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EQUATIONAL NOETHERICITY OF METABELIAN r-GROUPS

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Abstract: The author had earlier defined the concept of an r-group, generalizing the concept of a rigid (solvable) group. This article proves that every metabelian r-group is equationally Noetherian; i.e., each system of equations in finitely many variables with coefficients in the group is equivalent to some finite subsystem.

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1. Introduction

The author introduced the concept of a rigid (solvable) group more than a decade ago. Then he developed algebraic geometry and model theory for the class of rigid groups jointly with Myasnikov; see [1–8]. The recent article [9] defines some more general class of r-groups that includes, for instance, the solvable Baumslag–Solitar groups as well as studies the basic properties of r-groups. The author has also noted that many important properties of rigid groups are unlikely to carry over to r-groups. However, there is a hope for that in the metabelian case. Thus, the deeper study of metabelian r-groups has begun in [10], and the present article is a continuation. Algebraic geometry over rigid groups includes the fundamental result with a rather intricate proof: The equational Noethericity holds in an arbitrary rigid group [2]. Here we prove the following statement.

Theorem. *Every metabelian r-group is equationally Noetherian.*

Recall that a group is called *equationally Noetherian* whenever each system of equations over it is equivalent to some finite subsystem. The equational Noethericity of a group G is equivalent to the Noethericity of the Zariski topology on the affine space G^n for all n. The latter property is very important, and the above theorem makes it possible to develop algebraic geometry over metabelian r-groups. We should note that the equational Noethericity is sufficiently obvious for every finitely generated metabelian group G because then the coordinate group of the affine space G^n is a finitely generated metabelian group; consequently, it satisfies the maximality condition for normal subgroups, and so everything is straightforward. However, in case that the finite generation condition is dropped, the article [2] exhibits an example of a rather good metabelian group (length 2 nilpotent and torsion-free) which is not equationally Noetherian. In future, we intend to study algebraic geometry over the divisible metabelian r-groups that are mostly not finitely generated.

We prove the theorem along the lines of the proof of the author's theorem on the equational Noethericity of rigid groups. Many steps simplify because the group is metabelian, but some become more intricate since the group is not rigid in general.

2. Definitions and Auxiliary Statements

2.1. Assume that a group G has some normal series

$$G = G_1 > G_2 > \cdots > G_m > G_{m+1} = 1 \tag{1}$$

with abelian quotients G_i/G_{i+1}. The action of G on G_i by conjugation, $x \to x^g = g^{-1}xg$, determines on G_i/G_{i+1} the structure of a (right) module over the group ring $\mathbb{Z}[G/G_i]$. Denote by R_i the quotient

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ring of $\mathbb{Z}[G/G_i]$ by the annihilator of G_i/G_{i+1}, so that we can regard G_i/G_{i+1} as a right R_i-module. The group G is called an *r-group* whenever G_i/G_{i+1} is R_i-torsion-free and the canonical mapping $\mathbb{Z}[G/G_i] \to R_i$ is injective on G/G_i. As [9] shows, if series (1) exists then it is uniquely determined by G; the notation $G_i = \rho_i(G)$ was introduced for the terms of this r-series (rigid series). A subgroup of G is also an r-group; we obtain its r-series by intersecting with (1) and omitting repetitions. The ring R_i defined above is called the ring *associated* to the quotient G_i/G_{i+1} of (1). Note that R_i is a (left and right) Ore domain, and so it embeds into a skew field of fractions. The concept of a *divisible r-group G* was defined: every module G_i/G_{i+1} in G must be a divisible R_i-module and then we can regard the latter as a (right) vector space over the skew field of fractions of R_i.

