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# ON $\sigma$-SUBNORMAL SUBGROUPS OF FINITE GROUPS

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**Abstract:** Let $p$ be a prime and let $\sigma = \{\{p\}, \{p\}'\}$ be a partition of the set $\mathbb{P}$ of all primes. We prove the following conjecture by Skiba: If each complete Hall set of type $\sigma$ in a finite group $G$ is reducible to some subgroup $H$ of $G$ then $H$ is $\sigma$-subnormal in $G$.

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## 1. Introduction

Answering a question of Kegel [1] and Wielandt [2], Kleidman proved in [3] that a subgroup $H$ of a finite group $G$ is subnormal in $G$ if $H \cap P$ is a Sylow $p$-subgroup in $H$ for every Sylow $p$-subgroup $P$ in $G$ and every prime $p$.

This result led to the corresponding question for the $\sigma$-subnormal subgroups of a finite group which was posed by Skiba in [4] as Question 19.86 (see also Question 7.2 in [5]).

**Problem 1.** *Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes and let $G$ be a finite group having a Hall $\sigma_i$-subgroup for each $i \in I$. Let $H$ be a subgroup in $G$ such that $H \cap S_i$ is a Hall $\sigma_i$-subgroup in $H$ for each $i \in I$ and every Hall $\sigma_i$-subgroup $S_i$ in $G$. Is it true that $H$ is a $\sigma$-subnormal subgroup in $G$?*

It was Skiba who proposed in [7] the concept of $\sigma$-subnormal subgroup which develops the idea of a subnormal subgroup from [6]. This concept bases on the following definitions:

Let $\mathbb{P}$ be the set of primes, $\pi \subseteq \mathbb{P}$, and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is a natural then $\pi(n)$ is the set of all primes dividing $n$; in particular, $\pi(G) = \pi(|G|)$ is the set of all primes dividing the order $|G|$ of $G$. In what follows, $\sigma$ is always a partition of $\mathbb{P}$ into pairwise disjoint subsets $\sigma_i$ $(i \in I)$, i.e., $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$. Following [8], we say that a group $G$ is *$\sigma$-primary* if $G$ is a $\sigma_i$-subgroup for some $i \in I$.

A subgroup $H$ in a group $G$ is called *$\sigma$-subnormal* if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \dots \subseteq H_n = G$$

such that for each $i = 1, 2, \dots, n$ either the subgroup $H_{i-1}$ is normal in $H_i$ or the subgroup $H_i/\mathrm{Core}_{H_i}(H_{i-1})$ is $\sigma$-primary. Clearly, a subgroup $H$ is subnormal in $G$ if and only if it is $\sigma$-subnormal in $G$ for the *minimal* partition $\sigma = \{\{2\}, \{3\}, \{5\}, \dots\}$.

Problem 2 below is more general as compared with Problem 1. This is connected with existence of the subgroups having several classes of conjugate Hall subgroups.

Following [8], we say that a system $\Sigma = \{S_1, S_2, \dots, S_k\}$ of $\sigma$-primary Hall subgroups in a group $G$ is a *complete Hall set of type* $\sigma$ of $G$ provided that

(1) $(|S_i|, |S_j|) = 1$ for all $i \neq j \in \{1, 2, \dots, k\}$;

(2) $\pi(G) = \pi(S_1) \cup \pi(S_2) \cup \dots \cup \pi(S_k)$.

If $\Sigma = \{S_1, S_2, \dots, S_k\}$ is a complete Hall set of type $\sigma$ of $G$; then, obviously, $\Sigma^g = \{S_1^g, S_2^g, \dots, S_k^g\}$ is also a complete Hall set of type $\sigma$ in $G$ for every $g \in G$. A group $G$ is called *$\sigma$-complete* if $G$ possesses

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at least one Hall set of type $\sigma$ (clearly, for some partitions $\sigma$, there exist groups for which the set of all complete Hall sets of type $\sigma$ is empty).

We say that a complete Hall set $\Sigma$ of type $\sigma$ in a group $G$ is *reduced* to a subgroup $H$ of $G$ if $H \cap S_i$ is a Hall $\sigma_i$-subgroup in $H$ for each $i = 1, 2, \dots, k$ (it is possible that $H \cap S_i = 1$ for some $i = 1, 2, \dots, k$).

