ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 3, pp. 504–515. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Sibirskii Matematicheskii Zhurnal, 2020, Vol. 61, No. 3, pp. 641–653.

COHOMOLOGICALLY RIGID SOLVABLE LEIBNIZ ALGEBRAS WITH NILRADICAL OF ARBITRARY CHARACTERISTIC SEQUENCE

U. Kh. Mamadaliev and B. A. Omirov UDC 512.554

Abstract: We describe the $(n + s)$-dimensional solvable Leibniz algebras whose nilradical has characteristic sequence $(m_1, \dots, m_s)$, where $m_1 + \dots + m_s = n$. The completeness and cohomological rigidity of this algebra are proved.

DOI: 10.1134/S003744662003012X

Keywords: Leibniz algebra, solvable algebra, nilradical, rigid algebra, second cohomology group

§ 1. Introduction

Leibniz algebras were first introduced by Blokh (see [1]) under the title "D-algebras" in 1965. These algebras reappeared in 1993 in Loday's article [2], where the present-day term "Leibniz algebras" was established. Leibniz algebras are so natural generalization of Lie algebras that many properties of Lie algebras extend to Leibniz algebras.

In much the same way as other finite-dimensional algebras defined by identities, the Leibniz algebras of a given dimension define an algebraic variety. Since each algebraic variety is representable as the union of finitely many irreducible components, which are in turn described by open subsets, of importance in describing the varieties of finite-dimensional algebras is the description of the algebras whose orbits under the action of the linear group are open sets. Algebras with such orbits are called *rigid algebras*. Thus, the closures of the orbits of rigid algebras give the irreducible components of the variety. A sufficient condition for the rigidity of algebras is the triviality of the second cohomology group (see [3]). Note that the calculation of the second cohomology groups is rather difficult. Hochschild and Serre proved a theorem that significantly simplifies the calculation of cohomology groups for Lie algebras (see [4]). Unfortunately there is still no analog of this theorem by now for non-Lie Leibniz algebras. Therefore, we have to use the structure of the algebra in each particular case.

The present article provides some classification of solvable Leibniz algebras whose nilradical has a characteristic sequence equal to $(m_1, \dots, m_s)$ provided that the dimension of the complementary space of the nilradical has the maximal value. Moreover, we prove the completeness and cohomological rigidity of such Leibniz algebras. Note that these algebras are a Leibniz analog of the Lie algebras considered in [5].

§ 2. Preliminaries

Let us start with some notions and preliminary results.

Definition 2.1. An algebra L over a field F is called a *Leibniz algebra* if all $x, y, z \in L$ satisfy the identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

where $[\cdot, \cdot]$ is the multiplication in L.

Given an arbitrary Leibniz algebra L, define the lower central and derived series as

$$L^{[1]} = L, \quad L^{[k+1]} = [L^{[k]}, L^{[k]}]; \quad L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1.$$

[†)] To Academician Yuri Leonidovich Ershov on his 80th birthday.

Original article submitted March 13, 2019; revised January 13, 2020; accepted February, 19, 2020.

DEFINITION 2.2. A Leibniz algebra L is called *solvable* (respectively, *nilpotent*) if there exists $s \in \mathbb{N}$ such that $L^{[s]} = 0$ (respectively, $L^s = 0$).

The maximal nilpotent ideal of a Leibniz algebra L is called the *nilradical* of L.

Note that the operators of right multiplication by an element of an algebra (i.e., $\mathscr{R}_x(y) = [x, y]$) are derivations, which we will call *inner*.

DEFINITION 2.3. A Leibniz algebra L is called *complete* if the center of L is trivial and every derivation of L is inner.

The reader is referred to [2] for more details with the cohomology group of Leibniz algebras. Recall only that a 2-cocycle $\varphi \in ZL^2(L, M)$ and a 2-coboundary $f \in BL^2(L, M)$ of a Leibniz algebra L with coefficients in a module M are defined as follows:

$$(d^2\varphi)(a, b, c) = [a, \varphi(b, c)] - [\varphi(a, b), c] + [\varphi(a, c), b] + \varphi(a, [b, c]) - \varphi([a, b], c) + \varphi([a, c], b) = 0,$$
$$f(a, b) = [g(a), b] + [a, g(b)] - g([a, b])$$

for some linear mapping g. Moreover,

$$HL^2(L, M) = ZL^2(L, M)/BL^2(L, M).$$

Define the action of the group $GL_n(F)$ on the set of all n-dimensional Leibniz algebras $\mathrm{Leib}_n(F)$ as follows:

$$(g * \lambda)(x, y) := g(\lambda(g^{-1}(x), g^{-1}(y))), \quad g \in GL_n(F),\ \lambda \in \mathrm{Leib}_n(F).$$

Denote the orbit of an algebra L under this action by $\mathrm{Orb}(L)$. Recall that L is called *rigid* if $\mathrm{Orb}(L)$ is open in the Zariski topology.

The various rigidity criteria are available that express this initially topological notion in other terms. Most important in this connection is the sufficient rigidity criterion proved by Balavoine in [3]. Namely, the rigidity of the algebra follows from the triviality of the second cohomology group with coefficients in a module defined by the Leibniz regular representation.

DEFINITION 2.4. A Leibniz algebra L is called *cohomologically rigid* if $HL^2(L, L) = 0$.

Let L be a nilpotent Leibniz algebra and let $x \in L \backslash L^2$. Given the nilpotent operator of right multiplication $\mathscr{R}_x$, consider the decreasing sequence $C(x) = (m_1, \dots, m_s)$ consisting of the sizes of the Jordan cells of $\mathscr{R}_x$. Endow the set of such sequences with the lexicographic order.

DEFINITION 2.5. The sequence

$$C(L) = \max_{x \in L \backslash [L,L]} C(x)$$

is called the *characteristic sequence* of an algebra L.

§ 3. The Main Results

Consider a nilpotent Leibniz algebra with characteristic sequence $(m_1, \dots, m_s)$, where $m_1 \geq \dots \geq m_s \geq 1$, and multiplication table

$$\left[e_i^t, e_1^1\right] = e_{i+1}^t, \quad 1 \leq t \leq s,\ 1 \leq i \leq m_t - 1.$$

Denote this algebra by $N_{m_1,\dots,m_s}$. We will use the notation $R(N_{m_1,\dots,m_s}, s)$ for solvable Leibniz algebras with nilradical $N_{m_1,\dots,m_s}$ and complementary space of dimension s.

Theorem 3.1. *An arbitrary solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$ is isomorphic to the algebra*

$$\begin{cases} \left[e_i^t, e_1^1\right] = e_{i+1}^t, & 1 \leq t \leq s, 1 \leq i \leq m_t - 1, \\ \left[e_i^1, x_1\right] = ie_i^1, & 1 \leq i \leq m_1, \\ \left[e_i^t, x_1\right] = (i-1)e_i^t, & 2 \leq t \leq s, 2 \leq i \leq m_t, \\ \left[e_i^t, x_t\right] = e_i^t, & 2 \leq t \leq s, 1 \leq i \leq m_t, \\ \left[x_1, e_1^1\right] = -e_1^1, \end{cases}$$

where $\{x_1, \dots x_s\}$ is a basis for the complementary vector space.

The proof is carried out by the methods analogous to the demonstration of Theorem 3.2 in [6].

Describe the space of derivations of $R(N_{m_1,\dots,m_s}, s)$.