2.2. The following construction appears in [10]. Given a pair (p, q) of primes with p dividing $q^n - 1$ for some positive integer n and a cardinal $\alpha \geq 1$, we can construct a length 2 solvable periodic r-group to be denoted by $E(p, q, \alpha)$. Let F_q stand for the field with q elements and $\overline{F}_q$, for its algebraic closure. In the multiplicative group $\overline{F}_q^*$ choose the cyclic subgroup A of order p, which is unique. Consider the subring generated by A in $\overline{F}_q$; actually, this is the subfield F_{q^n}. Take the vector space T over F_{q^n} with basis of cardinality α. Put $E(p, q, \alpha)$ equal to the group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$. This group is an r-group with the r-series

$$\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} > \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} > 1.$$

In this example the ring associated to the first quotient of the series equals F_p, and to the second, F_{q^n}.

We need also the two propositions that are extracted from [10].

Proposition 1. *Each metabelian r-group embeds into a divisible metabelian r-group.*

Proposition 2. *Up to isomorphism, each divisible metabelian r-group is one of the following groups:*
(1) *an abelian group of prime period p;*
(2) *the direct sum of several copies of the additive group $\mathbb{Q}$ of rational numbers;*
(3) *$E(p, q, \alpha)$;*
(4) $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, *where A is a nontrivial group of type* (2) *lying in the multiplicative group R^* of a commutative integral domain R and, furthermore, R is generated by A as a $\mathbb{Z}$-module, while T is a nontrivial vector space over the fraction field F of R.*

Let us state another available property. The first claim here is quite obvious and the second is proved in [11].

Proposition 3. *The class of equationally Noetherian groups is closed under subgroups and finite extensions.*

Recall that by an *equation* in $x_1, \ldots, x_n$ over a group G we usually mean an expression of the form $v = 1$, where v lies in the free product of G and the free group $\langle x_1, \ldots, x_n \rangle$. However, often it is convenient to consider a more general situation. Refer as the *group of equations* over G in $x_1, \ldots, x_n$ to an arbitrary group D generated by G and $x_1, \ldots, x_n$ provided that D is such that each mapping $(x_1, \ldots, x_n) \to G^n$ extends to a G-epimorphism $D \to G$. Then we can take the expressions $v = 1$ with $v \in D$ as equations over G.

2.3. Assume in this subsection that the group $G = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$ is of type (4) in Proposition 2. Note that G is a length 2 solvable r-group and a semidirect product of its subgroups $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$, which are identified respectively with A and the additive group of T. Take a basis $\{t_k \mid k \in K\}$ for T. Call the set of variables $X = X_1 \cup X_2$, where $X_1 = \{x_{11}, \ldots, x_{1n}\}$ and $X_2 = \{x_{21}, \ldots, x_{2n}\}$, *special* whenever the variables x_{1i} take values only in A, whenever the variables x_{2i}, only in T. The usual variables

$x_1, \ldots, x_n$ can be expressed in terms of the special ones as $x_1 = x_{11}x_{21}, \ldots, x_n = x_{1n}x_{2n}$. Conversely, we can understand the special variables as the usual ones satisfying the additional equations $[x_{1i}, a] = 1$ and $[x_{2i}, b] = 1$, where a and b are fixed elements subject to the conditions $1 \neq a \in A$ and $1 \neq b \in \rho_2(G)$. Let us construct the group of equations over G in the special variables X.

Consider the direct product $C = A \times \langle x_{11} \rangle \times \cdots \times \langle x_{1n} \rangle$ of A and the free abelian group with basis X_1, the ring $R(X_1)$ of Laurent polynomials in X_1 with coefficients in R, and the ring $F(X_1)$ of Laurent polynomials in X_1 with coefficients in the field F. Consider the $R(X_1)$-module

$$T' = \sum_K t_k \cdot F(X_1) + x_{21} \cdot R(X_1) + \cdots + x_{2n} \cdot R(X_1),$$

