0 \subseteq z(2)\Lambda(2)_1\Lambda(2)_0 + (z(2)\Lambda(2)_0\Lambda(2)_0)\xi \subseteq I.$$

Analogously,

$$z(2)\Lambda(2)_1 \cdot \Lambda(2)_1\xi \subseteq (z(2)\Lambda(2)_1\Lambda(2)_1)\xi \subseteq z(2)\Lambda(2)_0\xi \subseteq I.$$

By (1) and Item 3

$$\Lambda(2)_1\xi \cdot z(2)\Lambda(2)_0\xi \subseteq \{\Lambda(2)_1, z(2)\Lambda(2)_0\}_2$$
$$\subseteq z(2)\{\Lambda(2)_1, \Lambda(2)_0\}_2 + D(\Lambda(2)_1)z(2)\Lambda(2)_0 \subseteq z(2))\Lambda(2)_1 \subseteq I.$$

Hence, I is an ideal of $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$.

Assume that $I = J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$. Then $z(2)t = (x_0 - x_1 + x_2)t = 1$ for some $t \in \Lambda(2)_1$. Since $\Lambda(2) = F[x_0, x_1] \oplus x_2 F[x_0, x_1]$; therefore, $t = f + x_2 g$, where $f, g \in F[x_0, x_1]$, whence

$$(x_0 - x_1)f + \bigl(1 - x_0^2 - x_1^2\bigr)g = 1, f + (x_0 - x_1)g = 0.$$

Hence,

$$\bigl(\bigl(1 - x_0^2 - x_1^2\bigr) - (x_0 - x_1)^2\bigr)g = 1.$$

It follows from here that $1 - 2x_0^2 - 2x_1^2 + 2x_0x_1$ is invertible in $F[x_0, x_1]$. Thus, I is a proper ideal of $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$. $\square$

Lemma 8. *The Jordan superalgebra $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$ is exceptional. In particular, $J(\Lambda(2), \{\,,\}_2)$ is an exceptional Jordan superalgebra.*

Proof. Put $x = x_0$, $y = x_1$, $z = x_2$, and $u = x - y + z$. Define the bracket $\langle\,,\rangle$ on $\Lambda(2)$ by putting $\langle a, b\rangle = \{a, b\}_2 - D(a)b + aD(b)$, where D is a derivation of $\{\,,\}_2$. Then $\langle \Lambda(2)_0, \Lambda(2)_0\rangle \subseteq \Lambda(2)_0$, and $\langle a, bc\rangle = b\langle a, c\rangle + \langle a, b\rangle c$ for $a, b, c \in \Lambda(2)$.

Let $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$ be a special superalgebra. Then the subsuperalgebra $J = \Lambda(2)_0 + (\Lambda(2)_0 u)\xi$ is special, and J is a superalgebra of the Jordan bracket $\{a, b\} = \{au, bu\}_2$ on $\Lambda(2)_0$. By Theorem 2, $\{\Lambda(2), u\}_2 = 0$. Thus, $\{\,,\}$ is a bracket of Poisson type. By [7], $\{\{a, b\}, c\} = 0$ for $a, b, c \in \Lambda(2)_0$. Since $\{\Lambda(2), u\}_2 = 0$; therefore,

$$\{a, b\} = \{au, bu\}_2 = \{a, b\}_2 u^2 - D(a)bu^2 + aD(b)u^2 = \langle a, b\rangle u^2$$

for $a, b \in \Lambda(2)_0$ by (1). Hence,

$$\begin{aligned} 0 = \{\{a, b\}, c\} &= \langle\langle a, b\rangle u^2, c\rangle u^2 \\ &= \langle\langle a, b\rangle, c\rangle u^4 + \langle a, b\rangle\langle u^2, c\rangle u^2 = \langle\langle a, b\rangle, c\rangle u^4 + 2\langle a, b\rangle D(c)u^4 \end{aligned}$$

for $a, b, c \in \Lambda(2)_0$. Thus, $\langle\langle a, b\rangle, c\rangle = -2\langle a, b\rangle D(c)$ for $a, b, c \in \Lambda(2)_0$.