Proposition 3.2. *The following linear transformations form a basis for* $\operatorname{Der}(R(N_{m_1,\dots,m_s},s))$*:*

$$\begin{cases} d_0(x_1)=-e_1^1,\ d_0\big(e_i^p\big)=e_{i+1}^p, & 1\le p\le s,\ 1\le i\le m_p-1,\\ d_1\big(e_i^1\big)=ie_i^1, & 1\le i\le m_1,\\ d_1\big(e_i^p\big)=(i-1)e_i^p, & 2\le p\le s,\ 2\le i\le m_p,\\ d_p\big(e_i^p\big)=e_i^p, & 2\le p\le s,\ 1\le i\le m_p. \end{cases}$$

Proof. Let d be a derivation of $R(N_{m_1,\dots,m_s},s)$. Put

$$d\big(e_1^p\big)=\sum_{t=1}^{s}\sum_{i=1}^{m_t}\alpha_{t,i}^pe_i^t+\sum_{i=1}^{s}\beta_{1,i}^px_i,\quad d(x_p)=\sum_{t=1}^{s}\sum_{i=1}^{m_t}\gamma_{t,i}^pe_i^t+\sum_{i=1}^{s}\beta_{2,i}^px_i,\quad 1\le p\le s.$$

The equality $d\big([e_1^1,x_1]\big)=\big[d\big(e_1^1\big),x_1\big]+\big[e_1^1,d(x_1)\big]$ gives the constraints

$$\begin{cases} \gamma_{1,1}^1=-\alpha_{1,2}^1,\ \beta_{2,1}^1=0,\\ \alpha_{1,i}^1=0, & 3\le i\le m_1,\\ \alpha_{t,i}^1=0, & 2\le t\le s,\ i=1,\ 3\le i\le m_t,\\ \beta_{1,i}^1=0, & 1\le i\le s. \end{cases}$$

Given $2\le p\le s$ and considering the equality $d\big([e_1^p,x_1]\big)=\big[d\big(e_1^p\big),x_1\big]+\big[e_1^p,d(x_1)\big]$, we obtain

$$\begin{cases} \alpha_{p,2}^p=\alpha_{1,2}^1,\\ \alpha_{1,i}^p=0, & 1\le i\le m_1,\\ \alpha_{t,i}^p=0, & 2\le t\le s,\ 2\le i\le m_t,\\ \beta_{2,p}^1=0. \end{cases}$$

If $1\le p\le s$ then the equality $0=d([x_p,x_1])=[d(x_p),x_1]+[x_p,d(x_1)]$ yields

$$\begin{cases} \gamma_{t,i}^p=0, & 1\le t\le s,\ 2\le i\le m_t,\ 1\le p\le s,\\ \gamma_{1,1}^p=0, & 2\le p\le s. \end{cases}$$

The equality $d\big([e_1^1,x_p]\big)=\big[d\big(e_1^1\big),x_p\big]+\big[e_1^1,d(x_p)\big]$ for $2\le p\le s$ implies $\alpha_{p,2}^1=\beta_{2,1}^p=0,\ 2\le p\le s$. Consequently,

$$d(e_1^1)=\alpha_{1,1}^1e_1^1+\alpha_{1,2}^1e_2^1,\quad d(x_p)=\sum_{t=2}^{s}\gamma_{t,1}^pe_1^p+\sum_{i=2}^{s}\beta_{2,i}^px_i,\quad 2\le p\le s.$$

For $1\le p\le s$, the equality $0=d\big([x_p,e_1^1]\big)=\big[d(x_p),e_1^1\big]+\big[x_p,d\big(e_1^1\big)\big]$ gives $\gamma_{t,1}^p=0,\ 2\le t\le s$. From the relation $d\big([e_1^p,x_j]\big)=\big[d\big(e_1^p\big),x_j\big]+\big[e_1^p,d(x_j)\big]$ for $2\le p,j\le s$ we have

$$\begin{cases} \alpha_{j,1}^p=0, & 2\le p,j\le s,\ p\ne j,\\ \beta_{t,j}^p=0, & 2\le p\le s,\ 1\le t\le 2,\ 1\le j\le s. \end{cases}$$

Therefore,

$$\begin{cases} d\big(e_1^p\big)=\alpha_{p,1}^pe_1^p+\alpha_{1,2}^1e_2^p, & 1\le p\le s,\\ d(x_1)=-\alpha_{1,2}^1e_1^1,\ d(x_p)=0, & 2\le p\le s. \end{cases}$$

Using the chain of equalities

$$d\big(e_i^p\big)=d\big(\big[e_{i-1}^p,e_1^1\big]\big)=\big[d\big(e_{i-1}^p\big),e_1^1\big]\big)+\big[e_{i-1}^p,d\big(e_1^1\big)\big],\quad 1\le p\le s,\ 2\le i\le m_p,$$

and the above constraints, it is not hard to see that

$$\begin{aligned} d\big(e_i^1\big)&=i\alpha_{1,1}^1e_i^1+\alpha_{1,2}^1e_{i+1}^1,\quad 1\le i\le m_1-1,\\ d\big(e_{m_1}^1\big)&=m_1\alpha_{1,1}^1e_{m_1}^1,\\ d\big(e_i^p\big)&=\big((i-1)\alpha_{1,1}^1+\alpha_{p,1}^p\big)e_i^p+\alpha_{1,2}^1e_{i+1}^p,\quad 2\le p\le s,1\le i\le m_p-1,\\ d\big(e_i^p\big)&=\big((m_p-1)\alpha_{1,1}^1+\alpha_{p,1}^p\big)e_{m_p}^p,\quad 2\le p\le s.\quad\square \end{aligned}$$

Theorem 3.1 and Proposition 3.2 imply the following result:

Corollary 3.3. *The Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$ is complete.*

Let L_2 be an ideal of a solvable Leibniz algebra R such that $L_1 \simeq R/L_2$ is a subalgebra in R. Denote the complementary spaces of $ZL^2(L_1, L_1)$ and $BL^2(L_1, L_1)$ to $ZL^2(R, R)$ and $BL^2(R, R)$ by $\overline{ZL^2(L_1, L_1)}$ and $\overline{BL^2(L_1, L_1)}$.

Turn to proving the main result of the article: Namely, demonstrate that $HL^2(R(N_{m_1,\dots,m_s}, s), R(N_{m_1,\dots,m_s}, s)) = 0$. Proceed by induction on s.

For $s = 1$, the algebra $R(N_{m_1}, 1)$ is solvable with zero-filiform nilradical. It is known from [7] that $HL^2(R(N_{m_1}, 1), R(N_{m_1}, 1)) = 0$. This gives the induction base.

Suppose that $HL^2(R(N_{m_1,\dots,m_k}, k), R(N_{m_1,\dots,m_k}, k)) = 0$ for all $k \leq s - 1$. Since the proof for an arbitrary s is very cumbersome, we will provide the proof for $s = 2$, from which it will be clear how the induction step for s is carried out.

For $s = 2$, the algebra $R(N_{m_1,m_2}, 2)$ has the multiplication table

$$R(N_{m_1,m_2}, 2) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq m_1 - 1, \\ [f_i, e_1] = f_{i+1}, & 1 \leq i \leq m_2 - 1, \\ [e_i, x_1] = ie_i, & 1 \leq i \leq m_1, \\ [f_i, x_1] = (i-1)f_i, & 1 \leq i \leq m_2, \\ [f_i, x_2] = f_i, & 1 \leq i \leq m_2, \\ [x_1, e_1] = -e_1, \end{cases}$$

where $\{e_1, \dots, e_{m_1}, f_1, f_2, \dots, f_{m_2}\}$ is a basis for the nilradical and $\{x_1, x_2\}$ is a basis for the complementary vector space.