which amounts to the direct sum of the free $F(X_1)$-module with basis $\{t_k \mid k \in K\}$ and the free $R(X_1)$-module with basis X_2. Clearly, T lies in T' and the module T' is generated by $T \cup X_2$. Furthermore, put $D = \begin{pmatrix} C & 0 \\ T' & 1 \end{pmatrix} \geqslant G$. Identify the variable x_{1i} with the matrix $\begin{pmatrix} x_{1i} & 0 \\ 0 & 1 \end{pmatrix}$, and the variable x_{2i}, with the matrix $\begin{pmatrix} 1 & 0 \\ x_{2i} & 1 \end{pmatrix}$. With this convention, D is generated by G and X. Moreover, each mapping $(x_{11}, \ldots, x_{1n}) \to A^n$, $(x_{21}, \ldots, x_{2n}) \to T^n$ extends to a G-epimorphism $D \to G$. Thus, D is a group of equations over G in the special variables X.

Observe that the affine space A^n is endowed with the Zariski topology defined by the group equations of the form $x_{11}^{m_1} \ldots x_{1n}^{m_n} a = 1$ with coefficients a is A. We also consider ring equations of the form $f(X_1) = 0$, where $f \in R(X_1)$, seeking the values of the variables x_{1i} in A. Without loss of generality, we may assume that f lies in the ring of polynomials $F[X_1]$. Classical algebraic geometry shows that A is equationally Noetherian. Observe that each ring equation $f(X_1) = 0$ is realized as a group equation over G; for instance, as $\begin{pmatrix} 1 & 0 \\ t \cdot f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $0 \neq t \in T$. Thus, A^n is endowed with both the group Zariski topology and the ring Zariski topology, and the second is stronger in general.

3. Proof of the Theorem

By Proposition 1, each metabelian r-group embeds into a divisible group, and once we prove that every divisible metabelian r-group is equationally Noetherian, Proposition 3 will imply that so are its subgroups.

Thus, suppose that G is a divisible metabelian r-group. Proposition 2 describes such groups. Abelian groups are known to be equationally Noetherian. Moreover, basing on Proposition 3, we may assert that so are all almost abelian groups. Since the group $E(p, q, \alpha)$ is almost abelian, the only remaining nontrivial case is of the group G of type (4) in Proposition 2. Below we assume that and use the construction of Section 2 of the group of equations D in the special variables X. Since the usual variables can be expressed in terms of the special ones, it suffices to show that G is Noetherian with respect to the latter.

Assume on the contrary that G is not Noetherian with respect to the special variables. Then there exists a system equations

$$\{d_l = 1 \quad (l \in L), \tag{2}$$

where $d_l = \begin{pmatrix} f_l & 0 \\ v_l & 1 \end{pmatrix} \in D$, which is not equivalent to any finite subsystem. This system is equivalent to the union of the two systems

$$\{f_l(X_1) = 1 \quad (l \in L), \tag{3}$$

$$\{v_l(X) = 0 \quad (l \in L). \tag{4}$$

Since A is equationally Noetherian, system (3) is equivalent to some finite subsystem and it determines in A^n a subset S which is algebraic in the group Zariski topology, and so algebraic in the ring Zariski topology. This subset is nonempty; otherwise, the set of solutions to system (2) would be empty, and (2) would be equivalent to some finite subsystem.

We may now assert that system (4) under the condition $X_1 \in S$ is not equivalent to any finite subsystem. Suppose that we have found in S a proper nonempty algebraic subset P **in the ring Zariski topology** such that system (4) under the condition $X_1 \in P$ is not equivalent to any finite subsystem. Then we replace S with P. Since the ring Zariski topology on A^n is Noetherian, the process of similar replacements stops in finitely many steps. Thus, we may assume that there exists a nonempty algebraic subset $S \subseteq A^n$ in the ring Zariski topology such that system (4) under the condition $X_1 \in S$ is not equivalent to any finite subsystem, but for every proper nonempty algebraic subset $P \subset S$ in the ring Zariski topology system (4) under the condition $X_1 \in P$ is equivalent to some finite subsystem. Below we arrive at a contradiction with this assumption, and therefore establish the theorem.