Let $a = x^2$, $b = y^2$, and $c = z^2$. Then

$$\langle a, b\rangle = 4xy\langle x, y\rangle = 4xy(1 - (y + z)y + x(-x + z)) = 4xyzu.$$

So,

$$\begin{aligned} \langle\langle a, b\rangle, c\rangle &= \langle 4xyzu, z^2\rangle = 8z^2\langle xyu, z\rangle = 8z^2u(x\langle y, z\rangle + \langle x, z\rangle y) + 8xyz^2\langle u, z\rangle \\ &= 8z^2u(x(1 - (-x + z)z + y(-x - y))) + y(1 - (y + z)z + x(-x - y)) + 8xyz^2uD(z) \\ &= 8z^2u^2(x^2 - y^2) - 8xyz^2u(x + y). \end{aligned}$$

On the other hand, $2\langle a, b\rangle D(c) = -16xyz^2u(x + y)$. Hence,

$$8z^2u^2(x^2 - y^2) - 8xyz^2u(x + y) = 16xyz^2u(x + y).$$

Since $\Lambda(2)$ has no zero divisors, $u(x - y) = 3xy$; a contradiction. Consequently, $J = \Lambda(2)_0 + (\Lambda(2)_0 u)\xi$ is an exceptional superalgebra.

Thus, $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$ is an exceptional superalgebra. Since $J(\Lambda(2)_0, \Lambda(2)_1, \{\,,\}_2)$ is a subsuperalgebra of $J(\Lambda(2), \{\,,\}_2)$; therefore, $J(\Lambda(2), \{\,,\}_2)$ is an exceptional superalgebra. $\square$

Lemma 9. *The algebra $\Lambda(n)$ may be equipped with a Leibniz bracket which is not Jordan if $n \geq 3$.*

Proof. Consider the Jordan bracket on $\Lambda(n)$ which was defined in Theorem 2. Namely,

$$\{x_i, x_i\}_n = 0, \quad \{x_i, x_j\}_n = 1, \quad 0 \leq i < j \leq n,$$

$$D(x_i)_n = \{x_i, 1\}_n = -\sum_{k=0}^{i-1} x_k + \sum_{k=i+1}^{n} x_k, \quad i = 0, \dots, n.$$

Let $J(a, b, c)_n$ be the Jacobian of $\{\,,\}_n$, and let $S(a, b, c)_n$ be the function S of the bracket $\{\,,\}_n$, which is defined by (2).

Define the Leibniz bracket on $\Lambda(n)$ by

$$\langle a, b\rangle = \{a, b\} - D(a)b + D(b)a.$$

Denote by $J(a, b, c)_{\langle,\rangle}$ and $S(a, b, c)_{\langle,\rangle}$ the Jacobian and the function S of $\langle\,,\rangle$. Since $\{\,,\}_n$ is a Jordan bracket then $J(1, a, b)_n = 0$ for all $a, b \in \Lambda(n)$. By Lemma 2

$$J(a, b, c)_{\langle,\rangle} = J(a, b, c)_n + S(a, b, c)_n.$$

Hence,

$$J(x_i, x_j, x_k)_{\langle,\rangle} = 2(D(x_i) - D(x_j) + D(x_k))$$

with $0 \le i < j < k \le n$. Inserting the values $D(x_i)$, $D(x_j)$, and $D(x_k)$, we get

$$\begin{aligned} D(x_i) - D(x_j) + D(x_k) &= -\sum_{l=0}^{i-1} x_l + \sum_{l=i+1}^{n} x_l + \sum_{l=0}^{j-1} x_l - \sum_{l=j+1}^{n} x_l - \sum_{l=0}^{k-1} x_l + \sum_{l=k+1}^{n} x_l \\ &= -\sum_{l=0}^{i-1} x_l + \sum_{l=i+1}^{j-1} x_l - \sum_{l=j+1}^{k-1} x_l + \sum_{l=k+1}^{n} x_l. \end{aligned}$$

Hence, $J(x_i, x_j, x_k)_{\langle,\rangle} \ne 0$.

