

ANALOGS OF THE LIOUVILLE PROPERTY FOR HARMONIC FUNCTIONS ON UNBOUNDED DOMAINS

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Abstract: We obtain some analogs of the Liouville property for the function that is harmonic on the exterior of a Jordan domain $G \subset \mathbb{C}$ and has constant boundary values of the function itself and its normal derivative. We show that these conditions cannot be relaxed in general.

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Introduction

It is known that every bounded analytic function on the complex plane \mathbb{C} is constant. This circumstance is called the Liouville property for analytic functions. In this assertion the analytic functions can be replaced by harmonic; and the complex plane \mathbb{C} , by an arbitrary Euclidean space \mathbb{R}^n . One of the first theorems of this kind asserts (see [1, § 24, Subsection 11]) that if a function f is harmonic on \mathbb{R}^n and bounded above (or below) then f is constant. Another theorem is a consequence of a more general result from [2, Part 3, Chapter 3, § 3.2] and concerns the harmonic functions on the exterior of a compact set in \mathbb{R}^n with connected complement. The theorem states that if such a function decreases as $|x| \rightarrow \infty$ faster than every negative degree of $|x|$ (here $|x|$ is the Euclidean norm) then it is zero, and this decay rate cannot be relaxed.

In some assertions of the type, the condition of harmonicity can be relaxed. In particular, behavior at infinity was studied for the functions satisfying the mean value equations over spheres (or balls) with constraints on the sets of centers and radii [2–5]. For example, in [2, Part 5, Chapter 5; 4] the sharp Liouville-type theorems were obtained for the functions on \mathbb{R}^n satisfying the Mean Value Theorem for all centers and the same radius. It is shown (see [2, Part 5, Chapter 5]) that this function class is much wider than the class of functions harmonic on \mathbb{R}^n . Note also an interesting result by Hansen [5, 6]: If f is a continuous bounded function on \mathbb{R}^2 such that

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r(x)e^{it}) dt,$$

where $r(x)$ is a positive function on \mathbb{R}^2 satisfying the condition

$$\overline{\lim}_{x \rightarrow \infty} (r(x) - |x|) < +\infty,$$

then f is a constant function.

Many authors studied the Liouville property for solutions to some differential equations and inequalities on various spaces (see [7–11] and the bibliography therein).

Another important trend in the topic under consideration is the study of a possible asymptotic behavior at infinity of mean-periodic functions. The impossibility of vanishing at infinity is an obvious property of nonzero periodic functions on the real axis. In the multidimensional case when different generalizations of the notion of periodicity are possible, the situation becomes much more complicated.

Developing the theory of mean-periodic functions, John obtained in [12, Chapter 6] the following result: If a continuous function f on \mathbb{R}^3 has zero integrals over all spheres of unit radius and

$$f(x) = o\left(\frac{1}{|x|}\right) \quad \text{as } x \rightarrow \infty$$

then $f \equiv 0$ (see also [13], where the n -dimensional case is considered). Examples show (see [12, 13]) that the condition of decay of f is sharp. John's Theorem was further developed and refined in various directions (see [14–23]).

Firstly, its generalizations were studied for the functions f satisfying the convolution equation $f * T = 0$, where T is a given nonzero distribution in \mathbb{R}^n with compact support. It turned out in particular that there are no nonzero solutions f belonging to $L^p(\mathbb{R}^n)$ for some $p \in [1, 2n/(n-1)]$, $n \geq 2$. Moreover, the exponent p cannot be increased (see [14–17]).

Secondly, the so-called “spectral” analogs of John's Theorem were proved. Like in the classical Liouville Theorem for entire functions the relationship is established in them between the behavior at infinity of solutions to the convolution equations and the set of nonzero coefficients in their Fourier expansions in the spherical harmonics [2, Part 3, Chapter 3; 18, Part 3, Chapter 14].

Thirdly, the problem of admissible growth (decay) was investigated for functions with zero spherical means on unbounded domains. The new interesting effects appeared here which are connected with the dependence on the form of the domain [2, Part 3, Chapter 3, § 3.3; 18, Part 3, Chapter 14, § 14.6; 19, Part 2, Chapter 1, § 1.6]. The various theorems of the Phragmén–Lindelöf type and theorems on the boundary behavior of analytic functions belong to the circle of ideas under consideration (see [20, 21]).

Fourthly, the analogs of John's Theorem on symmetric spaces were obtained [18, Part 3, Chapter 15; 22]. The methods for this development turned out useful in the study of similar problems for a distorted convolution equation on a complex Euclidean space and the Heisenberg group (see [23]).

In the present article, we obtain some analogs of the Liouville property for the harmonic functions on the interior of a Jordan domain $G \subset \mathbb{C}$ which have constant boundary values of the function itself and its normal derivative (see Theorems 1 and 2 below). Moreover, we prove that the hypotheses of Theorem 1 cannot be relaxed in general (see Theorem 3).