**Problem 2.** *Let $\sigma$ be a partition of the set $\mathbb{P}$ of all primes and let $\Sigma = \{S_1, S_2, \dots, S_k\}$ be a complete Hall set of type $\sigma$ in a finite group $G$. Let $H$ be a subgroup in $G$ such that $\Sigma^g$ is reduced to $H$ for every $g \in G$. Is it true that $H$ is a $\sigma$-subnormal subgroup in $G$?*

While Problem 1 requires that each complete Hall set $\Sigma$ of type $\sigma$ in $G$ be reduced to a subgroup $H$ in $G$, Problem 2 deals only with complete Hall sets $\Sigma^g$ ($g \in G$) for some given complete Hall set $\Sigma$ in $G$. Therefore, the affirmative solution of Problem 2 leads to the solution to Problem 1.

In this article, Problems 1 and 2 are solved for a partition $\sigma = \{\{p\}, \{p\}'\}$, where $p$ is a prime. Our main goal is to prove the following theorem:

**Theorem 1.1.** *Let $p$ be a prime, $\sigma = \{\{p\}, \{p\}'\}$, and let $\Sigma$ be a complete Hall set of type $\sigma$ in a finite group $G$. If $H$ is a subgroup in $G$ such that $\Sigma^g$ is reduced to $H$ for every $g \in G$ then $H$ is $\sigma$-subnormal in $G$.*

The key to proving Theorem 1.1 is given by Lemma 2.4 establishing the structure of a minimal counterexample for every partition $\sigma$, together with Kazarin's Theorem (see [9]) which describes simple nonabelian groups containing a Hall $p'$-subgroup.

## 2. Definitions and Preliminary Results

The present work deals only with finite groups and uses the definitions and notations of [10]. As regards the terminology of the theory of $\sigma$-subnormal subgroups, the reader is referred to [7, 8].

We use the following notations:

If $\pi$ is a set of primes then $\mathrm{Hall}_\pi(G)$ is the set of all Hall $\pi$-subgroups in a group $G$.

If $\Sigma = \{S_1, S_2, \dots, S_k\}$ is a complete Hall set of type $\sigma$ in $G$ and $N$ is a normal subgroup in $G$ then $\Sigma N/N = \{S_1N/N, S_2N/N, \dots, S_kN/N\}$.

If $n$ is a natural then $\sigma(n) = \{\sigma_i \cap \pi(n) \mid i \in I, \sigma_i \cap \pi(n) \neq \varnothing\}$.

$\sigma(G) = \sigma(|G|)$.

The basic properties of $\sigma$-subgroups are given as some lemmas whose proofs are straightforward.

**Lemma 2.1.** *Let $H$ and $N$ be subgroups in $G$, where $N$ is normal in $G$. Then*
(1) *if $H$ $\sigma$-subnormal in $G$ then the subgroup $HN/N$ is $\sigma$-subnormal in $G/N$;*
(2) *if $N \subseteq H$ then $H$ is $\sigma$-subnormal in $G$ if and only if $H/N$ is $\sigma$-subnormal in $G/N$.*

**Lemma 2.2.** *Let $H$ and $K$ be subgroups in $G$, where $H$ is $\sigma$-subnormal in $G$. Then*
(1) *if $K \subseteq H$ and $K$ is $\sigma$-subnormal in $H$ then $K$ is $\sigma$-subnormal in $G$;*
(2) *$K \cap H$ is $\sigma$-subnormal in $K$;*
(3) *if $H \subseteq K$ then $H$ is $\sigma$-subnormal in $K$.*

Following [8], we say that a group $G$ is *$\sigma$-nilpotent* (or *$\sigma$-decomposable*) if $G$ is the direct product of some $\sigma$-primary groups; i.e., $G$ is representable as a direct product of its Hall $\sigma_i$-subgroups for some $i \in I$.