Lemma 1. *Under the above assumption, S is an irreducible algebraic subset of A^n in the ring Zariski topology.*

PROOF. Suppose that S is the union of proper algebraic subsets S_1 and S_2. Then under the condition $X_1 \in S_i$ for $i = 1, 2$ system (4) is equivalent to some finite subsystem Σ_i. It is obvious that under the condition $X_1 \in S$ system (4) is equivalent to the subsystem $\Sigma_1 \cup \Sigma_2$; a contradiction. □

Denote by F_S the quotient ring $F(X_1)/\Theta(S)$, where $\Theta(S)$ is the annihilator of S in $F(X_1)$. Since S is an irreducible algebraic subset of A^n in the ring Zariski topology, F_S is an integral domain. Denote the image of $R(X_1)$ in F_S by $R_S = R(X_1)/(R(X_1) \cap \Theta(S))$. To avoid tricking notation, we denote the image of the set X_1 of variables by X_1 as well. Moreover, let C_S stand for the multiplicative subgroup of R_S^* generated by A and X_1. Consider the R_S-module

$$T_S = \sum_K t_k \cdot F_S + x_{21} \cdot R_S + \cdots + x_{2n} \cdot R_S,$$

which is the canonical image of the $R(X_1)$-module T', and the group $D_S = \begin{pmatrix} C_S & 0 \\ T_S & 1 \end{pmatrix}$.

The next lemma is easy.

Lemma 2. *The group D_S is a group of equations over G in the special variables X under the condition $X_1 \in S$, i.e., every mapping $X_1 \to S$, $X_2 \to T^n$, extends to a G-epimorphism $D_S \to G$.*

Resting on Lemma 2, we may assume that the left-hand sides of the equations in (4) lie in the R_S-module T_S. Denote by E the field of fractions of the ring R_S (or F_S); by $T_S \cdot E$, the natural extension of T_S to a vector space over the field E with basis $\{t_k \mid k \in K\} \cup X_2$, and by V the subspace in $T_S \cdot E$ generated by the left-hand sides v_l for $l \in L$ of the equations of (4).

Lemma 3. *We have*

$$\Big(\sum_K t_k \cdot E\Big) \cap V = 0.$$

PROOF. Assume on the contrary that $(\sum_K t_k \cdot E) \cap V \neq 0$. Then

$$\Big(\sum_K t_k \cdot R_S\Big) \cap \Big(\sum_L v_l \cdot R_S\Big) \neq 0.$$

Suppose for instance that

$$t_1 u_1 + \cdots + t_m u_m \in \sum_L v_l \cdot R_S,$$

where $0 \neq u_i \in R_S$. We infer that the equation $u_1(X_1) = 0$ under the condition $X_1 \in S$ is a corollary of some finite subsystem Σ_1 of (4). This equation selects in S a proper subset P. If it is empty then the set of solutions to (4) under the condition $X_1 \in S$ is empty and this system is equivalent to Σ_1, which contradicts the above assumption on S. Assume that P is nonempty. Then (4) under the condition $X_1 \in P$ is equivalent to some finite subsystem Σ_2. This implies that (4) under the condition $X_1 \in S$ is equivalent to $\Sigma_1 \cup \Sigma_2$; again we arrive at a contradiction. □

Lemma 3 implies that the projection of V to the space $x_{21} \cdot E + \cdots + x_{2n} \cdot E$ is an injective mapping; and, in particular, the space V is finite-dimensional. Take a basis $v_1, \dots, v_r$ for V. Refer as *elementary transformations* over R_S of the tuple $(v_1, \dots, v_r)$ to the following operations:

(1) the transposition $v_i \leftrightarrow v_j$;

(2) the replacement of v_i by $v_i\alpha + v_j\beta$, where $\alpha, \beta \in R_S$, $\alpha \neq 0$, $i \neq j$.