§ 1. Statements of the Main Results

Let Γ be a closed smooth Jordan curve in the complex plane \mathbb{C} , let G be a bounded domain in \mathbb{C} with boundary Γ , and $\overline{G} = G \cup \Gamma$. The curve Γ is endowed in a standard manner with some measure μ which is equal every open subset of Γ to the sum of the arcs that compose the subset Γ (see [24, Chapter 10, § 1]).

As usual, the symbol $\frac{\partial}{\partial n}$ stands for the derivative in the direction of the outward normal to Γ (with respect to G).

Given a continuous function f on the circle $\gamma_r = \{z \in \mathbb{C} : |z| = r\}$, we put

$$M_r(f) = \int_{\gamma_r} |f(z)| |dz|. \quad (1)$$

Let us formulate the main results of the article:

Theorem 1. Suppose that $f \in C(\mathbb{C} \setminus G)$ is harmonic on $\mathbb{C} \setminus \overline{G}$ and the following are fulfilled for some $c_1, c_2 \in \mathbb{C}$:

- (1) $f = c_1$ on Γ ;
- (2) $\frac{\partial f}{\partial n} = c_2$ μ -almost everywhere on Γ ;
- (3) $\lim_{r \rightarrow \infty} \frac{1}{r^3} M_r(f) = 0$.

If G is not a disk then $f = c_1$ in $\mathbb{C} \setminus G$. If G is a disk then

$$f(z) = c_1 + c_2 R \log \frac{|z - z_0|}{R}, \quad z \in \mathbb{C} \setminus G, \quad (2)$$

where z_0 and R are the center and radius of G .

The proof of Theorem 1 is based on applying the conformal mapping of the unit disk onto the domain $\mathbb{C} \setminus \overline{G}$. This mapping enables us to reduce the initial problem to the overdetermined boundary value problem for the Laplace equation on the exterior of the disk in which the main difficulty is the inhomogeneity of the boundary condition for the normal derivative. For studying this condition, some subtle results on the boundary values of the function performing the above conformal mapping are needed as well as some properties of the Hardy classes H_p in the unit disk (see [24, Chapters 9 and 10]). The corresponding auxiliary constructions and assertions are collected in § 2. Note that the absence of a relevant machinery in the multidimensional case leaves open the question of the existence of an analog of Theorem 1 in \mathbb{R}^n for $n > 2$.

Theorem 1 immediately implies

Corollary 1. *Suppose that $g \in C(\mathbb{C} \setminus G)$ is holomorphic on $\mathbb{C} \setminus \overline{G}$ and $f = \operatorname{Re} g$ satisfies (1)–(3) of Theorem 1. Then g is a constant function.*

Furthermore, if we put $c_2 = 0$ in Theorem 2 then f must be identically constant independently of the form of G . As is shown by the following theorem, this assertion remains valid under weaker assumptions.

Theorem 2. *Suppose that $f \in C(\mathbb{C} \setminus G)$ is harmonic on $\mathbb{C} \setminus \overline{G}$ and the following are fulfilled for some $c \in \mathbb{C}$:*

- (1) $f = c$ on Γ ;
- (2) $\frac{\partial f}{\partial n} = 0$ on some set $E \subset \Gamma$ of positive measure μ ;
- (3) $\lim_{r \rightarrow \infty} \frac{1}{r^\alpha} M_r(f) = 0$ for some $\alpha > 0$.

Then $f = c$ in $\mathbb{C} \setminus G$.

The following result shows the necessity of conditions (1) and (2) as well as the sharpness of (3) in Theorem 1.

Theorem 3. *There exist a bounded central-symmetric domain $G \subset \mathbb{C}$ other than a disk with smooth Jordan boundary Γ and functions f_1 , f_2 , and f_3 continuous on $\mathbb{C} \setminus G$ and harmonic in $\mathbb{C} \setminus \overline{G}$ such that*

- (1) f_1 satisfies (1) and (3) of Theorem 1 and is not identically constant;
- (2) f_2 satisfies (3) of Theorem 1 and

$$\frac{\partial f_2}{\partial n} = 1 \quad \text{everywhere on } \Gamma;$$

- (3) $f_3 = 0$ and $\frac{\partial f_3}{\partial n} = 1$ everywhere on Γ ; moreover,

$$f_3(z) = O(|z|^2) \quad \text{as } z \rightarrow \infty.$$

In particular, $M_r(f_3) = O(r^3)$ as $r \rightarrow \infty$.

The proof of Theorems 1–3 will appear in § 3.