A straightforward check shows that the class $\mathfrak{N}_\sigma$ of all $\sigma$-nilpotent groups is a hereditary Fitting formation. This implies in particular that every group $G$ contains some least normal subgroup the quotient group by which is $\sigma$-nilpotent. This subgroup is denoted by $G^{\mathfrak{N}_\sigma}$ and called the *$\sigma$-nilpotent residual* (or the *$\mathfrak{N}_\sigma$-residual*) of $G$.

**Lemma 2.3.** *A subgroup $H$ is $\sigma$-subnormal in $G$ if one of the following is fulfilled:*
(1) *$G$ is $\sigma$-nilpotent;*
(2) *$G^{\mathfrak{N}_\sigma} \subseteq H$;*
(3) *$|\sigma(G)| = 1$.*

Suppose that $H$ is a subgroup in a $\sigma$-complete group $G$, while $\sigma(G) = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$, and $\Sigma = \{S_1, S_2, \dots, S_k\}$ is a complete Hall set of type $\sigma$ in $G$. We say that a pair $(G, H)$ is a *counterexample to Problem* 2 if for every $g \in G$ the complete Hall set $\Sigma^g$ is reduced to $H$ but the subgroup $H$ is not $\sigma$-subnormal in $G$. If, moreover, the pair $(G, H)$ is such that the sum $|G| + |H|$ is minimal; then we refer to the counterexample $(G, H)$ as a *minimal counterexample to Problem* 2.

**Lemma 2.4.** *Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes. If $(G, H)$ is a minimal counterexample to Problem 2 then $G$ and $H$ are simple nonabelian groups.*

Proof. Let $N$ be a minimal normal subgroup in $G$ and let $\Sigma = \{S_1, S_2, \dots, S_k\}$ be a complete Hall set of type $\sigma$ in $G$. By Lemma 2.3, $k = |\sigma(G)| \geq 2$. By hypothesis, $H \cap S_i^g$ is a Hall $\sigma_i$-subgroup in $H$ for all $g \in G$ and $i = 1, 2, \dots, k$. Since $N \trianglelefteq G$, we infer that $N \cap S_i^g \trianglelefteq S_i^g$ and $N \cap S_i^g$ is a Hall $\sigma_i$-subgroup in $N$. It is easy to conclude from this that $S_i^g \cap HN$ is a Hall $\sigma_i$-subgroup in $HN$. Now, since $S_i^g N/N \cap HN/N = \big(S_i^g \cap HN\big)N/N$; therefore, $S_i^g N/N \cap HN/N$ is a Hall $\sigma_i$-subgroup in $HN/N$. Thus, the complete Hall set $\Sigma^g N/N$ of type $\sigma$ in $G/N$ is reduced to $HN/N$ for every $g \in G$. Hence, by the minimality of the counterexample, $HN/N$ is $\sigma$-subnormal in $G/N$. But then $HN$ is a $\sigma$-subnormal subgroup in $G$ by Lemma 2.1. Therefore, $N$ is not in $H$; in particular, $\operatorname{Core}_G(H) = 1$. Clearly, $H \cap S_i^x$ is a Hall $\sigma_i$-subgroup in $H$ for all $x \in HN$, and $\big\{HN \cap S_1^x, HN \cap S_2^x, \dots, HN \cap S_k^x\big\}$ is a complete Hall set of type $\sigma$ in $HN$ which is reduced to $H$. If $|HN| < |G|$; then the subgroup $H$ is $\sigma$-subnormal in $HN$ by the minimality of the counterexample. Then $H$ is a $\sigma$-subnormal subgroup in $G$ by Lemma 2.2, which contradicts the assumption.

Thus, we assume in what follows that $G = HN$.

Suppose first that $N$ is an elementary abelian $p$-subgroup for some $p \in \pi(G)$. Then the index $|G : H|$ is a power of $p$. This and $k \geq 2$ imply that there exists $\sigma_i \in \sigma$ for which $\sigma_i \cap \pi(G) \in \sigma(G)$ and $p \notin \sigma_i$. Then $H$ includes each Hall $\sigma_i$-subgroup $S_i^g$ for every $g \in G$ and also $S_i^g \neq 1$. Hence, $\big\langle S_i^g \mid g \in G\big\rangle \subseteq H$. Since $1 \neq \big\langle S_i^g \mid g \in G\big\rangle \trianglelefteq G$; therefore, $\operatorname{Core}_G(H) \neq 1$. This is impossible.