The proof of the triviality of $HL^2(R(N_{m_1,m_2}, 2), R(N_{m_1,m_2}, 2))$ will be implemented by induction on m_2.

For $m_2 = 1$, the algebra $R(N_{m_1,1}, 2)$ has the following multiplication table:

$$R(N_{m_1,1}, 2) : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq m_1 - 1, \\ [e_i, x_1] = ie_i, & 1 \leq i \leq m_1, \\ [f_1, x_2] = f_1, \\ [x_1, e_1] = -e_1. \end{cases}$$

The multiplication table shows that $J = \operatorname{span}\{f_1, x_2\}$ is an ideal and the quotient algebra $R(N_{m_1,1}, 2)/J \simeq R(N_{m_1}, 1)$ embeds isomorphically into the Leibniz algebra $R(N_{m_1,1}, 2)$; the triviality if the 2-cohomology group of $R(N_{m_1}, 1)$ is known from [7].

Since the triviality of the second cohomology group of $R(N_{m_1,1}, 2)$ is equivalent to the equality

$$\dim \overline{ZL^2(R(N_{m_1}, 1), R(N_{m_1}, 1))} = \dim \overline{BL^2(R(N_{m_1}, 1), R(N_{m_1}, 1))},$$

we will prove it in the next two propositions and their corollaries.

Proposition 3.4. *The following 2-cocycles form a basis for $\overline{ZL^2(R(N_{m_1}, 1), R(N_{m_1}, 1))}$:*

$$\begin{array}{llll}
\varphi_{i,1}(e_i, e_1) = f_1, & \varphi_{i,1}(e_{i+1}, x_1) = (i+1)f_1, & \varphi_{i,1}(e_{i+1}, x_2) = -f_1, & 1 \leq i \leq m_1 - 1, \\
\varphi_{i,2}(e_i, e_1) = x_2, & \varphi_{i,2}(e_{i+1}, x_1) = (i+1)x_2, & \varphi_{i,2}(f_1, e_{i+1}) = f_1, & 1 \leq i \leq m_1 - 1, \\
\psi_1(f_1, e_1) = e_1, & \psi_1(f_1, x_2) = x_1, & \psi_1(e_i, f_1) = -ie_i, & 1 \leq i \leq m_1, \\
\psi_2(x_1, f_1) = -e_1, & \psi_2(e_i, f_1) = e_{i+1}, & \psi_{2,2}(x_2, x_2) = f_1, & 1 \leq i \leq m_1, \\
\psi_i(f_1, e_1) = e_i, & \psi_i(f_1, x_1) = (i-1)e_{i-1}, & \psi_i(f_1, x_2) = -e_{i-1}, & 2 \leq i \leq m_1 - 1, \\
\psi_{1,1}(f_1, e_1) = f_1, & \psi_{1,1}(x_1, e_1) = x_2, & \psi_{1,1}(e_1, x_1) = -x_2, & \\
\xi_1(x_2, e_1) = e_1, & \xi_1(e_1, x_2) = -e_1, & \xi_1(e_i, x_2) = -ie_i, & 2 \leq i \leq m_1, \\
\xi_2(e_1, x_2) = e_2, & \xi_2(e_i, x_2) = e_{i+1}, & \xi_2(x_1, x_2) = -e_1, & 2 \leq i \leq m_1, \\
\xi_i(x_2, e_1) = e_i, & \xi_i(x_2, x_1) = (i-1)e_{i-1}, & \eta(f_1, x_2) = f_1, & 2 \leq i \leq m_1, \\
\phi_1(x_1, e_1) = f_1, & \phi_1(e_1, x_2) = f_1, & \phi_2(x_1, x_2) = f_1, & \\
\phi_3(f_1, x_2) = -x_2, & \phi_3(f_1, f_1) = f_1, & \phi_4(x_2, x_1) = e_{m_1}, & \\
\chi_1(f_1, x_1) = e_{m_1}, & \chi_1(f_1, x_2) = -\frac{1}{m_1}e_{m_1}, & \chi_2(f_1, x_1) = f_1. &
\end{array}$$

PROOF. Let $\varphi \in \overline{ZL^2(R(N_{m_1},1),R(N_{m_1},1))}$. Then it is easy to see that $\varphi(a,b) \in J$ for all $a,b \in R(N_{m_1},1)$. Put

$$\varphi(e_i,e_1) = \xi_{i,1}f_1 + \xi_{i,2}x_2, \quad 1 \le i \le m_1,$$

$$\varphi(x_1,e_1) = \xi^2_{1,1}f_1 + \xi^2_{1,2}x_2, \varphi(x_1,x_1) = \nu_1 f_1 + \nu_2 x_2,$$

$$\varphi(f_1,e_1) = \sum_{i=1}^{m_1} \alpha^1_{1,i}e_i + \alpha^1_1 f_1 + \alpha^1_{3,1}x_1 + \alpha^1_{3,2}x_2,$$

$$\varphi(e_1,f_1) = \sum_{i=1}^{m_1} \alpha^2_{1,i}e_i + \alpha^2_1 f_1 + \alpha^2_{3,1}x_1 + \alpha^2_{3,2}x_2,$$

$$\varphi(e_1,x_2) = \sum_{i=1}^{m_1} \delta_{1,i}e_i + \delta_1 f_1 + \delta^1_{3,1}x_1 + \delta^1_{3,2}x_2,$$

$$\varphi(x_2,e_1) = \sum_{i=1}^{m_1} \delta_{2,i}e_i + \delta_2 f_1 + \delta^2_{3,1}x_1 + \delta^2_{3,2}x_2,$$

$$\varphi(f_1,x_1) = \sum_{i=1}^{m_1} \beta_{1,i}e_i + \beta_1 f_1 + \beta^3_{1,1}x_1 + \beta^3_{1,2}x_2,$$

$$\varphi(x_1,f_1) = \sum_{i=1}^{m_1} \beta_{2,i}e_i + \beta_2 f_1 + \beta^3_{2,1}x_1 + \beta^3_{2,2}x_2,$$

$$\varphi(f_1,x_2) = \sum_{i=1}^{m_1} \beta^i_{1,2}e_i + \beta^2_2 f_1 + \beta^{1,1}_{1,2}x_1 + \beta^{1,2}_{1,2}x_2,$$

$$\varphi(x_2,f_1) = \sum_{i=1}^{m_1} \beta^i_{2,1}e_i + \beta^{2,2}_{2,1} f_1 + \beta^{1,1}_{2,1}x_1 + \beta^{1,2}_{2,1}x_2,$$

$$\varphi(f_1,f_1) = \sum_{i=1}^{m_1} \alpha^2_{2,i}e_i + \alpha_{2,2} f_1 + \alpha^3_{2,1}x_1 + \alpha^3_{2,2}x_2,$$

$$\varphi(x_1,x_2) = \sum_{i=1}^{m_1} \gamma^i_{1,2}e_i + \gamma^{1,2}_{2,1} f_1 + \gamma^{1,2}_{3,1}x_1 + \gamma^{1,2}_{3,2}x_2,$$

$$\varphi(x_2,x_1) = \sum_{i=1}^{m_1} \gamma^i_{2,1}e_i + \gamma^{2,1}_{2,1} f_1 + \gamma^{2,1}_{3,1}x_1 + \gamma^{2,1}_{3,2}x_2,$$

$$\varphi(x_2,x_2) = \sum_{i=1}^{m_1} \gamma^i_{2,2}e_i + \gamma^{2,2}_{2,1} f_1 + \gamma^{2,2}_{3,1}x_1 + \gamma^{2,2}_{3,2}x_2.$$