§ 2. Auxiliary Constructions and Assertions

Suppose that $A = \{z \in \mathbb{C} : |z| > 1\}$ and a function u is continuous on $\overline{A} = \{z \in \mathbb{C} : |z| \geq 1\}$. For every fixed $\rho \geq 1$, to the function $u(\rho e^{i\varphi})$ there corresponds the Fourier series

$$u(\rho e^{i\varphi}) = \sum_{n=-\infty}^{\infty} u_n(\rho) e^{in\varphi}, \quad \varphi \in [0, 2\pi], \quad (3)$$

where

$$u_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\varphi}) e^{-in\varphi} d\varphi. \quad (4)$$

Let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplace operator. If $u \in C^2(A)$ then easy calculations with the use of (4) show that

$$\Delta(u_n(\rho)e^{in\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} (\Delta u)(\rho e^{it}) e^{in(\varphi-t)} dt \quad (5)$$

for all $\rho > 1$, $\varphi \in [0, 2\pi]$, $n \in \mathbb{Z}$.

Moreover,

$$\Delta(u_n(\rho)e^{in\varphi}) = \left(u_n''(\rho) + \frac{u_n'(\rho)}{\rho} - \frac{n^2}{\rho^2} u_n(\rho) \right) e^{in\varphi}. \quad (6)$$

By the Riemann Mapping Theorem, there exists a unique function $w = \psi(z)$ holomorphic on A that sends A conformally onto the domain $\mathbb{C} \setminus G$ under the conditions

$$\psi(\infty) = \infty, \quad \psi'(\infty) = a > 0. \quad (7)$$

The first condition in (7) shows that the function $w = \psi(z)$ takes the point $z = \infty$ to the point $w = \infty$, and the second condition means that

$$\lim_{z \rightarrow \infty} \frac{\psi(z)}{z} = a > 0. \quad (8)$$

Conditions (7) and (8) imply that the function ψ , holomorphic on A , has a simple pole at $z = \infty$; therefore, the Laurent expansion of ψ in A looks as

$$\psi(z) = az + \sum_{n=0}^{\infty} \psi_n z^{-n}, \quad a > 0. \quad (9)$$

By the boundary correspondence principle for conformal mappings, ψ can be extended by continuity to \bar{A} . Preserve the notation $w = \psi(z)$ for this extension.

Put $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We can show (see [24, Chapter 10, § 1, Theorem 2]) that to every set of positive Lebesgue measure on \mathbb{T} under ψ there corresponds a μ -measurable set of positive measure in Γ and vice versa. Similarly, to every set of measure zero on \mathbb{T} there corresponds a set of measure zero μ in Γ and vice versa.

As usual, denote by $H_p(\mathbb{D})$, $p > 0$, the class of functions f holomorphic on \mathbb{D} and such that, for each of them, the integral

$$\int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi \quad (10)$$

is bounded for $0 \leq r < 1$. Given $z \in \mathbb{D}$, put

$$h(z) = a - \sum_{n=0}^{\infty} n\psi_n z^{-n-1} = \psi' \left(\frac{1}{z} \right). \quad (11)$$

Note that, by the univalence of ψ , the function h has no zeros in \mathbb{D} . Below we will need some auxiliary assertions about the properties of h and ψ .

Lemma 1. *h and $1/h$ belong to $H_p(\mathbb{D})$ for all $p > 0$.*

PROOF. Let $b \in G$. Denote by $d(b)$ the distance from a point b to Γ . Then, for every $z \in \mathbb{D}$, we have

$$\left| \psi \left(\frac{1}{z} \right) - b \right| \geq d(b) > 0.$$

This inequality and (9) imply that there exists a constant $c_1 > 0$ such that

$$\left| z \left(\psi \left(\frac{1}{z} \right) - b \right) \right| > c_1 \quad (12)$$

for all $z \in \mathbb{D}$. Moreover, from (9) and the definition of ψ we obtain that

$$\left| z \left(\psi \left(\frac{1}{z} \right) - b \right) \right| < c_2, \quad z \in \mathbb{D}, \quad (13)$$

where $c_2 > 0$ does not depend on z . Furthermore, for $z \in \mathbb{D}$, from (11) we find

$$h(z) = -z^2 \left(\psi \left(\frac{1}{z} \right) - b \right)^2 \lambda'(z), \quad (14)$$

where

$$\lambda(z) = \left(\psi \left(\frac{1}{z} \right) - b \right)^{-1}.$$

Reckoning with (12) and (13), for all $p > 0$ and $r \in [0, 1)$, from (14) we get

$$\int_0^{2\pi} (|h(re^{i\varphi})|^p + |h(re^{i\varphi})|^{-p}) d\varphi < c_2^{2p} \int_0^{2\pi} |\lambda'(re^{i\varphi})|^p d\varphi + c_1^{-2p} \int_0^{2\pi} |\lambda'(re^{i\varphi})|^{-p} d\varphi. \quad (15)$$

The function λ sends \mathbb{D} conformally and univalently onto some bounded domain with smooth Jordan boundary. By the Lindelöf Theorem (see [24, Chapter 10, § 1, Theorem 4]), for an appropriate choice of the argument, the functions $\arg \lambda'(z)$ and $\arg 1/\lambda'(z)$ can be extended to continuous functions on $\overline{\mathbb{D}}$. This implies (see [24, Chapter 10, § 1, Theorem 5]) that the functions λ and $1/\lambda$ belong to $H_p(\mathbb{D})$ for every $p > 0$. Thus, the integrals on the right-hand side of (15) are bounded in r . This and (15) give the claim of Lemma 1. \square