Consequently, $N$ is a direct product of isomorphic simple nonabelian groups. Put $K = H \cap N$. Obviously, $K \trianglelefteq H$. Therefore, the complete Hall set $\Sigma^g$ of type $\sigma$ in $G$ is reduced to $K$ for every $g \in G$. Thus, $(G, K)$ satisfies the hypothesis of the lemma. If $K = H$ then $G = N$ is a simple nonabelian group. Hence, $K \neq H$. By the minimality of the counterexample, $K$ is $\sigma$-subnormal in $G$. But then, by Lemma 2.2, $K$ is $\sigma$-subnormal in $N$. Hence, there exists a chain of subgroups

$$K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_{n-1} \subseteq K_n = N$$

such that, for each $i = 1, 2, \dots, n$, either $K_{i-1}$ is normal in $K_i$ or $K_i / \operatorname{Core}_{K_i}(K_{i-1})$ is $\sigma$-primary.

Suppose that $K \neq 1$ and consider the three possible cases:

Case 1. Let $N$ be a simple nonabelian group. Then $K_{n-1}$ is not normal in $N$. Thus, $N/\operatorname{Core}_N(K_{n-1}) = N/1 = N$ is a $\sigma_i$-group for some $i \in \{1, \dots, k\}$. Since $G = HN$ and $k \geq 2$; it follows that $H$ contains all Hall $\sigma_j$-subgroups $S_j^g$ $(j \neq i)$ for all $g \in G$. Consequently $\operatorname{Core}_G(H) \neq 1$, which is impossible.

Case 2. Let $N = N_1 \times N_2$, where $N_1$ and $N_2$ are isomorphic simple groups. If $K_{n-1}$ is not normal in $N$ then $N/\operatorname{Core}_N(K_{n-1})$ is a $\sigma_i$-group for some $i = 1, 2, \dots, k$. Clearly, either $\operatorname{Core}_N(K_{n-1}) = 1$ or $\operatorname{Core}_N(K_{n-1}) \in \{N_1, N_2\}$. If $\operatorname{Core}_N(K_{n-1}) = 1$ then $N/\operatorname{Core}_N(K_{n-1}) = N$ is a $\sigma_i$-group. If $\operatorname{Core}_N(K_{n-1}) \in \{N_1, N_2\}$; then, due to the isomorphism $N/\operatorname{Core}_N(K_{n-1}) \simeq N_1$, the subgroup $N_1$ is a $\sigma_i$-group, which implies that $N$ is a $\sigma_i$-group too. Next, as in the case when $N$ is a simple nonabelian group, we get a contradiction to $\operatorname{Core}_G(H) \neq 1$.

If $K_{n-1}$ is normal in $N$ then $K_{n-1} \in \{N_1, N_2\}$, i.e., $K_{n-1}$ is a simple nonabelian group. If $K \subset K_{n-1}$; then, as in Case 1, we get a contradiction. Consequently, $K_{n-1} = K$. Then $K = H \cap N \trianglelefteq N$ and $K = H \cap N \trianglelefteq H$. Therefore, $K \trianglelefteq \langle N, H\rangle = HN = G$. Since $K \subseteq H$, this yields $\operatorname{Core}_G(H) \neq 1$. We get a contradiction once again.

Case 3. Suppose that $N = N_1 \times N_2 \times \dots \times N_t$, where $t \geq 3$ and $N_1$, $N_2$, $\dots$, $N_t$ are isomorphic simple groups. Inducting on $t$, as in Cases 1 and 2, we conclude that either $N$ is a $\sigma$-primary group or $\operatorname{Core}_G(H) \neq 1$. This contradicts the above.

