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CONSTRUCTION AND APPLICATIONS OF AN ADDITIVE BASIS FOR THE RELATIVELY FREE ASSOCIATIVE ALGEBRA WITH THE LIE NILPOTENCY IDENTITY OF DEGREE 5

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Abstract: We construct an additive basis for the relatively free associative algebra $F^{(5)}(K)$ with the Lie nilpotency identity of degree 5 over an infinite domain K containing $\frac{1}{6}$. We prove that approximately half of the elements in $F^{(5)}(K)$ are central. We also prove that the additive group of $F^{(5)}(\mathbb{Z})$ lacks the elements of simple degree ≥ 5. We find an asymptotic estimation of the codimension of T-ideal, which is generated by the commutator $[x_1, x_2, \dots, x_5]$ of degree 5.

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Introduction

We consider only *associative* algebras over an infinite domain K that contains $\frac{1}{6}$. In what follows, we use the notations:

$F = F_{\mathrm{Ass}}[X]$ is the free associative K-algebra over an infinite countable set $X = \{x_1, x_2, \dots\}$ of free generators;

$X_n = \{x_1, \dots, x_n\}$;

$[x_1, \dots, x_n]$ is a right-normed commutator of degree $n \geq 2$, i.e., $[x_1, x_2] = x_1x_2 - x_2x_1$ and $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$;

$\mathrm{LN}(n) : [x_1, \dots, x_n] = 0$ is the identity of left nilpotency of degree n;

$T^{(n)}$ and $V^{(n)}$ are the T-ideal and T-space of F, which are generated by the commutator $[x_1, \dots, x_n]$; if $S \subset F$ then $(S)^T$ and $(S)^V$ denote the T-ideal and T-space that are generated by S;

$F^{(n)} = F/T^{(n)}$ is the relatively free algebra of countable rank with the identity $\mathrm{LN}(n)$;

$P_n(A)$ is the space of multilinear polynomials over X_n with respect to a free algebra A;

$Z^*(A)$ is the kernel of A (the greatest ideal of A which lies in the center $Z(A)$ of A).

The study of algebras with the Lie nilpotency identity was initiated in [1–3].

The codimensions of $T^{(3)}$ and $T^{(4)}$ are known (see [4–6]):

$$c_n(T^{(3)}) = 2^{n-1}, \quad c_n(T^{(4)}) = 2^{n-1} + 2\binom{n}{4} + 2\binom{n}{3}.$$

Moreover, in [6] some algebra was distinguished that generates the variety of Lie nilpotent algebras of degree 4.

The state of the art in the theory of Lie nilpotent algebras is rather well-detailed in the introductions of [7–10]. The proper central polynomials for the algebras $F^{(5)}$ and $F^{(6)}$ over a field of characteristic 0 were studied in [7, 8]. Furthermore, some hypotheses were stated in [7] about the center and the kernel that are confirmed in particular in [8]. The model algebra $E^{(2)}$ and some auxiliary superalgebras play an important role in these articles (see [7, 8]).

In [9], the case was elaborated of relatively free algebras with the identity $\mathrm{LN}(n)$ in two and three generators over a ring K.

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The plan of construction of an additive basis for $F^{(5)}$ was outlined in [10]. It was supposed to consider a sequence of T-ideals $T^{(4)} \supset H \supset H'$ of $F^{(5)}$, where H and H' are the T-ideals generated by the Hall element $[[x_1, x_2]^2, x_3]$ and the weak Hall element $[[x_1, x_2]^2, x_2]$, respectively. Some additive basis for the algebra modulo H' was presented and it was proved that H' coincides with the ideal of identities of the model algebra $E^{(2)}$ and with the kernel $Z^*(F^{(5)})$ of $F^{(5)}$.

In this article we realize the above plan of constructing an additive basis for $F^{(5)}$ and show some applications of the so-constructed basis that were announced in [10]. The article consists of five sections. In § 1 some available results we will need are contained. In § 2 we construct an additive basis for the ideal H'. In § 3 we obtain an asymptotic estimation of the codimension of the T-ideal $T^{(5)}$; i.e.,

$$c_n(T^{(5)}) \approx n^2 \cdot 2^{n-2}, \quad \text{i.e.,} \ \lim_{n\to\infty} \frac{c_n}{n^2 \cdot 2^{n-2}} = 1.$$

Note that it is impossible to obtain this result by the methods of [8], since [8] used the technique that is based on the application of either the skew-symmetric elements or the superalgebras generated by one odd element.