Corollary 2. *The function h has finite limit values along nontangent paths almost everywhere on \mathbb{T} that form a boundary function $h(e^{i\varphi})$, $\varphi \in (0, 2\pi)$. Moreover, $h(e^{i\varphi}) \in L^p(0, 2\pi)$ for any $p > 0$ and*

$$h(e^{i\varphi}) \neq 0 \quad \text{for almost all } \varphi \in (0, 2\pi). \quad (16)$$

PROOF. The assertion about the existence of the limit function $h(e^{i\varphi})$ and the membership of the latter in $L^p(0, 2\pi)$ follows from Lemma 1 and the well-known analogous property for every function in $H_p(\mathbb{D})$ (see, for example, [24, Chapter 9, § 4]). Applying this property to the function $1/h$ and using Lemma 1, we obtain (16). \square

Corollary 3. *Suppose that $\alpha, \beta \in \mathbb{C}$ and $|\alpha| \geq |\beta| > 0$. Then the function*

$$h_1(z) = \frac{h(z)}{(\alpha + \beta z)^2}$$

belongs to $H_p(\mathbb{D})$ for all $0 < p < \frac{1}{4}$.

PROOF. Suppose first that $|\beta| = |\alpha|$. In this event, $\alpha + \beta z = \beta(z - e^{i\gamma})$ for some $\gamma \in [0, 2\pi]$. Let $0 < p < \frac{1}{4}$. Choose $q > 1$ so that $qp < \frac{1}{4}$. Using Hölder's inequality and Lemma 1, for any $r \in [0, 1)$, we have

$$\begin{aligned} \int_0^{2\pi} |h_1(re^{i\varphi})|^p d\varphi &= \int_0^{2\pi} |h(re^{i\varphi})|^p |\beta(re^{i\varphi} - e^{i\gamma})|^{-2p} d\varphi \\ &\leq \left(\int_0^{2\pi} |h(re^{i\varphi})|^{\frac{pq}{q-1}} d\varphi \right)^{\frac{q-1}{q}} \left(\int_0^{2\pi} |\beta(re^{i\varphi} - e^{i\gamma})|^{-2pq} d\varphi \right)^{\frac{1}{q}} \leq c \left(\int_0^{2\pi} |1 - re^{i(\varphi-\gamma)}|^{-2pq} d\varphi \right)^{\frac{1}{q}}, \end{aligned} \quad (17)$$

where the constant $c > 0$ does not depend on r . Since $|1 - re^{i(\varphi-\gamma)}| \geq 1 - |\cos(\varphi - \gamma)|$ and $pq < \frac{1}{4}$, from (17) we obtain the desired assertion. For $|\alpha| > |\beta| > 0$, the assertion is obvious by Lemma 1 and the inequality $|\alpha + \beta z| \geq |\alpha| - |\beta|$, $z \in \mathbb{D}$. \square

Lemma 2. For almost all $\varphi \in [0, 2\pi]$ (with respect to the Lebesgue measure), we have

$$\lim_{\varepsilon \rightarrow +0} \psi'((1+\varepsilon)e^{i\varphi}) = \lim_{\varepsilon \rightarrow +0} \frac{\psi((1+\varepsilon)e^{i\varphi}) - \psi(e^{i\varphi})}{\varepsilon e^{i\varphi}}, \quad (18)$$

where both limits exist and are finite.

PROOF. From (11) we have $\psi'((1+\varepsilon)e^{i\varphi}) = h((1+\varepsilon)^{-1}e^{-i\varphi})$ for all $\varepsilon > 0$ and $\varphi \in [0, 2\pi]$. This and Corollary 2 imply the existence of a finite limit on the left-hand side of (18) for almost all $\varphi \in [0, 2\pi]$. By the Mean Value Theorem, these values of φ satisfy the equality

$$\int_1^{1+\varepsilon} \psi'(\rho e^{i\varphi}) d\rho = \varepsilon \psi'(\xi e^{i\varphi})$$

for every $\varepsilon > 0$ and some $\xi \in (1, 1+\varepsilon)$ depending on ε . Therefore, we have the finite limit

$$\lim_{\varepsilon \rightarrow +0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \psi'(\rho e^{i\varphi}) d\rho$$

which is equal to the limit on the left-hand side of (18). Since

$$\int_1^{1+\varepsilon} \psi'(\rho e^{i\varphi}) d\rho = \frac{\psi((1+\varepsilon)e^{i\varphi}) - \psi(e^{i\varphi})}{e^{i\varphi}},$$

this gives Lemma 2. \square

Lemma 3. For almost all $\varphi \in (0, 2\pi)$ there exists $\delta = \delta(\varphi) > 0$ such that, for all $\varepsilon \in (0, \delta(\varphi))$, the disk

$$K_{\varepsilon, \varphi} = \left\{ z \in \mathbb{C} : |z - \psi(e^{i\varphi}) - \varepsilon e^{i\varphi} \psi'(e^{i\varphi})| \leq \frac{\varepsilon}{2} |\psi'(e^{i\varphi})| \right\} \quad (19)$$

is disjoint from Γ .