Considering the equalities $(d^2\varphi)(e_i,e_1,x_1) = (d^2\varphi)(x_1,e_1,x_1) = 0$, we obtain

$$\varphi(e_i,x_1) = i\varphi(e_{i-1},e_1),\ 2 \le i \le m_1, \quad \varphi(e_1,x_1) = -\varphi(x_1,e_1).$$

Inserting all possible basis elements $x,y,z \in R(N_{m_1,1},2)$ in the identity $(d^2\varphi)(x,y,z) = 0$ and using the fact that $e_2,\dots,e_{m_1},f_1 \in \mathrm{Ann}_r(R(N_{m_1,1},2))$, we obtain the constraints on the parameters defining φ, where triples are given on the left in the implications, and the constraints are on the right:

$$\{f_1,e_1,x_2\} \Rightarrow \beta^{1,1}_{1,2} = \alpha^1_{1,1}, \quad \beta^i_{1,2} = -\alpha^1_{1,i+1},\ 1 \le i \le m_1 - 1,$$

$$\delta^1_{3,2} = \alpha^1_{3,1} = \alpha^1_{3,2} = 0,$$
$$\{f_1, e_1, x_1\} \Rightarrow \beta_{1,i} = i\alpha^1_{1,i+1},\ \xi^2_{1,2} = \alpha^1_1, \quad 1 \le i \le m_1 - 1,$$
$$\{x_2, e_1, x_2\} \Rightarrow \gamma^i_{2,2} = \gamma^{2,2}_{3,1} = \delta_2 = 0, \quad 1 \le i \le m_1 - 1,$$
$$\{f_1, x_2, x_2\} \Rightarrow \gamma^{2,2}_{3,2} = 0,$$
$$\{x_2, x_2, x_1\} \Rightarrow \gamma^{m_1}_{2,2} = \gamma^{2,1}_{2,1} = 0,$$
$$\{x_2, x_1, e_1\} \Rightarrow \gamma^i_{2,1} = i\delta_{2,i+1}, \quad \gamma^{2,1}_{3,1} = \delta^2_{3,1} = \delta^2_{3,2} = 0, \quad 1 \le i \le m_1 - 1,$$
$$\{f_1, x_1, e_1\} \Rightarrow \beta^3_{1,1} = 0,$$
$$\{f_1, x_2, x_1\} \Rightarrow \gamma^{2,1}_{3,2} = \beta^3_{1,2} = 0,\ \beta^{m_1}_{1,2} = -\frac{1}{m_1}\beta_{1,m_1},$$
$$\{x_1, x_2, x_1\} \Rightarrow \gamma^i_{1,2} = \nu_1 = 0, \quad \gamma^1_{1,2} = -\delta_{2,2}, \quad 2 \le i \le m_1,$$
$$\{f_1, x_1, x_1\} \Rightarrow \nu_2 = 0,$$
$$\{e_1, x_1, x_2\} \Rightarrow \delta_{1,i} = \gamma^{1,2}_{3,1} = \delta^1_{3,1} = \delta^1_{3,2} = 0, \quad 3 \le i \le m_1,$$
$$\delta_{1,2} = \delta_{2,2}, \delta_1 = \xi^2_{1,1},$$
$$\{f_1, x_1, x_2\} \Rightarrow \gamma^{1,2}_{3,2} = 0,$$
$$\{x_1, x_2, e_1\} \Rightarrow \delta_{1,1} = -\delta_{2,1},$$
$$\{x_1, f_1, x_1\} \Rightarrow \beta_{2,1} = -\alpha^1_{1,2}, \beta_{2,i} = 0, \quad 2 \le i \le m_1,$$
$$\{x_1, f_1, x_2\} \Rightarrow \beta^3_{2,1} = \beta^3_{2,2} = 0,$$
$$\{x_2, f_1, x_2\} \Rightarrow \beta^{1,1}_{2,1} = \beta^{1,2}_{2,1} = \beta^i_{2,1} = 0, \quad 1 \le i \le m_1,$$
$$\{x_2, x_2, f_1\} \Rightarrow \beta^{2,2}_{2,1} = 0,$$
$$\{x_1, x_2, f_1\} \Rightarrow \beta_2 = 0,$$
$$\{f_1, f_1, x_2\} \Rightarrow \beta^{1,2}_{1,2} = -\alpha_{2,2}, \quad \alpha^2_{2,i} = \alpha^3_{2,1} = \alpha^3_{2,2} = 0, \quad 1 \le i \le m_1,$$
$$\{e_1, f_1, x_1\} \Rightarrow \alpha^2_{1,2} = \alpha^1_{1,2}, \quad \alpha^2_{1,i} = \alpha^2_1 = \alpha^2_{3,1} = \alpha^2_{3,2} = 0, \quad 3 \le i \le m_1,$$
$$\{e_1, f_1, x_2\} \Rightarrow \alpha^2_{1,1} = -\alpha^1_{1,1}.$$

Considering the identity $(d^2\varphi)(x, y, z) = 0$ for the following triples of elements:

$$\{e_i, e_1, f_1\}, \quad \{e_1, e_i, e_1\}, \quad \{e_1, e_1, e_i\}, \quad \{e_i, e_1, e_j\}, \quad \{f_1, e_i, e_1\},$$
$$\{x_1, e_i, e_1\}, \quad \{e_i, e_1, x_2\}, \quad \{e_{m_1}, e_1, x_1\}, \quad \{x_2, e_i, e_1\}$$

for $1 \le i, j \le m_1$, we recurrently obtain

$$\varphi(e_i, f_1) = -i\alpha^1_{1,1}e_i + \alpha^1_{1,2}e_{i+1}, \quad \varphi(f_1, e_i) = -\xi_{i-1,2}f_1, \quad 2 \le i \le m_1,$$
$$\varphi(e_i, x_2) = -i\delta_{2,1}e_i + \delta_{2,2}e_{i+1} - \xi_{i-1,1}f_1, \quad 2 \le i \le m_1,$$
$$\varphi(e_{m_1}, e_1) = \varphi(e_i, e_j) = \varphi(x_1, e_j) = \varphi(x_2, e_j) = 0, \quad 1 \le i \le m_1,\ 2 \le j \le m_1.$$

Thus, every element $\varphi \in \overline{ZL^2(R(N_{m_1}, 1), R(N_{m_1}, 1))}$ has the form

$$\varphi(e_i, e_1) = \xi_{i,1}f_1 + \xi_{i,2}x_2, \quad \varphi(f_1, e_1) = \sum_{i=1}^{m_1} \alpha^1_{1,i}e_i + \alpha^1_1 f_1,\ 1 \le i \le m_1 - 1,$$

$$\varphi(x_1, e_1) = \xi^2_{1,1}f_1 + \alpha^1_1 x_2, \quad \varphi(x_2, e_1) = \sum_{i=1}^{m_1} \delta_{2,i}e_i, \quad \varphi(x_2, x_2) = \gamma^{2,2}_{2,1}f_1,$$

$$\varphi(f_1,f_1)=\alpha_{2,2}f_1,\quad \varphi(f_1,x_1)=\sum_{i=1}^{m_1-1} i\alpha^1_{1,i+1}e_i+\beta_{1,m_1}e_{m_1}+\beta_1 f_1,$$

$$\varphi(f_1,x_2)=-\sum_{i=1}^{m_1-1}\alpha^1_{1,i+1}e_i-\frac{1}{m_1}\beta_{1,m_1}e_{m_1}+\beta^2_2 f_1+\alpha^1_{1,1}x_1-\alpha_{2,2}x_2,$$