Using elementary transformations, we can obtain from the tuple $(v_1, \dots, v_r)$ a tuple $(w_1, \dots, w_r)$ such that the matrix of coefficients of the expansions of w_i with respect to $x_{21} \dots, x_{2n}$ modulo $\sum_K t_k \cdot E$, up to a permutation of its columns, is of the form

$$\begin{pmatrix} u & 0 & \dots & 0 & * & \dots & * \\ 0 & u & \dots & 0 & * & \dots & * \\ . & . & \dots & . & . & \dots & . \\ 0 & 0 & \dots & u & * & \dots & * \end{pmatrix}, \quad 0 \neq u = u(X_1) \in R_S.$$

By construction, the equations $w_1 = 0, \dots, w_r = 0$ under the condition $X_1 \in S$ are corollaries of the system Σ_1 consisting of the equations $v_1 = 0, \dots, v_r = 0$. Represent an arbitrary element v_l for $l \in L$ modulo $\sum_K t_k \cdot F_S$ as $x_{21}v_{l1} + \cdots + x_{2n}v_{ln}$, where $v_{l1}, \dots, v_{ln} \in R_S$. Since $\{w_1, \dots, w_r\}$ is a basis for V, we have $v_l u = w_1 v_{l1} + \cdots + w_r v_{lr}$. Suppose that

$$x_{11} = a_1, \dots, x_{1n} = a_n, x_{21} = h_1, \dots, x_{2n} = h_n,$$

where $(a_1, \dots, a_n) \in S$ and $h_i \in T$, is a solution to the system Σ_1 and $u(a_1, \dots, a_n) \neq 0$. Then it is also a solution to every equation $v_l(X) = 0$ of (4). Therefore, (4) under the conditions $X_1 \in S$ and $u(X_1) \neq 0$ is equivalent to the system Σ_1. The equation $u(X_1) = 0$ selects in S a proper subset P. If it is empty then (4) under the condition $X_1 \in S$ is equivalent to Σ_1. If it is nonempty then by assumption (4) under the condition $X_1 \in P$ is equivalent to some finite subsystem Σ_2. We infer that (4) under the condition $X_1 \in S$ is equivalent to its finite subsystem $\Sigma_1 \cup \Sigma_2$; this is a contradiction.

The proof of the theorem is complete. $\square$

References

1. Romanovskii N. S., “Divisible rigid groups,” Algebra and Logic, vol. 47, no. 6, 426–434 (2008).
2. Romanovskii N. S., “Equational Noetherianness of rigid soluble groups,” Algebra and Logic, vol. 48, no. 2, 147–160 (2009).
3. Romanovskii N. S., “Irreducible algebraic sets over divisible decomposed rigid groups,” Algebra and Logic, vol. 48, no. 6, 449–464 (2009).
4. Myasnikov A. and Romanovskii N., “Krull dimension of solvable groups,” J. Algebra, vol. 324, no. 10, 2814–2831 (2010).
5. Romanovskiy N. S., “Hilbert’s Nullstellensatz in algebraic geometry over rigid soluble groups,” Izv. Math., vol. 79, no. 5, 1051–1063 (2015).
6. Romanovskii N. S., “Divisible rigid groups. Algebraic closedness and elementary theory,” Algebra and Logic, vol. 56, no. 5, 395–408 (2017).
7. Myasnikov A. G. and Romanovskii N. S., “Divisible rigid groups. II. Stability, saturation, and elementary submodels,” Algebra and Logic, vol. 57, no. 1, 29–38 (2018).
8. Romanovskii N. S., “Divisible rigid groups. III. Homogeneity and quantifier elimination,” Algebra and Logic, vol. 57, no. 6, 478–489 (2018).
9. Romanovskii N. S., “Generalized rigid groups: Definitions, basic properties, and problems,” Sib. Math. J., vol. 59, no. 4, 705–709 (2018).
10. Romanovskii N. S., “Generalized rigid metabelian groups,” Sib. Math. J., vol. 60, no. 1, 148–152 (2019).
11. Baumslag G., Myasnikov A., and Roman’kov V., “Two theorems about equationally Noetherian groups,” J. Algebra, vol. 194, no. 2, 654–664 (1997).

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