PROOF. Suppose that for some $\varphi \in (0, 2\pi)$ there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of positive reals such that $\overline{\lim}_{j \rightarrow \infty} \varepsilon_j = 0$ and the disk $K_{\varepsilon_j, \varphi}$ intersects Γ for each j . Denote by φ_j one of the points of the half-interval $[0, 2\pi)$ for which $\psi(e^{i\varphi_j}) \in K_{\varepsilon_j, \varphi} \cap \Gamma$. Then

$$|\psi(e^{i\varphi_j}) - \psi(e^{i\varphi}) - \varepsilon_j e^{i\varphi} \psi'(e^{i\varphi})| \leq \frac{\varepsilon_j}{2} |\psi'(e^{i\varphi})|. \quad (20)$$

From this inequality and the univalence of ψ we conclude that $\varphi_j \rightarrow \varphi$ as $j \rightarrow \infty$.

Furthermore, $\psi(e^{it})$ is absolutely continuous on $[0, 2\pi]$ and

$$\frac{d}{dt} \psi(e^{it}) = i e^{it} \psi'(e^{it}) \quad (21)$$

for almost all $t \in [0, 2\pi]$ (see [24, Chapter 10, § 1, Theorem 1]).

Suppose now that $\psi'(e^{i\varphi}) \neq 0$ and

$$\psi(e^{i\varphi}) - \psi(e^{i\varphi_j}) = i \psi'(e^{i\varphi}) e^{i\varphi} (\varphi - \varphi_j) + o(\varphi - \varphi_j) \quad \text{as } j \rightarrow \infty. \quad (22)$$

Corollary 2 and (21) imply that these requirements are fulfilled for almost all $\varphi \in (0, 2\pi)$. Comparing (22) and (20) and taking into account that $\psi'(e^{i\varphi}) \neq 0$, we arrive at the inequality

$$|\varepsilon_j + i(\varphi - \varphi_j) + o(\varphi - \varphi_j)| \leq \frac{\varepsilon_j}{2} \quad \text{as } j \rightarrow \infty.$$

For j sufficiently large, the last inequality is contradictory, which implies Lemma 3. \square

Lemma 4. Suppose that f is harmonic on the disk $K = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < r\}$ and

$$M = \sup_{\zeta \in K} |f(\zeta)| < +\infty. \quad (23)$$

Then, for any $\zeta \in K$, we have the estimate

$$|f(\zeta) - f(\zeta_0)| \leq \frac{2M|\zeta - \zeta_0|}{r - |\zeta - \zeta_0|}. \quad (24)$$

PROOF. We obtain from the hypothesis that $w(z) = f(rz + \zeta_0)$ is harmonic on \mathbb{D} and $|w(z)| \leq M$ for every $z \in \mathbb{D}$. It follows (see, for example, [24, Chapter 9, § 2, Corollary 2]) that w has nontangent finite limit values almost everywhere on \mathbb{T} . As usual, we preserve the notation $w(e^{it})$ for the corresponding limit function defined almost everywhere on \mathbb{T} . Then (23) yields $|w(e^{it})| \leq M$. Moreover, for every $z = \rho e^{i\varphi}$, $0 \leq \rho < 1$, the Poisson formula

$$w(z) - w(0) = \frac{1}{\pi} \int_0^{2\pi} w(e^{it}) \sum_{n=1}^{\infty} \rho^n \cos(n(\varphi - t)) dt \quad (25)$$

holds (see [24, Chapter 9, § 2, Theorem 3]). The integral on the right-hand side does not exceed the expression

$$\frac{1}{\pi} \int_0^{2\pi} |w(e^{it})| \sum_{n=1}^{\infty} \rho^n dt \leq \frac{2M\rho}{1-\rho};$$

therefore, (24) stems from (25). \square

Lemma 5. The following holds for almost all $\varphi \in (0, 2\pi)$: If f is harmonic on $\mathbb{C} \setminus \overline{G}$ and continuous on $\mathbb{C} \setminus G$ and there exists $\frac{\partial f}{\partial n}(\psi(e^{i\varphi}))$ then

$$\lim_{\varepsilon \rightarrow +0} \frac{f(\psi((1+\varepsilon)e^{i\varphi})) - f(\psi(e^{i\varphi}))}{\varepsilon} = \frac{\partial f}{\partial n}(\psi(e^{i\varphi})) |\psi'(e^{i\varphi})|.$$