$$\varphi(x_1,x_2)=-\delta_{2,2}e_1+\gamma^{1,2}_{2,1}f_1,\varphi(x_2,x_1)=\sum_{i=1}^{m_1-1} i\delta_{2,i+1}e_i+\gamma^{m_1}_{2,1}e_{m_1},$$

$$\varphi(e_1,x_2)=-\delta_{2,1}e_1+\delta_{2,2}e_2+\xi^2_{1,1}f_1,\quad \varphi(x_1,f_1)=-\alpha^1_{1,2}e_1,$$

$$\varphi(e_i,f_1)=-i\alpha^1_{1,1}e_i+\alpha^1_{1,2}e_{i+1},\quad 1\le i\le m_1,$$

$$\varphi(e_1,x_1)=-\varphi(x_1,e_1),\quad \varphi(e_i,x_1)=i\varphi(e_{i-1},e_1),\quad 2\le i\le m_1,$$

$$\varphi(f_1,e_i)=-\xi_{i-1,2}f_1,\quad \varphi(e_i,x_2)=-i\delta_{2,1}e_i+\delta_{2,2}e_{i+1}-\xi_{i-1,1}f_1,\quad 2\le i\le m_1,\quad \square$$

Corollary 3.5. $\dim \overline{ZL^2(R(N_{m_1},1),R(N_{m_1},1))}=4m_1+7$.

By analogy with Proposition 3.4, we prove

Proposition 3.6. *The following 2-coboundaries form a basis for* $\overline{BL^2(R(N_{m_1},1),R(N_{m_1},1))}$:

$$\begin{array}{llll}
\widetilde{\varphi}_{1,1}(x_1,e_1)=f_1, & \widetilde{\varphi}_{i,1}(e_i,x_1)=-if_1, & \widetilde{\varphi}_{i,1}(e_i,x_2)=f_1, & 1\le i\le m_1,\\
\widetilde{\varphi}_{i,2}(e_{i-1},e_1)=-x_2, & \widetilde{\varphi}_{i,1}(e_{i-1},e_1)=-f_1, & \widetilde{\varphi}_{i,2}(f_1,e_i)=f_1, & 2\le i\le m_1,\\
\widetilde{\varphi}_{1,2}(f_1,e_1)=f_1, & \widetilde{\varphi}_{1,2}(x_1,e_1)=x_2, & \widetilde{\varphi}_{i,2}(e_i,x_1)=-ix_2, & 1\le i\le m_1,\\
\widetilde{\xi}_1(e_i,f_1)=e_{i+1}, & \widetilde{\xi}_i(f_1,x_2)=-e_i, & \widetilde{\xi}_i(f_1,x_1)=ie_i, & 1\le i\le m_1,\\
\widetilde{\xi}_1(x_1,f_1)=-e_1, & \widetilde{\xi}_i(f_1,e_1)=e_{i+1}, & & 1\le i\le m_1-1,\\
\widetilde{\xi}_{1,1}(f_1,x_2)=-x_1, & \widetilde{\xi}_{1,1}(f_1,e_1)=-e_1, & \widetilde{\xi}_{1,1}(e_i,f_1)=ie_i, & 1\le i\le m_1,\\
\widetilde{\xi}_{1,2}(f_1,x_2)=-x_2, & \widetilde{\xi}_{1,2}(f_1,f_1)=f_1, & \widetilde{\xi}_{1,3}(f_1,x_2)=f_1, & \\
\widetilde{\psi}_1(x_1,x_2)=-e_1, & \widetilde{\psi}_1(e_i,x_2)=e_{i+1}, & \widetilde{\psi}_i(x_2,e_1)=e_{i+1}, & 1\le i\le m_1-1,\\
\widetilde{\psi}_i(x_2,x_1)=ie_i, & & & 1\le i\le m_1,\\
\widetilde{\phi}_1(x_2,e_1)=-e_1, & \widetilde{\phi}_1(e_i,x_2)=-ie_i, & & 1\le i\le m_1,\\
\widetilde{\phi}_2(f_1,x_1)=f_1, & \widetilde{\phi}_3(x_1,x_2)=f_1, & \widetilde{\phi}_4(x_2,x_2)=f_1. &
\end{array}$$

Corollary 3.7. $\dim \overline{BL^2(R(N_{m_1},1),R(N_{m_1},1))}=4m_1+7$.

Assertions 3.5–3.7 imply the following result:

Theorem 3.8. $HL^2(R(N_{m_1,1},2),R(N_{m_1,1},2))=0$.

Prove that $HL^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))=0$ for $m_2\ge 2$. Since $R(N_{m_1,m_2},2)$ contains no ideal J for which the quotient algebra $R(N_{m_1,m_2},2))/J$ embeds isomorphically into the Leibniz algebra $R(N_{m_1,m_2},2)$, calculate the dimension of the spaces of 2-cocycles and 2-coboundaries of $R(N_{m_1,m_2},2)$ by counting the number of independent parameters defining arbitrary elements of these spaces.

Proposition 3.9. $\dim \mathrm{ZL}^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))=(m_1+m_2)^2+4(m_1+m_2)+1$.

Proof. Let $\varphi\in \mathrm{ZL}^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))$. Put

$$\varphi(e_k,e_1)=\sum_{i=1}^{m_1}\alpha^k_{1,i}e_i+\sum_{i=1}^{m_2}\alpha^k_{2,i}f_i+\alpha_{k,1}x_1+\alpha_{k,2}x_2,\quad 1\le k\le m_1,$$

$$\varphi(f_k,e_1)=\sum_{i=1}^{m_1}\beta^k_{1,i}e_i+\sum_{i=1}^{m_2}\beta^k_{2,i}f_i+\beta_{k,1}x_1+\beta_{k,2}x_2,\quad 1\le k\le m_2,$$

$$\varphi(x_1,e_1)=\sum_{i=1}^{m_1}\delta^1_{1,i}e_i+\sum_{i=1}^{m_2}\delta^1_{2,i}f_i+\delta^1_1x_1+\delta^1_2x_2,$$
$$\varphi(e_1,x_1)=\sum_{i=1}^{m_1}\delta^2_{1,i}e_i+\sum_{i=1}^{m_2}\delta^2_{2,i}f_i+\delta^2_1x_1+\delta^2_2x_2,$$
$$\varphi(x_2,e_1)=\sum_{i=1}^{m_1}\delta^3_{1,i}e_i+\sum_{i=1}^{m_2}\delta^3_{2,i}f_i+\delta^3_1x_1+\delta^3_2x_2,$$
$$\varphi(e_1,x_2)=\sum_{i=1}^{m_1}\delta^4_{1,i}e_i+\sum_{i=1}^{m_2}\delta^4_{2,i}f_i+\delta^4_1x_1+\delta^4_2x_2,$$
$$\varphi(e_1,f_1)=\sum_{i=1}^{m_1}\alpha^{1,2}_{1,i}e_i+\sum_{i=1}^{m_2}\alpha^{1,2}_{2,i}f_i+\alpha^{1,2}_1x_1+\alpha^{1,2}_2x_2,$$
$$\varphi(f_1,f_1)=\sum_{i=1}^{m_1}\alpha^{2,2}_{1,i}e_i+\sum_{i=1}^{m_2}\alpha^{2,2}_{2,i}f_i+\alpha^{2,2}_1x_1+\alpha^{2,2}_2x_2,$$
$$\varphi(x_i,x_j)=\sum_{t=1}^{m_1}\gamma^{i,j}_{1,t}e_t+\sum_{t=1}^{m_2}\gamma^{i,j}_{2,t}f_t+\gamma^{i,j}_1x_1+\gamma^{i,j}_2x_2,\quad 1\le i,j\le 2,$$
$$\varphi(x_t,f_1)=\sum_{i=1}^{m_1}\xi^t_{1,i}e_i+\sum_{i=1}^{m_2}\xi^t_{2,i}f_i+\xi^t_1x_1+\xi^t_2x_2,\quad 1\le t\le 2,$$
$$\varphi(f_1,x_1)=\sum_{i=1}^{m_1}\xi^3_{1,i}e_i+\sum_{i=1}^{m_2}\xi^3_{2,i}f_i+\xi^3_1x_1+\xi^3_2x_2,$$
$$\varphi(f_1,x_2)=\sum_{i=1}^{m_1}\xi^4_{1,i}e_i+\sum_{i=1}^{m_2}\xi^4_{2,i}f_i+\xi^4_1x_1+\xi^4_2x_2.$$