Thus, $K = H \cap N = 1$, and so $G = N \rtimes H$ is the semidirect product of $N$ and $H$. If $S_i \in \Sigma$ then $S_i = \widetilde{N} \rtimes \widetilde{H}$, where $\widetilde{N} \in \mathrm{Hall}_{\sigma_i}(N)$ and $\widetilde{H} \in \mathrm{Hall}_{\sigma_i}(H)$. Consider the group $N \rtimes \widetilde{H}$ which includes $S_i$. By hypothesis, $H \cap S_i^x$ are Hall $\sigma_i$-subgroups in $H$ for all $x \in N\widetilde{H}$. Hence, $S_i^x \cap H \subseteq N\widetilde{H} \cap H = (N \cap H)\widetilde{H} = \widetilde{H}$. Consequently, $S_i^x \cap H = \widetilde{H}$, whence we infer that $\widetilde{H} \subseteq S_i^x$ for all $x \in N\widetilde{H}$. Hence, $\widetilde{H} \subseteq O_{\sigma_i}(N\widetilde{H})$. If $O_{\sigma_i}(N) \neq 1$ then $N$ is a $\sigma_i$-group. As shown above, this is impossible. Thus, $O_{\sigma_i}(N) = 1$, and the Hall $\sigma_i$-subgroup $\widetilde{H}$ in $H$ centralizes $N$. Since this holds for each $i \in \{1, \dots, k\}$; therefore, $G = N \times H$ and $H \trianglelefteq G$, which is impossible. Thus, $G$ is a simple nonabelian group.

Show that $H$ is simple. Suppose that $H$ contains a proper normal subgroup $L \neq 1$. Obviously, $(G, L)$ satisfies the hypothesis of the lemma. Since $|G| + |L| < |G| + |H|$, the subgroup $L$ is $\sigma$-subnormal in $G$. This means that there exists some chain of subgroups

$$L = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{n-1} \subseteq L_n = G$$

such that, for each $i = 1, 2, \dots, n$, either $L_{i-1}$ is normal in $L_i$ or $L_i / \mathrm{Core}_{L_i}(L_{i-1})$ is $\sigma$-primary. This and the simplicity of $G$ imply that $G$ is a $\sigma_i$-group for some $i \in \{1, \dots, k\}$. Since $k \geq 2$, this is impossible. Consequently, $H$ is a simple nonabelian group. The lemma is proved.

Remark. As follows from [3], the structure of a minimal counterexample to the Kegel–Wielandt conjecture is the same as in Lemma 2.4, i.e., $G$ and $H$ are simple nonabelian groups.

The simplicity of $G$ of the minimal counterexample to Problem 2 implies

**Lemma 2.5.** *Let $(G, H)$ be a minimal counterexample to Problem 2 and let $\sigma(G) = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$. Then $\pi(H) \nsubseteq \sigma_i$ for each $i \in \{1, 2, \dots, k\}$.*

Additional information about the structure of a minimal counterexample to Problem 2 is given by the following proposition, which is of interest in its own right. Note only that if $H$ is a subgroup of $G$ and $p$ is a prime; then, by [12], the notation $H \leq_p G$ means that $H$ is $p$-subnormal in $G$, i.e., for every Sylow $p$-subgroup $P$ in $G$, the intersection $P \cap H$ is a Sylow $p$-subgroup in $H$.

**Proposition 2.6.** *For every partition $\sigma$, any group $G$ in a minimal counterexample $(G, H)$ to Problem 2 cannot be an alternating group.*

Proof. Suppose that $G \simeq A_n$, $n \geq 5$. Since $H$ is a simple nonabelian group, $n \geq 6$. By [11], $G$ is $\sigma$-complete for a nonminimal partition $\sigma$ if and only if either $n = p$ is a prime or $n = 8$.

Let $n = 8$. Then $G$ has a Hall $\{2, 3\}$-subgroup and is $\sigma$-complete only for those nonminimal partitions $\sigma$ for which $\sigma(G) = \{\{2, 3\}, \{5\}, \{7\}\}$. Since $H$ is a simple nonabelian group, $\pi(H)$ contains at least one of the numbers 5 or 7. Let $5 \in \pi(H)$. Then $H \leq_5 G$ and, by [12, Theorem 1.4], $n = s \cdot 5^a > 5$, where $1 \leq s < 5$. Since $n = 8$, this is impossible. It is demonstrated similarly that the case of $7 \in \pi(H)$ is impossible either.