PROOF. Corollary 2 and Lemma 2 imply that, for almost all $\varphi \in (0, 2\pi)$, there exists a nonzero limit on the left-hand side of (18) and

$$\psi((1+\varepsilon)e^{i\varphi}) = \psi(e^{i\varphi}) + \varepsilon e^{i\varphi} \psi'(e^{i\varphi}) + o(\varepsilon) \quad \text{as } \varepsilon \rightarrow +0. \quad (26)$$

Hence, for such φ , the point $\psi((1+\varepsilon)e^{i\varphi})$ is contained in $K_{\varepsilon, \varphi}$ (see (19)) for all sufficiently small $\varepsilon > 0$. Apply Lemma 4 on supposing that

$$\zeta_0 = \psi(e^{i\varphi}) + \varepsilon e^{i\varphi} \psi'(e^{i\varphi}), \quad r = \frac{\varepsilon}{2} |\psi'(e^{i\varphi})|, \quad \zeta = \psi((1+\varepsilon)e^{i\varphi}),$$

where $\varepsilon \in (0, \delta(\varphi))$ is sufficiently small. Since $K_{\varepsilon, \varphi}$ lies in the disk $\{z \in \mathbb{C} : |z| \leq |\psi(e^{i\varphi})| + \frac{3}{2}\delta(\varphi) |\psi'(e^{i\varphi})|\}$, there exists a constant M_φ independent of ε such that

$$\sup_{K_{\varepsilon, \varphi}} |f| \leq M_\varphi \quad \text{for all } \varepsilon \in (0, \delta(\varphi)).$$

Then (26) and Lemma 4 yield

$$f(\psi((1+\varepsilon)e^{i\varphi})) - f(\psi(e^{i\varphi}) + \varepsilon e^{i\varphi} \psi'(e^{i\varphi})) = o(\varepsilon) \quad \text{as } \varepsilon \rightarrow +0. \quad (27)$$

The definition of normal derivative gives

$$\frac{\partial f}{\partial n}(\psi(e^{i\varphi})) = \lim_{\varepsilon \rightarrow +0} \frac{f(\psi(e^{i\varphi}) + \varepsilon e^{i\varphi} \psi'(e^{i\varphi})) - f(\psi(e^{i\varphi}))}{\varepsilon |\psi'(e^{i\varphi})|}.$$

This equality and (27) imply Lemma 5. \square

Lemma 6. Suppose that f is harmonic on $\mathbb{C} \setminus G$ and

$$\lim_{r \rightarrow \infty} \frac{M_r(f)}{r^\alpha} = 0 \quad (28)$$

for some $\alpha > 1$. Then

$$f(z) = \sum_{0 < k < \alpha-1} a_k z^k + \sum_{1-\alpha < k < 0} b_k (\bar{z})^{-k} + O(\log |z|) \quad \text{as } z \rightarrow \infty,$$

where $a_k, b_k \in \mathbb{C}$ and the sums with the empty index set are assumed equal to zero.

PROOF. Let B be an open disk in \mathbb{C} centered at zero and including \overline{G} . For $z = \rho e^{i\varphi} \in \mathbb{C} \setminus \overline{B}$, we have the equality

$$f(z) = \sum_{n=-\infty}^{\infty} f_n(\rho) e^{in\varphi}, \quad (29)$$

where

$$f_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\varphi}) e^{-in\varphi} d\varphi. \quad (30)$$

Relation (30) and the harmonicity of f imply that series (29) converges locally uniformly in $\mathbb{C} \setminus \overline{B}$ and the functions $f_n(\rho) e^{in\varphi}$ are harmonic on $\mathbb{C} \setminus \overline{B}$ (see (3)–(5)). Using (6), we obtain

$$f_0(\rho) = a_0 + b_0 \log \rho, \quad f_n(\rho) = a_n \rho^n + b_n \rho^{-n} \quad \text{for } n \neq 0, \quad (31)$$

where $a_n, b_n \in \mathbb{C}$. Equalities (30), (31), and (1) show that

$$|a_n r^n + b_n r^{-n}| \leq \frac{M_r(f)}{2\pi r} \quad \text{for } r > 0, \quad n \neq 0.$$

Therefore, (28) implies that $a_n = 0$ for $n \geq \alpha - 1$ and $b_n = 0$ for $n < 1 - \alpha$. This together with (31) and (29) gives the desired assertion. \square

REMARK 1. The far-going generalizations of Lemma 6 connected with the behavior at infinity of solutions to the convolution equation on symmetric spaces were obtained in [18, Chapter 15] (for the case of \mathbb{R}^n , see also [2, Part 3, Chapter 3]).