Inserting in the identity

$$\begin{aligned}(d^2\varphi)(x,y,z)&=[x,\varphi(y,z)]-[\varphi(x,y),z]+[\varphi(x,z),y]\\&+\varphi(x,[y,z])-\varphi([x,y],z)+\varphi([x,z],y)=0\end{aligned}$$

different elements $x,y,z\in R(N_{m_1,m_2},2)$, we obtain constraints on the parameters defining the 2-cocycle φ.

The equality $(d^2\varphi)(e_i,e_1,e_1)=0$ gives

$$\varphi(e_i,e_2)=-i\alpha_{1,1}e_1-\alpha^1_{1,1}e_{i+1},\quad 1\le i\le m_1.$$

The equalities $(d^2\varphi)(e_i,e_1,e_j)=0$ and $(d^2\varphi)(e_1,e_i,e_j)=0$ for $1\le i\le m_1$ and $2\le j\le m_1$ imply

$$\varphi(e_i,e_j)=-i\alpha_{j-1,1}e_i+\big(\alpha_{j-2,1}-\alpha^{j-1}_{1,1}\big)e_{i+1},\quad 1\le i\le m_1,\ 3\le j\le m_1.$$

Considering the equality $(d^2\varphi)(f_i,e_j,e_1)=0$ for $1\le i\le m_2$ and $1\le j\le m_1$, we conclude

$$\varphi(f_i,e_2)=-((i-1)\alpha_{1,1}+\alpha_{1,2})f_i-\alpha^1_{1,1}f_{i+1},\quad 1\le i\le m_2,$$

$$\varphi(f_i,e_j)=-((i-1)\alpha_{j-1,1}+\alpha_{j-1,2})f_i+\big(\alpha_{j-2,1}-\alpha^{j-1}_{1,1}\big)f_{i+1},\ 1\le i\le m_2,\ 3\le j\le m_1.$$

Inserting different elements $x,y,z\in L$ in the identity $(d^2\varphi)(x,y,z)=0$, we obtain constraints on the parameters defining φ, where the implications contain equalities on the left and constraints on the right:

$$(d^2\varphi)(x_1,f_1,x_1)=0\Rightarrow \xi^3_{1,1}=-\xi^1_{1,1},\xi^1_{1,i}=0,\ 2\le i\le m_1,\quad \xi^1_{2,j}=0,\quad 2\le j\le m_2,$$

$$
\begin{aligned}
(d^2\varphi)(x_1,e_1,f_1)=0 \Rightarrow\ & \xi_1^1=\alpha_{2,1}^{1,2}=\alpha_1^{1,2}=\alpha_2^{1,2}=0, \quad \alpha_{1,i}^{1,2}=0, \quad 3\le i\le m_1,\\
& \xi_{1,1}^1=-\alpha_{1,2}^{1,2}, \quad \xi_{2,1}^1=-\alpha_{2,2}^{1,2}, \quad \alpha_{2,i}^{1,2}=0, \quad 3\le i\le m_2,\\
(d^2\varphi)(e_1,x_2,f_1)=0 \Rightarrow\ & \xi_{1,1}^2=\xi_1^2=\alpha_{2,2}^{1,2}=0,\\
(d^2\varphi)(f_1,f_1,x_1)=0 \Rightarrow\ & \alpha_{2,2}^{2,2}=\alpha_{1,2}^{1,2}, \quad \xi_2^3=\alpha_{1,i}^{2,2}=0, \quad 1\le i\le m_1,\\
& \alpha_{2,j}^{2,2}=0,\ 3\le j\le m_2,\\
(d^2\varphi)(f_1,x_1,f_1)=0 \Rightarrow\ & \xi_2^1=0,\\
(d^2\varphi)(e_1,x_1,x_1)=0 \Rightarrow\ & \gamma_{1,1}^{1,1}=\gamma_1^{1,1}=0,\\
(d^2\varphi)(f_1,x_1,x_1)=0 \Rightarrow\ & \gamma_2^{1,1}=0,\\
(d^2\varphi)(x_1,e_1,x_1)=0 \Rightarrow\ & \delta_{1,2}^2=\delta_{1,2}^1,\ \delta_{2,2}^2=0,\ \gamma_{1,i}^{1,1}=i\delta_{1,i+1}^1-\delta_{1,i+1}^2,\ 1\le i\le m_1-1,\\
& \delta_{2,1}^2=-\delta_{2,1}^1, \quad \delta_2^2=-\delta_2^1,\\
& \delta_1^2=-\delta_1^1, \quad \gamma_{2,i}^{1,1}=(i-1)\delta_{2,i+1}^1-\delta_{2,i+1}^2, \quad 2\le i\le m_2-1,\\
(d^2\varphi)(x_1,x_1,x_2)=0 \Rightarrow\ & \gamma_{2,1}^{1,1}=\gamma_{1,i}^{1,2}=0, \quad \gamma_{2,m_2}^{1,2}=\frac{1}{m_2-1}\gamma_{2,m_2}^{1,1}, \quad 2\le i\le m_1,\\
& \gamma_{2,i}^{1,2}=\delta_{2,i+1}^1-\frac{1}{i-1}\delta_{2,i+1}^2, \quad 2\le i\le m_2-1,\\
(d^2\varphi)(e_1,e_1,x_1)=0 \Rightarrow\ & \delta_{1,1}^2=-\delta_{1,1}^1,\\
(d^2\varphi)(e_1,x_1,x_2)=0 \Rightarrow\ & \delta_1^4=\delta_2^4=\gamma_1^{1,2}=0, \quad \gamma_{1,1}^{1,2}=-\delta_{1,2}^4, \quad \delta_{1,i}^4=0, \quad 3\le i\le m_1,\\
& \delta_{2,1}^4=\delta_{2,1}^1, \quad \delta_{2,i}^4=\frac{1}{i-2}\delta_{2,i}^2, \quad 3\le i\le m_2,\\
(d^2\varphi)(x_2,e_1,x_2)=0 \Rightarrow\ & \gamma_1^{2,2}=\delta_{2,1}^3=\gamma_{1,i}^{2,2}=0, \quad 1\le i\le m_1-1,\\
& \gamma_{2,i}^{2,2}=\delta_{2,i+1}^3, \quad 1\le i\le m_2-1,\\
(d^2\varphi)(x_1,x_2,e_1)=0 \Rightarrow\ & \delta_{1,1}^4=-\delta_{1,1}^3, \quad \gamma_{2,1}^{1,2}=\delta_{2,2}^1-\delta_{2,2}^4,\\
(d^2\varphi)(x_2,x_1,e_1)=0 \Rightarrow\ & \gamma_1^{2,1}=\gamma_{2,1}^{2,1}=0, \quad \gamma_{1,i}^{2,1}=i\delta_{1,i+1}^3, \quad 1\le i\le m_1-1,\\
& \delta_1^3=\delta_2^3=0, \quad \gamma_{2,i}^{2,1}=(i-1)\delta_{2,i+1}^3, \quad 2\le i\le m_2-1,\\
(d^2\varphi)(f_1,x_2,x_1)=0 \Rightarrow\ & \gamma_2^{2,1}=\xi_1^3=0, \quad \xi_{1,i}^4=-\frac{1}{i}\xi_{1,i}^3, \quad 1\le i\le m_1,\\
& \delta_{1,2}^3=\xi_{2,2}^4, \quad \xi_{2,i}^4=0, \quad 3\le i\le m_2,\\
(d^2\varphi)(f_1,x_1,x_2)=0 \Rightarrow\ & \gamma_2^{1,2}=0, \quad \delta_{1,2}^4=\delta_{1,2}^3,\\
(d^2\varphi)(e_1,f_1,x_2)=0 \Rightarrow\ & \xi_{1,1}^3=\alpha_{1,2}^{1,2}, \quad \xi_1^4=-\alpha_{1,1}^{1,2},\\
(d^2\varphi)(f_1,f_1,x_2)=0 \Rightarrow\ & \xi_2^4=-\alpha_{2,1}^{2,2}, \quad \alpha_1^{2,2}=\alpha_2^{2,2}=0,\\
(d^2\varphi)(x_2,x_2,f_1)=0 \Rightarrow\ & \xi_{2,i}^2=0, \quad 1\le i\le m_2,\\
(d^2\varphi)(x_2,f_1,x_1)=0 \Rightarrow\ & \xi_{1,i}^2=0, \quad 2\le i\le m_1,\\
(d^2\varphi)(x_2,f_1,x_2)=0 \Rightarrow\ & \xi_2^2=0,\\
(d^2\varphi)(f_1,x_2,x_2)=0 \Rightarrow\ & \gamma_2^2=\gamma_{1,1}^{2,2}=0,\\
(d^2\varphi)(x_2,x_2,x_1)=0 \Rightarrow\ & \gamma_{1,m_1}^{2,2}=0, \quad \gamma_{2,m_2}^{2,2}=\frac{1}{m_2-1}\gamma_{2,m_2}^{2,1}.
\end{aligned}
$$