On the other hand, $\langle a, 1\rangle = 0$ for $a \in \Lambda(n)$; i.e., $S(x_i, x_j, x_k)_{\langle,\rangle} = 0$. Thus, $J(x_i, x_j, x_k)_{\langle,\rangle} \ne S(x_i, x_j, x_k)_{\langle,\rangle}$, and $\langle,\rangle$ is not a Jordan bracket. □

§ 3. The Simple Abelian Jordan Superalgebras Defined by the n-Sphere

A superalgebra $J = A + M$ is *abelian* provided that A is an associative algebra, and M is an associative A-module. In [22] the following assertion was proved:

Let $J = A+M$ be a simple abelian Jordan superalgebra not isomorphic to a superalgebra of a bilinear form. Put $D_{x,y}(a) = (ax)y - a(xy)$, where $a \in A$ and $x, y \in M$. Then A is a simple differential algebra with respect to the set of derivations $\Delta = \{D_{x,y} \mid x, y \in M\}$ of A. Moreover, J is a unital superalgebra, and M is a finitely generated projective A-module of rank 1.

The following theorem was proved in [18]:

Theorem 3. *Let $J = A + M$ be a unital simple special Jordan superalgebra not isomorphic to a superalgebra of a bilinear form. Then there exist $x_1, \dots, x_n \in M$ such that $M = Ax_1 + \dots + Ax_n$, and the product in M is given by*

$$ax_i \cdot bx_j = \gamma_{ij} ab + D_{ij}(a)b - aD_{ji}(b), \quad i, j = 1, \dots, n,$$

where $\gamma_{ij} \in A$, and D_{ij} is a derivation of A. The algebra A is a simple differential algebra with respect to the set of derivations $\Delta = \{D_{ij} \mid i, j = 1, \dots, n\}$. Moreover, J is a subalgebra of the Jordan superalgebra of vector type $J(\Gamma, D)$.

In [18–21], some examples were constructed of unital simple special abelian Jordan superalgebras whose odd part is generated as a module by two elements and is not generated by one element. Thus, the constructed examples of superalgebras are not isomorphic to a superalgebra of vector type $J(\Gamma, D)$.

In this section, we construct some examples of unital simple exceptional abelian Jordan superalgebras $J = A + M$ whose even part has an arbitrary number of generators as an A-module.

Theorem 4. *Let F be a field of characteristic 0, and let n be odd. Then the Jordan superalgebra $J(\Lambda(n), \{\,,\,\}_n)$ is simple. In particular, $J(\Lambda(n)_0, \Lambda(n)_1, \{\,,\,\}_n)$ is a simple superalgebra.*

Proof. Denote $\{\,,\,\}_n$ by $\{\,,\,\}$. By Theorem 2

$$\{x_i, x_i\} = 0, \quad \{x_i, x_j\} = 1, \quad 0 \le i < j \le n,$$

$$D(x_i) = -\sum_{j=0}^{i-1} x_j + \sum_{j+1}^{n} x_j, \quad i = 0, \dots, n.$$

Let $D_{u,v}(a) = \{au, v\} - a\{u, v\}$ and $\mathscr{D} = \{D_{u,v} \mid u, v \in \Lambda(n)\}$. By Lemma 3, it suffices to prove that $\Lambda(n)$ is a simple differential algebra with respect to the derivations in $\mathscr{D}$.

Assume that $\Lambda(n)$ is not a simple differential algebra. Then the set of ideals of $\Lambda(n)$, which are invariant under the derivations in $\mathscr{D}$, has a maximal ideal I. By [31], I is a simple ideal. Since $D_{u,v}(I) \in I$ for all $u, v \in \Lambda(n)$, we have

$$\{I, u\} \subseteq D_{1,u}(I) + I\{1, u\} \subseteq I.$$

Hence, $\{I, \Lambda(n)\} \subseteq I$. Note that $x_i \notin I$, since otherwise $\{x_i, x_j\} = \pm 1 \in I$ and $I = \Lambda(n)$.