§ 3. Proofs of the Main Results

PROOF OF THEOREM 1. Suppose that a function f satisfies the conditions of Theorem 1. Consider the function $u(z) = f(\psi(z))$, where ψ is the function defined in § 2. The condition of the theorem implies that u is harmonic on A and continuous on \overline{A} ; moreover,

$$u(e^{i\varphi}) = c_1 \quad \text{for every } \varphi \in [0, 2\pi]. \quad (32)$$

Furthermore, condition (3) of Theorem 1, (9), and Lemma 6 imply that

$$u(z) = O(|z|) \quad \text{as } z \rightarrow \infty. \quad (33)$$

For every fixed $\rho \geq 1$, to the function $u(\rho e^{i\varphi})$ there corresponds the Fourier series (3) in which $u_n(\rho)$ are continuous on $[1, +\infty)$ (see (4)). The harmonicity of u and (5) give that $u_n(\rho) e^{in\varphi}$ are harmonic on A for all n . By (6), this means that

$$u_0(\rho) = a_0 + b_0 \log \rho, \quad u_n(\rho) = a_n \rho^n + b_n \rho^{-n} \quad \text{for } n \neq 0,$$

where $\rho \geq 1$; while a_n and b_n are complex constants. Equalities (32) and (4) imply that $u_0(1) = c_1$ and $u_n(1) = 0$ for $n \neq 0$. Therefore,

$$a_0 = 0 \quad \text{and} \quad a_n + b_n = 0 \quad \text{for } n \neq 0. \quad (34)$$

Next, from (4) and (33) we obtain

$$u_n(\rho) = O(\rho) \quad \text{as } \rho \rightarrow +\infty.$$

The last equality means that $a_n = 0$ and $b_{-n} = 0$ for $n \geq 2$. Comparing this with (34), we conclude that, for $|z| \geq 1$,

$$u(z) = b_0 \log |z| + a_1 \left(z - \frac{1}{\bar{z}} \right) + a_{-1} \left(\frac{1}{z} - \bar{z} \right), \quad (35)$$

where the bar stands for complex conjugation. This equality yields

$$\lim_{\varepsilon \rightarrow +0} \frac{u((1+\varepsilon)e^{i\varphi}) - u(e^{i\varphi})}{\varepsilon} = b_0 + 2a_1 e^{i\varphi} - 2a_{-1} e^{-i\varphi}$$

for all $\varphi \in [0, 2\pi]$. From this, using Lemma 5 and (2) of Theorem 1, we conclude that

$$b_0 + 2a_1 e^{-i\varphi} - 2a_{-1} e^{i\varphi} = c_2 |h(e^{i\varphi})| \quad (36)$$

for almost all $\varphi \in [0, 2\pi]$. If $c_2 = 0$ then the last equality implies that $b_0 = a_1 = a_{-1} = 0$. By (35), this means that $f = c_1$.

Suppose that $c_2 \neq 0$. Then (36) yields

$$c_2^{-1}(b_0 + 2a_1 e^{-i\varphi} - 2a_{-1} e^{i\varphi}) \geq 0. \quad (37)$$

By the Fejér–Riesz Theorem (see [25, Appendix 5]), the nonnegative trigonometric polynomial on the left-hand side of (37) is representable as

$$c_2^{-1}(b_0 + 2a_1 e^{-i\varphi} - 2a_{-1} e^{i\varphi}) = |\alpha + \beta e^{i\varphi}|^2, \quad \varphi \in [0, 2\pi], \quad (38)$$

where the complex constants α and β are such that

$$\alpha + \beta z \neq 0 \quad \text{for } z \in \mathbb{D}. \quad (39)$$

Prove that $\beta = 0$. Suppose the contrary; then from (39) we have $|\alpha| \geq |\beta| > 0$. Applying Lemma 1 and Corollary 3, we conclude that the values

$$h_1(z) = \frac{h(z)}{(\alpha + \beta z)^2}, \quad h_2(z) = \frac{1}{h_1(z)}$$

belong to $H_p(\mathbb{D})$ for $p \in (0, 1/4)$. Moreover, (36) and (39) imply that

$$|h_1(e^{i\varphi})| = |h_2(e^{i\varphi})| = 1$$

for almost all $\varphi \in (0, 2\pi)$. By the Smirnov Theorem (see [24, Chapter 9, § 4, Theorem 4]), this means that $|h_1(z)| = 1$ for all $z \in \mathbb{D}$. Consequently, $h(z) = \gamma(\alpha^2 + 2\alpha\beta z + \beta^2 z^2)$, where $\gamma \in \mathbb{C}$, $|\gamma| = 1$. Reckoning with (11), we obtain $\beta = 0$, which contradicts our assumption. This argument and formulas (36) and (38) show that $\beta = a_1 = a_{-1} = 0$ and $|h(e^{i\varphi})| = c_2^{-1}b_0$ for almost all $\varphi \in (0, 2\pi)$. As above, from here and the Smirnov Theorem, we conclude that h is identically constant. Reckoning with (11) and (9), we have that $h(z) = a$ in \mathbb{D} and $\psi(z) = az + \psi_0$ in A . Consequently, G is a circle centered at $z_0 = \psi_0$ and of radius $R = a$. In this case, from (35) we obtain (2), which completes the proof of Theorem 1. \square