In § 4 the centers of the free Lie nilpotent algebras $F^{(3)}, F^{(4)}$, and the free metabelian algebra $F_{(2)}$ are described. It is proved that

$$c_n(F^{(3)}) = 2\xi_n(F^{(3)}), \quad n \geq 2,$$

where $\xi_n(F^{(3)}) = \dim_K(Z(F^{(3)}) \cap P_n(F^{(3)}))$ and $c_n(F^{(3)}) = \dim_K P_n(F^{(3)})$.

Moreover, we prove some asymptotic relations $\xi_n(F^{(4)})/c_n(F^{(4)}) \to \frac{1}{2}$ and $\xi_n(F_{(2)})/c_n(F_{(2)}) \to 0$ as $n \to \infty$.

In § 5 we describe the center of $F^{(5)}$ and prove that if the main field K is of characteristic 0 then $Z(F^{(5)})$ as a T-space is generated by the following elements:

$$[x_1, x_2, x_3, x_4], \quad [[x_1, x_2, x_3] \cdot x_4, x_5], \quad [[x_1, x_2]^2, x_2].$$

If $p = \mathrm{char}(K) \geq 5$ then the center $Z(F^{(5)})$ is generated by

$$x^p, \quad [x_1, x_2, x_3, x_4], \quad [[x_1, x_2, x_3] \cdot x_4, x_5], \quad [[x_1, x_2]^2, x_2].$$

In the center, some essential part of an additive basis was distinguished, and it was proved that

$$\xi_n(F^{(5)})/c_n(F^{(5)}) \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Thus, about half of the elements in $F^{(n)}$ ($n = 3, 4, 5$) are central. Note that even for $F^{(3)}$ this result is new.

Note that an additive basis for $Z(F^{(5)})$ is unknown.

§ 1. The Main Notions and Available Results

1.1. Proper polynomials. Let $A^{\#}$ be an algebra obtained from an algebra A by externally adjoining the unity. A variety $\mathfrak{M}$ is *unitarily closed* provided that $\mathfrak{M}$ contains $A^{\#}$ for all $A \in \mathfrak{M}$.

Let $F = F_{\mathfrak{M}}[X]$ be the relatively free algebra of countable rank of an unitarily closed variety $\mathfrak{M}$. The set of free generators $X = \{x_1, \dots, x_n, \dots\}$ is assumed to be ordered by increasing indices.

A subalgebra of F generated by the Lie monomials (the commutators in generators) of degree ≥ 2 is the *subalgebra of proper polynomials.*

A variety is unitarily closed if and only if it can be defined by some set of proper identities (see [11, 12]).

Let $\Gamma_n(F)$ be the space of the proper multilinear polynomials of F which depend on the variables in X_n. We write $\Gamma_n(\mathfrak{M})$ instead of $\Gamma_n(F_{\mathfrak{M}})$ as well.

1.2. Codimensions of some *T*-ideals. Together with $\Gamma_n(F)$ consider the space $P_n(F)$ generated by the multilinear polynomials in X_n.

If F is a free associative algebra $\mathrm{Ass}[X]$ and T is its T-ideal (or the verbal ideal) then we put $T_n = P_n \cap T$. The sequence of codimensions $c_n(T)$ is an important numerical characteristic of the T-ideal T. It is known (see [4–6]) that

$$c_n(T^{(3)}) = 2^{n-1}, \quad c_n(T^{(4)}) = 2^{n-1} + 2\binom{n}{4} + 2\binom{n}{3}.$$

In [13], the codimensions were found of the ideal of identities $T^0_{(2)}$ of the variety of metabelian associative algebras:

$$c_n\big(T^0_{(2)}\big) = (n-1)2^{n-1} + 2\binom{n}{4} - \binom{n}{2} + 1, \quad n \geq 4.$$

There were also pointed out the codimensions of the ideal of identities $T_{(2)}$ of the variety of metabelian alternative algebras:

$$c_n(T_{(2)}) = (n^2 - n - 1)2^{n-4} + 2\binom{n}{4} - 2\binom{n}{3} + \binom{n}{2} - 2\binom{n}{1} + 1, \quad n \geq 5.$$

1.3. Some available results.

Latyshev's Lemma [3]. $[x, y, z, a][a, t] = 0$ *in* $F^{(5)}$.

Volichenko's Lemma [5]. $(T^{(3)})^2 \subseteq T^{(5)}$.

Denote by $x \circ y = xy + yx$ the Jordan product of x and y. Recall that $(a, b, c)^+ = (a \circ b) \circ c - a \circ (b \circ c)$ stands for the Jordan associator of a, b, and c.