Since $\Lambda(n) = F[x_0, \ldots, x_{n-1}] + x_n F[x_0, \ldots, x_{n-1}]$ and $\Lambda(n)$ has no zero divisors, $F[x_0, , \ldots, x_{n-1}] \cap I \neq 0$.

Take $u \in \Lambda(n)$. The mapping $d : a \mapsto \{u, a\} - D(u)a$ is a derivation of $\Lambda(n)$ by (1). Then

$$d\big(x_0^{i_0} \ldots x_{n-1}^{i_{n-1}}\big) = \sum_{j=0}^{n-1} d\big(x_j^{i_j}\big) x_0^{i_0} \ldots x_{j-1}^{i_{j-1}} x_{j+1}^{i_{j+1}} \ldots x_{n-1}^{i_{n-1}}$$
$$= \sum_{j=0}^{n-1} i_j d(x_j) x_0^{i_0} \ldots x_{j-1}^{i_{j-1}} x_j^{i_j - 1} x_{j+1}^{i_{j+1}} \ldots x_{n-1}^{i_{n-1}}.$$

Hence,

$$\big\{u, x_0^{i_0} \ldots x_{n-1}^{i_{n-1}}\big\} = \sum_{j=0}^{n-1} i_j \{u, x_j\} x_0^{i_0} \ldots x_{j-1}^{i_{j-1}} x_j^{i_j - 1} x_{j+1}^{i_{j+1}} \ldots x_{n-1}^{i_{n-1}}$$
$$- D(u)\left(\sum_{j=0}^{n-1} i_j - 1\right) x_0^{i_0} \ldots x_{n-1}^{i_{n-1}}.$$

Take $f \in F[x_0, x_1, \ldots, x_{n-1}] \cap I$, $f \neq 0$. Then $f = f_k + f_{k-1} + \cdots + f_0$, where every nonzero f_i is a homogeneous polynomial in $F[x_0, x_1, \ldots, x_{n-1}]$ of degree i, $i = 0, \ldots, k$. Let the minimal degree of f be k.

Assume that $k > 0$. Let $z(n-1) = \sum_{i=0}^{n-1} (-1)^i x_i$. By Item 2 of Theorem 2, we have $\{z(n-1), x_i\} = \{z(n-1), x_i\}_n = 0$, $i = 0, \ldots, n-1$. Then

$$D(z(n-1)) = \{z(n-1), 1\} = \sum_{i=0}^{n} \{z(n-1), x_i\} x_i = \{z(n-1), x_n\} x_n = x_n.$$

Hence,

$$\big\{z(n-1), x_0^{i_0} \ldots x_{n-1}^{i_{n-1}}\big\} = -x_n \left(\sum_{j=0}^{n-1} i_j - 1\right) x_0^{i_0} \ldots x_{n-1}^{i_{n-1}}.$$

Thus, $\{z(n-1), f_i\} = -(i-1) x_n f_i$, where $i = 0, \ldots, k$, whence

$$\{z(n-1), f\} = \{z(n-1), f_k + f_{k-1} + \cdots + f_0\} = (-(k-1) f_k - (k-2) f_{k-1} + \cdots + f_0) x_n.$$

Since $\{z(n-1), f\} \in I$, I is a simple ideal and $x_n \notin I$; therefore, $g = -(k-1) f_k - (k-2) f_{k-1} + \cdots + f_0 \in I$. Then

$$(k-1) f + g = f_{k-1} + 2 f_{k-2} + \cdots + k f_0 \in F[x_0, x_1, \ldots, x_{n-1}] \cap I$$

and the degree of $(k-1)f + g$ is less than k. By the choice of f, we may assume that f is a homogeneous polynomial of degree k.