PROOF OF THEOREM 2. Suppose that f satisfies the conditions of Theorem 2. Repeating the arguments of the proof of Theorem 1, we see that the function $u(z) = f(\psi(z))$ enjoys (32)–(34). Using Lemma 6 and (3) of Theorem 2 for $\alpha > 1$, we as above arrive at the equality

$$u(z) = c_1 + b_0 \log |z| + \sum_{0 < k < \alpha - 1} (a_k(z^k - (\bar{z})^{-k}) + a_{-k}(z^{-k} - \bar{z}^k)). \quad (40)$$

Equality (40), Lemma 5, and condition (2) imply that

$$b_0 + 2 \sum_{0 < k < \alpha - 1} (a_k e^{ik\varphi} - a_{-k} e^{-ik\varphi}) = 0$$

for some set of points $\varphi \in [0, 2\pi]$ of positive Lebesgue measure. This means that $b_0 = 0$ and $a_k = 0$ for $0 < |k| < \alpha - 1$. From this, (40), and the definition of u we infer that $f = c_1$. \square

PROOF OF THEOREM 3. Obviously, for all sufficiently small $\varepsilon \in (0, 1)$, we have the inequality

$$\frac{2}{3} + \frac{1}{9(1-\varepsilon)^2} < (1-\varepsilon)^2. \quad (41)$$

For these ε , put $A_\varepsilon = \{z \in \mathbb{C} : |z| > 1 - \varepsilon\}$. Consider the function

$$\Phi(z) = z - \frac{2}{3z} - \frac{1}{27z^3}, \quad z \in A_\varepsilon. \quad (42)$$

For all $z_1, z_2 \in A_\varepsilon$, we have the estimates

$$|z_1 z_2| > (1 - \varepsilon)^2, \quad |z_1^{-2} + (z_1 z_2)^{-1} + z_2^{-2}| < \frac{3}{(1 - \varepsilon)^2}. \quad (43)$$

Moreover, from (42) we find

$$\Phi(z_1) - \Phi(z_2) = (z_1 - z_2) \left(1 + \frac{1}{z_1 z_2} \left(\frac{2}{3} + \frac{1}{27} (z_1^{-2} + (z_1 z_2)^{-1} + z_2^{-2}) \right) \right).$$

Reckoning with (41) and (43), from the last relation we conclude that $\Phi(z_1) \neq \Phi(z_2)$ for $z_1 \neq z_2$. Thus, Φ is univalent in A_ε . Put

$$\Phi(A_\varepsilon) = \{z \in \mathbb{C} : z = \Phi(\zeta), \zeta \in A_\varepsilon\}$$

and denote by g the inverse function of Φ that acts from $\Phi(A_\varepsilon)$ onto A_ε . It follows from (42) that, for all $z \in A_\varepsilon$, we have the inequalities

$$|z| - \frac{2}{3(1-\varepsilon)} - \frac{1}{27(1-\varepsilon)^3} < |\Phi(z)| < |z| + \frac{2}{3(1-\varepsilon)} + \frac{1}{27(1-\varepsilon)^3}.$$

These and the definition of g yield

$$|z| - \frac{2}{3(1-\varepsilon)} - \frac{1}{27(1-\varepsilon)^3} < |g(z)| < |z| + \frac{2}{3(1-\varepsilon)} + \frac{1}{27(1-\varepsilon)^3} \quad (44)$$

for all $z \in \Phi(A_\varepsilon)$.

Put $G = \mathbb{C} \setminus \overline{\Phi(A)}$, where $\Phi(A) = \{z \in \mathbb{C} : z = \Phi(\zeta), \zeta \in A\}$. Owing to the oddity and univalence of Φ , the set G is a central-symmetric bounded domain with the smooth Jordan boundary $\Gamma = \{z \in \mathbb{C} : z = \Phi(\zeta), \zeta \in \mathbb{T}\}$. Moreover, since

$$\Phi(1) = -\Phi(-1) = \frac{8}{27} \quad \text{and} \quad \Phi(i) = -\Phi(-i) = \frac{44}{27}i;$$

therefore, G is not a disk.

Consider the functions

$$f_1 = \log |g|, \quad f_2 = \frac{10}{9} \log |g| - \frac{1}{3} \operatorname{Re} \left(\frac{1}{g^2} \right), \quad f_3 = \frac{10}{9} \log |g| + \frac{1}{6} \operatorname{Re} \left(g^2 - \frac{1}{g^2} \right).$$

The definition of g implies that g is holomorphic on $\Phi(A_\varepsilon)$ and does not vanish. Hence, the functions f_1 , f_2 , and f_3 are harmonic on $\mathbb{C} \setminus \overline{G}$ and continuous on $\mathbb{C} \setminus G$. Using also (44), we conclude that f_1 , f_2 , and f_3 satisfy all hypotheses of Theorem 3. \square

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