The equality $(d^2\varphi)(e_i, e_1, f_j) = 0$ for $1 \le i \le m_1$ and $1 \le j \le m_2$ yields

$$\varphi(e_i, f_1) = i\alpha_{1,1}^{1,2} e_i + \alpha_{1,2}^{1,2} e_{i+1},\ \varphi(e_i, f_2) = -i\beta_{1,1} e_i - \big(\beta_{1,1}^1 + \alpha_{1,1}^{1,2}\big) e_{i+1}, \quad 1 \le i \le m_1,$$
$$\varphi(e_i, f_j) = -i\beta_{j-1,1} e_i + \big(\beta_{j-2,1} - \beta_{1,1}^{j-1}\big) e_{i+1}, \quad 1 \le i \le m_1,\ 3 \le j \le m_2.$$

The equalities $(d^2\varphi)(f_i, e_1, f_j) = 0$ and $(d^2\varphi)(f_i, f_j, e_1) = 0$ for $1 \le i, j \le m_2$ imply

$$\varphi(f_i, f_1) = \big((i-1)\alpha_{1,1}^{1,2} + \alpha_{2,1}^{2,2}\big) f_i + \alpha_{1,2}^{1,2} f_{i+1},$$
$$\varphi(f_i, f_2) = -((i-1)\beta_{1,1} + \beta_{1,2}) f_i - \big(\beta_{1,1}^1 + \alpha_{1,1}^{1,2}\big) f_{i+1},$$
$$\varphi(f_i, f_j) = -((i-1)\beta_{j-1,1} + \beta_{j-1,2}) f_i + \big(\beta_{j-2,1} - \beta_{1,1}^{j-1}\big) f_{i+1}, \quad 3 \le j \le m_2.$$

Considering $(d^2\varphi)(x, y, z) = 0$ for the triples of elements $\{x_1, e_i, e_1\}$, $\{e_i, e_1, x_1\}$, $\{x_2, e_i, e_1\}$, and $\{e_i, e_1, x_2\}$ for $1 \le i \le m_1 - 1$, we consecutively find by recursion that

$$\varphi(x_1, e_2) = \alpha_{1,1}^1 e_1, \varphi(x_1, e_i) = \big(\alpha_{1,1}^{i-1} - \alpha_{i-2,1}\big) e_1, 3 \le i \le m_1, \quad \varphi(x_2, e_i) = 0,\ 2 \le i \le m_1,$$

$$\varphi(e_i, x_1) = \sum_{t=1}^{i-2} (i-t) \left(\sum_{j=1}^{t} \alpha_{1,j}^{i-t+j-1} - \alpha_{i-t-1,1} \right) e_t + \left(\sum_{j=1}^{i-1} \alpha_{1,j}^{j} - \frac{(i+1)(i-2)}{2} \delta_1^1 \right) e_{i-1}$$
$$-i\delta_{1,1}^1 e_i + \sum_{j=i+1}^{m_1} \left(\delta_{j-i+1}^2 - (j-i) \sum_{t=1}^{i-1} \alpha_{1,j-i+t+1}^t \right) e_j + \sum_{t=1}^{i-1} \left((i-t+1) \sum_{j=1}^{t} \alpha_{2,j}^{i+j-t-1} \right) f_t$$
$$+ \sum_{j=i}^{m_2} \left((i+1-j) \sum_{t=1}^{i-1} \alpha_{2,j-i+t+1}^t + \delta_{2,j-i+1}^2 \right) f_j + i(\alpha_{i-1,1} x_1 + \alpha_{i-1,2} x_2), \quad 2 \le i \le m_1,$$

$$\varphi(e_i, x_2) = -i\delta_{1,1}^3 e_i + \delta_{1,2}^3 e_{i+1} - \sum_{t=1}^{i-1} \left(\sum_{j=1}^{t} \alpha_{2,j}^{i+j-t-1} \right) f_t + \left(\delta_{2,1}^1 - \sum_{t=1}^{i-1} \alpha_{2,t+1}^t \right) f_i$$
$$+ \left(\delta_{2,2}^4 - \sum_{t=1}^{i-1} \alpha_{2,t+2}^t \right) f_{i+1} + \sum_{j=i+2}^{m_2} \left(\frac{1}{j-i-1} \delta_{2,j-i+1}^2 - \sum_{t=1}^{i-1} \alpha_{2,j+t-i+1}^t \right) f_j, \quad 2 \le i \le m_1.$$