Let

$$f = \sum_{i_0 + \cdots + i_{n-1} = k} \alpha_{i_0 \ldots i_{n-1}} x_0^{i_0} \ldots x_{n-1}^{i_{n-1}},$$

where $\alpha_{i_0 \dots i_{n-1}} \in F$. Let x_j be the variable in f with the greatest index. Without loss of generality we may assume that $x_j = x_{n-1}$. Put $u = x_{n-1} - x_n$. Then $\{u, x_i\} = 0$, $i = 0, \dots, n-2$, and $\{u, x_{n-1}\} = 1$. Since $f \in I, \{u, f\} \in I$ and

$$\{u, f\} = \sum_{i_0+\dots+(i_{n-1}-1)=k-1} i_{n-1}\alpha_{i_0,\dots,i_{n-1}} x_0^{i_0} \dots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}-1} - (k-1) f D(u);$$

therefore,

$$\sum_{i_0+\dots+(i_{n-1}-1)=k-1} i_{n-1}\alpha_{i_0,\dots,i_{n-1}} x_0^{i_0} \dots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}-1} \in I.$$

By the choice of f, we get

$$\sum_{i_0+\dots+(i_{n-1}-1)=k-1} i_{n-1}\alpha_{i_0,\dots,i_{n-1}} x_0^{i_0} \dots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}-1} = 0,$$

whence

$$\sum_{i_0+\dots+i_{n-2}=k-i_{n-1}} \alpha_{i_0,\dots,i_{n-1}} x_0^{i_0} \dots x_{n-2}^{i_{n-2}} = 0$$

if $i_{n-1} \neq 0$. Thus, x_{n-1} does not appear in f; a contradiction. Hence, $k = 0$, $f \in F$, and $I = \Lambda(n)$.

Thus, the superalgebra $J(\Lambda(n), \{\,,\}_n)$ is simple. By Theorem 1, we infer that $J(\Lambda(n)_0, \Lambda(n)_1, \{\,,\}_n)$ is a simple superalgebra. $\square$

Let $X = S^n \times \mathbb{A}^m$ be the Cartesian product of the n-dimensional sphere and the m-dimensional affine space. Then X is an irreducible variety. So, the coordinate algebra $F[X] = \Lambda(n, m)$ of X has no zero divisors. Furthermore,

$$\Lambda(n, m) = F[x_0, x_1, \dots, x_n, y_1, \dots, y_m]/(S^n(x_0, \dots, x_n))$$

is the quotient of $F[x_0, x_1, \dots, x_n, y_1, \dots, y_m]$ by $S^n(x_0, \dots, x_n)F[x_0, x_1, \dots, x_n, y_1, \dots, y_m]$. So, $\Lambda(n, m)$ $= \Lambda(n, m)_0 + \Lambda(n, m)_1$ is a Z_2-graded algebra, where $\Lambda(n, m)_0$ is a subalgebra of $\Lambda(n, m)$ generated by the monomials of even degree, and $\Lambda(n, m)_1 = \Lambda(n, m)_0 x_0 + \dots + \Lambda(n, m)_0 x_n$. By Lemma 5, $\Lambda(n, m)_1$ is a projective $\Lambda(n, m)_0$-module of rank 1. Note that $\Lambda(n)$ is a subalgebra of $\Lambda(n, m)$.

Lemma 10. *The algebra $\Lambda(n, m)$ may be equipped with a Jordan bracket $\{\,,\}_{n,m}$, which is consistent with the Z_2-grading of $\Lambda(n, m) = \Lambda(n, m)_0 + \Lambda(n, m)_1$ for all $n \geq 1$ and m. Moreover, $(\Lambda(n), \{\,,\}_n)$ is a subalgebra of $(\Lambda(n, m), \{\,,\}_{n,m})$.*

PROOF. Define the bracket $\{\,,\}_{n,m}$ on $\Lambda(n, m)$ by putting $\{a, b\}_{n,m} = \{a, b\}_n$ on $\Lambda(n)$, and by

$$\{x_i, y_j\} = 1, \quad \{y_j, y_j\} = 0, \quad \{y_j, y_k\} = 1, \quad j < k,$$

$$D(y_j) = \{y_j, 1\} = -(x_0 + \dots + x_n), \quad i = 0, \dots, n, \ j, k = 1, \dots, m,$$

at the generators $y_1, \dots, y_m$.