Let $n = 7$. Then $G$ is $\sigma$-complete only for those nonminimal partitions $\sigma$ for which $\sigma(G) = \{\{2, 3\}, \{5\}, \{7\}\}$ or $\sigma(G) = \{\{2, 3, 5\}, \{7\}\}$. The first case is excluded like for $n = 8$. In the second case, in view of Lemma 2.5, we have $7 \in \pi(H)$ and $H \leq_7 G$. Then, by Theorem 1.4 from [12], $7 = s \cdot 5^a > 7$, where $1 \leq s < 7$, which is impossible.

If $n = p \geq 11$ then $G$ is $\sigma$-complete only for those nonminimal partitions $\sigma$ for which $\sigma(G) = \{\pi((p-1)!), \{p\}\}$. Obviously, $p \in \pi(H)$ and $H \leq_p G$. By [12, Theorem 1.4], $p = s \cdot p^a > p$, where $1 \leq s < p$, which is impossible. The proposition is proved.

We will also need the following number-theoretic result from [13] establishing that if $a$ and $b$ are naturals such that $a \geq 2$, $b \geq 3$, and $(a, b) \neq (2, 6)$ then there is a prime $r$ that divides $a^b - 1$ but does not divide $a^l - 1$ for all $l = 1, 2, \dots, b-1$. Such $r$ is called *primitive* with respect to the pair $(a, b)$.

## 3. Proof of Theorem 1.1

Let $(G, H)$ be a minimal counterexample. Then, by Lemma 2.4, $G$ and $H$ are simple nonabelian groups. Since $G$ is $\sigma$-complete, $G$ has a Hall $p'$-subgroup $M$. By [9, Theorem 7], one of the following cases is possible:

(a) $G = A_p$ and $M \simeq A_{p-1}$;

(b) $G = PSL_n(q)$, where $q = r^m$, $m \geq 1$, $r$ is a prime, and $M$ is a parabolic subgroup in $G$; moreover, $|G : M| = (q^n - 1)/(q - 1) = p^k$ and $n$ is a prime;

(c) $G = PSL_2(11)$, $p = 11$, and $M \simeq A_5$;

(d) $G = M_{11}$, $p = 11$, and $M \simeq M_{10}$;

(e) $G = M_{23}$, $p = 23$, and $M \simeq M_{22}$.

By Proposition 2.6, the alternating group $A_p$ cannot be a counterexample to Problem 2.

Consider case (b). Suppose first that $n \geq 3$. If $(q, n) \neq (2, 6)$ then there is a prime $t$ primitive with respect to $(q, n)$. By Fermat's Theorem, $t \geq n + 1$, and so $t \geq 5$. Since $M$ is a parabolic subgroup, $(|M|, t) = 1$. Lemma 2.5 implies that $H \leq_t G$. This case is excluded by [12, Theorem 1.4]. Thus, $(q, n) = (2, 6)$ and $G = SL_6(2)$. Since 6 is not a prime, the group $G = SL_6(2)$ does not satisfy condition (b).

Thus, $G = PSL_2(q)$, where $q + 1 = p^k$. By [13, Lemma 1.2], only the following cases are possible.

Suppose first that $r = 2$. Then $2^m + 1 = p^k$. The two cases are possible:

(1) $k = 1$ and $p$ is a Fermat prime, where $m$ is a power of 2. Then $G = PSL_2(2^2) \simeq A_5$ for $m = 2$. This case was considered above. For $m = 2^2 = 4$, we obtain $p = 17$. By Lemma 2.5, $H \leq_t G$ where $t \geq 17$. This case is impossible since it is excluded by [12, Theorem 1.4].

(2) $m = 3$, $p = 3$, $k = 2$, and $G = PSL_2(9)$. Since $9 + 1 = 10$ is not a prime power, this case is impossible too.

Thus, $r$ is an odd prime and $r^m + 1 = 2^k$. In this case $m = 1$ and $r = 2^k - 1$ is a Mersenne prime. Since $G = PSL_2(r)$, we have $H \simeq A_5$ (if such a subgroup exists). Since $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_2 = \{2\}$, the group $A_5$ contains a subgroup of order 15, which is impossible.

The analysis of cases (c), (d), and (e) shows that they are impossible in view of Lemma 2.5 and [12, Theorem 1.4]. The theorem is proved.

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