Similarly, inserting the triples of elements $\{x_1, f_i, e_1\}$, $\{f_i, e_1, x_1\}$, $\{f_i, e_1, x_2\}$, $\{x_2, f_i, e_1\}$, $1 \le i \le m$, in the equality $(d^2\varphi)(x, y, z) = 0$, we consecutively find by recursion that

$$\varphi(x_1, f_2) = \big(\alpha_{1,2}^{1,2} + \beta_{1,1}^1\big) e_1, q\varphi(x_1, f_i) = \big(\beta_{1,1}^{i-1} - \beta_{i-2,1}\big) e_1, \quad 3 \le i \le m_2,$$

$$\varphi(f_i, x_1) = \sum_{t=1}^{i-2} (i-t-1) \left(\sum_{j=1}^{t} \beta_{1,j}^{i-t+j-1} - \beta_{i-t-1,1} \right) e_t + \left(\alpha_{1,2}^{1,2} - \sum_{t=1}^{i-1} \beta_{1,t+1}^t \right) e_i$$
$$+ \left(\xi_{1,2}^3 - 2\sum_{t=1}^{i-1} \beta_{1,t+2}^t \right) e_{i+1} - \sum_{j=i+2}^{m_1} \left((j-i+1) \sum_{t=1}^{i-1} \beta_{1,j-i+t+1}^t \right) e_j + \sum_{t=1}^{i-2} \left((i-t) \sum_{j=1}^{t} \beta_{2,j}^{i+j-t-1} \right) f_t$$
$$\left(\sum_{t=1}^{i-1} \beta_{2,t}^t - \frac{(i-2)(i-1)}{2} \delta_1^1 - (i-1)\delta_2^1 \right) f_{i-1} + (\xi_{2,1}^3 - (i-1)\delta_{1,1}^1) f_i$$
$$+ \sum_{j=i+1}^{m_2} \left(\xi_{2,j-i+1}^3 - (j-i) \sum_{t=1}^{i-1} \beta_{2,j+t-i+1}^t \right) f_j + (i-1)\big(\beta_{i-1,1} x_1 + \beta_{i-1,2} x_2\big), \quad 2 \le i \le m_2,$$

$$\varphi(f_i, x_2) = \sum_{j=1}^{i-2} \left(\sum_{t=1}^{j} \beta_{1,t}^{i+t-j-1} - \beta_{i-j-1,1} \right) e_j + \left(\sum_{t=1}^{i-1} \beta_{1,t}^t + \alpha_{1,1}^{1,2} \right) e_{i-1}$$

$$+\Big(\sum_{t=1}^{i-1}\beta_{1,t+1}^{t}-\alpha_{1,2}^{1,2}\Big)e_i+\sum_{j=i+1}^{m_1}\Big(\sum_{t=1}^{i-1}\beta_{1,j+t-i+1}^{t}-\frac{1}{j-i+1}\xi_{1,j-i+1}^{3}\Big)e_j$$
$$+\big(\xi_{2,1}^{4}-(i-1)\delta_{1,1}^{3}\big)f_i+\xi_{2,2}^{4}f_{i+1}+\beta_{i-1,1}x_1+\beta_{i-1,2}x_2,\quad 2\le i\le m_2,$$

$$\varphi(x_2,f_i)=0,\quad 1\le i\le m_2.$$

The equalities $(d^2\varphi)(e_{m_1},e_1,x_1)=(d^2\varphi)(f_{m_2},e_1,x_1)=(d^2\varphi)(f_{m_2},e_1,x_2)=0$ imply

$$\alpha_{1,1}^{m_1}=\alpha_{m_1-1,1},\quad \alpha_{2,1}^{m_1}=\alpha_{m_1,1}=\alpha_{m_1,2}=0,$$

$$\alpha_{1,i}^{m_1}=\alpha_{m_1-i,1}-\sum_{j=1}^{i-1}\alpha_{1,j}^{m_1+j-i},\quad 2\le i\le m_1-1,$$

$$\alpha_{2,i}^{m_1}=-\sum_{j=1}^{i-1}\alpha_{2,j}^{m_1+j-i},\ 2\le i\le m_2,\quad \delta_1^1=\frac{2}{(m_1+2)(m_1-1)}\sum_{i=1}^{m_1}\alpha_{1,i}^{i},$$

$$\beta_{m_2,1}=\beta_{m_2,2}=\beta_{2,1}^{m_2}=0,\quad \beta_{1,1}^{m_2}=\beta_{m_2-1,1},\quad \beta_{1,i}^{m_2}=\beta_{m_2-i,1}-\sum_{j=1}^{i-1}\beta_{1,j}^{m_2-i+j},$$

$$\beta_{2,i}^{m_2}=-\sum_{j=1}^{i-1}\beta_{2,j}^{m_2+j-i},\delta_2^1=-\frac{m_2-1}{(m_1+2)(m_1-1)}\sum_{i=1}^{m_1}\alpha_{1,i}^{i}+\frac{1}{m_2}\sum_{i=1}^{m_2}\beta_{2,i}^{i},$$

$$\xi_{1,2}^{3}=2\sum_{t=1}^{m_2}\beta_{1,t+2}^{t},\quad \alpha_{1,2}^{1,2}=\sum_{t=1}^{m_2}\beta_{1,t+1}^{t},\quad \alpha_{1,1}^{1,2}=-\sum_{t=1}^{m_2}\beta_{1,t}^{t},\quad 2\le i\le m_2-1.$$

The description of an arbitrary $\varphi\in \mathrm{ZL}^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))$ involves $(m_1+m_2)^2+4(m_1+m_2)+1$ independent parameters; therefore,

$$\dim \mathrm{ZL}^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))=(m_1+m_2)^2+4(m_1+m_2)+1.\quad\square$$

It is not hard to conclude from the definition of the space $\mathrm{BL}^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))$ and Proposition 3.2 for $s=2$ that $\dim \mathrm{BL}^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))=(m_1+m_2+2)^2-3$.

The definition of the second cohomology group gives

$$HL^2(R(N_{m_1,m_2},2),R(N_{m_1,m_2},2))=0. \tag{1}$$

Generalizing the process of proving (1) yields

Theorem 3.10. $HL^2(R(N_{m_1,\dots,m_s},s),R(N_{m_1,\dots,m_s},s))=0$.

REMARK 3.11. In calculating the second cohomology group of $R(N_{m_1,\dots,m_s},s)$, we found that the dimensions of the spaces of 2-cocycles and 2-coboundaries are equal to $(m_1+m_2+\dots+m_s+s)^2-(s+1)$.

Corollary 3.12. *$R(N_{m_1,\dots,m_s},s)$ is a rigid algebra in the variety of Leibniz algebras of dimension $m_1+\dots+m_s+s$.*

References

1. Blokh A., "On a generalization of the concept of Lie algebras," Dokl. Akad. Nauk SSSR, vol. 165, no. 3, 471–473 (1965).
2. Loday J.-L., "Une version non commutative des algèbres de Lie: les algèbres de Leibniz," Enseign. Math., vol. 39, no. 3–4, 269–293 (1993).
3. Balavoine D., "Déformations et rigidité géométrique des algebras de Leibniz," Comm. Algebra, vol. 24, no. 3, 1017–1034 (1996).
4. Hochschild G. and Serre J-P., "Cohomology of Lie algebras," Ann. Math., vol. 57, 591–603 (1953).
5. Ancochea Bermúdez J. M. and Campoamor-Stursberg R., "Cohomologically rigid solvable Lie algebras with a nilradical of arbitrary characteristic sequence," Linear Algebra Appl., vol. 488, 135–147 (2016).
6. Khalkulova Kh. A. and Abdurasulov K. K., "Solvable Lie algebras with maximal dimension of complementary space to nilradical," Uzbek Math. J., vol. 1, 90–98 (2018).
7. Ancochea Bermúdez J. M. and Campoamor-Stursberg R., "On a complete rigid Leibniz non-Lie algebra in arbitrary dimension," Linear Algebra Appl., vol. 438, no. 8, 3397–3407 (2013).

U. Kh. Mamadaliev; B. A. Omirov
National University of Uzbekistan, Tashkent, Uzbekistan
E-mail address: `mamadaliyevuktamjon@mail.ru; omirovb@mail.ru`