Using (1), $\{\,,\}_{n,m}$ may be extended to $\Lambda(n, m)$. The bracket $\{\,,\}_{n,m}$ is given correctly and is consistent with the Z_2-grading of $\Lambda(n, m) = \Lambda(n, m)_0 + \Lambda(n, m)_1$. Clearly, $(\Lambda(n), \{\,,\}_n)$ is a subalgebra of $(\Lambda(n, m), \{\,,\}_{n,m})$.

Show that (7) holds. By Lemma 1, it suffices to verify (7) for the generators of $\Lambda(n, m)$. By Theorem 2, $J(a, b, c) = S(a, b, c)$ for $a, b, c \in \Lambda(n)$.

Let $0 \leq i < j \leq n$ and $1 \leq k \leq m$. Then

$$J(x_i, x_j, y_k) = \{x_i, 1\}_n - \{x_j, 1\}_n + \{y_k, 1\}_n = D(x_i) - D(x_j) + D(y_k).$$

However, $S(x_i, x_j, y_k) = D(y_k) + D(x_i) - D(x_j)$. Hence, $J(x_i, x_j, y_k) = S(x_i, x_j, y_k)$. Analogously, $J(x_i, y_j, y_k) = S(x_i, y_j, y_k)$ and $J(y_j, y_k, y_l) = S(y_j, y_k, y_l)$ for $0 \leq i \leq n$ and $1 \leq j < k < l \leq m$.

By Lemma 1

$$J(1,a,b) = J\bigg(\sum_{i=0}^{n} x_i^2, a, b\bigg) = \sum_{i=0}^{n} 2x_i J(x_i,a,b) - J(1,a,b)$$

for $a,b \in \Lambda(n,m)$. Thus, $J(1,a,b) = \sum_{i=0}^{n} x_i J(x_i,a,b)$. Let $a,b \in \{x_0,\dots,x_n,y_1,\dots,y_m\}$. Then by the above $J(x_i,a,b) = S(x_i,a,b)$, $i = 0,\dots,n$. Hence,

$$J(1,a,b) = \sum_{i=0}^{n} x_i J(x_i,a,b) = \sum_{i=0}^{n} x_i S(x_i,a,b) = \frac{1}{2} S(1,a,b)$$

by Lemma 1. Since $S(1,a,b) = 0$, we have $J(1,a,b) = 0$.

Thus, (7) holds, i.e., $\{\,,\}_{n,m}$ is a Jordan bracket. $\square$

Theorem 5. *Let F be a field of characteristic 0, and let n be even. Then the Jordan superalgebra $J(\Lambda(n,1),\{\,,\}_{n,1})$ is simple. In particular, $J(\Lambda(n,1)_0,\Lambda(n,1)_1,\{\,,\}_{n,1})$ is a simple superalgebra. If $n = 2$ then $J(\Lambda(n,1)_0,\Lambda(n,1)_1,\{\,,\}_{n,1})$ is an exceptional superalgebra.*

Proof. Denote the bracket $\{\,,\}_{n,1}$ by $\{\,,\}$. By Lemma 10

$$\{x_i,x_j\} = 1,\ i<j, \quad \{x_i,y_1\} = 1,\ i,j = 0,\dots,n,$$

$$D(x_i) = -\sum_{j=0}^{i-1} x_j + \sum_{j+1}^{n} x_j, i = 0,\dots,n, D(y_1) = -(x_0+\dots+x_n).$$

Repeating the arguments of Theorem 4, we may assume that $\Lambda(n,1)$ has a simple ideal I such that $\{I,\Lambda(n,1)\} \subseteq I$ and $x_i \notin I$.

Since $\Lambda(n,1) = F[x_0,\dots,x_{n-1},y_1] + x_n F[x_0,\dots,x_{n-1},y_1]$ and $\Lambda(n,1)$ has no zero divisors, $F[x_0,,\dots,x_{n-1},y_1] \cap I \neq 0$.

Let $z(n) = \sum_{i=0}^{n} (-1)^i x_i$. By Item 2 of Theorem 2, we have $\{z(n),a\} = \{z(n),a\}_n = 0$ for every $a \in \Lambda(n)$. Hence, the mapping $\varphi : \Lambda(n,1) \mapsto \Lambda(n,1)$, defined by the rule $\varphi(a) = \{z(n),a\}$, is a derivation. Furthermore, $\phi(I) = \{z(n),I\} \subseteq I$.

Let $f \in F[x_0,\dots,x_{n-1},y_1] \cap I$ and $f \neq 0$. Then

$$f = f_0 y_1^k + f_1 y_1^{k-1} + \dots + f_k,$$

where $f_i \in F[x_0,\dots,x_{n-1}]$, $i = 0,\dots,k$. Assume that f is a polynomial of minimal degree $k \geq 1$. Then

$$\phi(f) = \big(k f_0 y_1^{k-1} + (k-1) f_1 y_1^{k-2} + \dots + f_{k-1}\big)\phi(y_1) \in I,$$

because of $\phi(f_i) = 0$, $i = 0,\dots,k$. Since I is a simple ideal, we get $\varphi(y_1) \in I$ by the choice of f. Hence, $I = \Lambda(n,1)$, because of $\varphi(y_1) = \{z(n),y_1\} = 1$. Thus, $f \in F[x_0,\dots,x_{n-1}] \cap I$.

Let $f \in F[x_0,x_1,\dots,x_{n-1}] \cap I$, $f \neq 0$. Then $f = f_k + f_{k-1} + \dots + f_0$, where each nonzero f_i is a homogeneous polynomial in $F[x_0,x_1,\dots,x_{n-1}]$ of degree i, $i = 0,\dots,k$. Let f be of minimal degree k.

Assume that $k > 0$. Let $u = x_n - y_1$. Then $\{u,x_i\} = 0$, $i = 0\dots,n-1$, and $D(u) = x_n$. Repeating the arguments of Theorem 4, we get

$$\big\{u, x_0^{i_0}\dots x_{n-1}^{i_{n-1}}\big\} = -\bigg(\sum_{j=0}^{n-1} i_j - 1\bigg) D(u) x_0^{i_0}\dots x_{n-1}^{i_{n-1}} = -\bigg(\sum_{j=0}^{n-1} i_j - 1\bigg) x_n x_0^{i_0}\dots x_{n-1}^{i_{n-1}}.$$

Thus, $\{u,f_i\} = -(i-1)x_n f_i$, where $i = 0,\dots,k$, whence

$$\{u,f\} = (-(k-1)f_k - (k-2)f_{k-1} + \dots + f_0)x_n.$$

Since $\{u, f\} \in I$, I is the simple ideal, and $x_n \notin I$; therefore,

$$g = -(k-1)f_k - (k-2)f_{k-1} + \dots + f_0 \in F[x_0, x_1, \dots, x_{n-1}] \cap I.$$

Then $(k-1)f + g \in I$, and the degree of $(k-1)f + g$ is less than k. By the choice of f, we may assume that f is a homogeneous polynomial of degree k.

Repeating the arguments of Theorem 4, we get $k = 0$, $f \in F$, and $I = \Lambda(n, 1)$.

Thus, the superalgebra $J(\Lambda(n, 1), \{\,,\}_{n,1})$ is simple. Then $J(\Lambda(n, 1)_0, \Lambda(n, 1)_1, \{\,,\}_{n,1})$ is a simple superalgebra by Theorem 1. By Lemmas 8 and 10, $J(\Lambda(2, 1)_0, \Lambda(2, 1)_1, \{\,,\}_{2,1})$ is an exceptional superalgebra. □

The $\Lambda(n)_0$-module $\Lambda(n)_1$ of the Z_2-graded algebra $\Lambda(n)$ is equal to $\Lambda(n)_0x_0+\dots+\Lambda(n)_0x_n$; i.e., $\Lambda(n)_1$ is generated by $n + 1$ elements. The question arises of the number of generators of $\Lambda(n)_1$. To this end, we have

Theorem 6 [32]. *Let R be the field of reals, $\Lambda(n) = R[x_0, \dots, x_n]/(S^n(x_0, \dots, x_n))$, let $\Lambda(n)_0$ be the subalgebra of $\Lambda(n)$ generated by the monomials of even degree, and let $\Lambda(n)_1 = \Lambda(n)_0x_0 + \dots + \Lambda(n)_0x_n$. Then the $\Lambda(n)_0$-module $\Lambda(n)_1$ cannot be generated by less than $n + 1$ elements.*

Let C be the field of complexes, and put $\Lambda(n) = C[x_0, \dots, x_n]/(S^n(x_0, \dots, x_n))$. Then, as shown in [30], the module $\Lambda(n)_1$ is generated either by $\frac{n+1}{2}$ elements if n is odd or by $\frac{n}{2} + 1$ elements if n is even. Namely, $u_k = x_{2k} + ix_{2k+1}$, $k = 0, \dots, \frac{n+1}{2} - 1$ if n is odd, and $u_k = x_{2k} + ix_{2k+1}$, $k = 0, \dots, \frac{n}{2} - 1$, $u_{\frac{n}{2}} = x_n$ if n is even. Thus, the $\Lambda(n)_0$-module $\Lambda(n)_1$ cannot be generated by at most $\left[\frac{n}{2}\right] + 1$ elements.

This leads to

Theorem 7. *Let $F = R$ (C) be the field of reals (complexes), and let $J = A + M$ be a Jordan superalgebra. Assume that $J = J(\Lambda(n)_0, \Lambda(n)_1, \{\,,\}_n)$ if n is odd, and $J = J(\Lambda(n, 1)_0, \Lambda(n, 1)_1, \{\,,\}_{n,1})$ if n is even. Then J is a simple superalgebra. The number of generators of M as an A-module is at least $n + 1$ if $F = R$, and it is at least $\left[\frac{n}{2}\right] + 1$ if $F = C$. The superalgebra $J(\Lambda(2, 1)_0, \Lambda(2, 1)_1, \{\,,\}_{2,1})$ is exceptional. Moreover, if $n > 1$ or $F = R$ then J is not isomorphic to a superalgebra of a Jordan bracket.*

Proof. Prove the last assertion. Let $n > 1$. Assume that $J = A + M$ is isomorphic to the superalgebra of the Jordan bracket $J(\Gamma, \{\,,\})$ where $\Gamma = \Gamma_0 + \Gamma_1$ is an associative commutative superalgebra. Then $\Gamma_0 + \Gamma_1\xi$ is the even part and $\Gamma_1 + \Gamma_0\xi$ is the odd part of $J(\Gamma, \{\,,\})$. Given $a \in \Gamma_1$, we have $a\xi \cdot a = -a^2\xi = 0$. Hence, the $(\Gamma_0 + \Gamma_1\xi)$-module $\Gamma_1 + \Gamma_0\xi$ is a module with torsion. Since the A-module M is torsion free, $\Gamma_1 = 0$. Thus, the odd part $\Gamma_0\xi$ of $J(\Gamma, \{\,,\})$ is a one-generated Γ_0-module. Then the A-module M is generated by one element; a contradiction. The case $F = R$ is proved analogously.

Thus, J is not isomorphic to a superalgebra of the Jordan bracket. □

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