Lemma 1.1 [7]. *In* $F^{(5)}$*, the Hall polynomials possess the properties* $h \in Z(F^{(5)})$ *and* $h' \in Z^*(F^{(5)})$.

Lemma 1.2 [8]. *In* $F^{(5)}$*, the properties hold*:
(a) $[x, a, a, y][a, z] \neq 0$ *is skew-symmetric in* x, y, *and* z;
(b) $(u, v, t)^+ = 0$ *if two elements in* u, v, *and* t *are commutators.*

1.4. Auxiliary results. Recall some notations and results that were proved in [10]:
$h = [[x_1, x_2]^2, x_3]$ and $h' = [[x_1, x_2]^2, x_2]$ are Hall polynomials;
$\mathfrak{H} = \mathrm{var}\langle \mathrm{LN}(5), h\rangle$ and $\mathfrak{E} = \mathrm{var}\langle \mathrm{LN}(5), h'\rangle$ are Hall varieties;
$A = F_{\mathfrak{E}}[X]$ is the free algebra of the variety $\mathfrak{E} = \mathrm{var}(E^{(2)})$.

Agree that we symmetrize the variables that are marked with a bar; i.e., if $f(x_1, x_2, x_3)$ is a multilinear polynomial then we put

$$f(\overline{x_1}, \overline{x_2}, \overline{x_3}) = \sum_{\sigma \in S(3)} f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}),$$

where $S(3)$ is the symmetric group of degree 3.

Also, we put

$$\varphi(a, x, y, b) = [a, \bar{x}] \circ [\bar{y}, b], \quad \psi(a, b, x, y, z) = [\varphi(a, b, x, y), z];$$

$$\Phi = \varphi(X, X, X, X), \quad \Psi = \psi(X, X, X, X), \quad U = [X, X].$$

Denote by H_i the linear spans of proper polynomials:

$$H_1 = \sum_{m \geq 0} K \cdot [\Phi, X]U^m, \quad H_2 = \sum_{m \geq 0} K \cdot \Phi U^m,$$

$$H_3 = \sum_{m \geq 0} K \cdot V^{(3)}U^m, \quad H_4 = \sum_{m \geq 0} K \cdot V^{(4)}U^m.$$

It was proved in [10] that an additive basis $\Gamma_n(A) \cap T^{(4)}$ consists of the following elements:

(1) the right ψ-words; i.e., the elements that lie in $[\Phi, X]U^m$;
(2) the right φ-words; i.e., the elements that lie in ΦU^m;
(3) the right η-words; i.e., the elements that lie in $V^{(3)}U^m$;
(4) the right $V^{(4)}$-words; i.e., the elements that lie in $V^{(4)}U^m$.

Moreover, it is proved that the elements of types (1) and (4) are central. The remaining basis elements are linearly independent modulo the center. Below, we also need the eight remarks that are given in [10] under the same numbers.

Remark 1. $\dim_K(\Gamma_4(\mathfrak{H}) \cap T^{(4)}) = 6$.

Remark 2. $\dim_K(\Gamma_5(\mathfrak{H}) \cap T^{(4)}) = 10$. Furthermore, the following 10 elements are linearly independent modulo $Z(A)$:

$$a_{12} = [x_3\overline{x_1}x_4][\overline{x_2}x_5], \quad a_{13} = [x_2\overline{x_1}x_4][\overline{x_3}x_5],$$

$$a_{14} = [x_2\overline{x_1}x_3][\overline{x_4}x_5], \quad a_{15} = [x_2\overline{x_1}x_3][\overline{x_5}x_4],$$

$$b_{12} = [x_3\overline{x_1}x_5][\overline{x_2}x_4], \quad b_{13} = [x_2\overline{x_1}x_5][\overline{x_3}x_4],$$

$$b_{14} = [x_2\overline{x_1}x_5][\overline{x_4}x_3], \quad b_{15} = [x_2\overline{x_1}x_4][\overline{x_5}x_3],$$

$$c_{12} = [x_4\overline{x_1}x_3][\overline{x_2}x_5], \quad c_{13} = [x_4\overline{x_1}x_2][\overline{x_3}x_5].$$

Remark 3. $\dim_K(\Gamma_5(\mathfrak{C}) \cap H_1) = 5$.

Remark 4. $\dim_K(\Gamma_4(\mathfrak{C}) \cap K \cdot \Phi) = 2$.

Remark 5. $\dim_K(\Gamma_{2m} \cap H_2) = (2m-3)m$, $m \geq 3$.

Remark 6. $\dim_K(\Gamma_{2m+3} \cap \Psi) = 2\binom{2m}{1} + \binom{2m}{2} = 2m^2 + 3m$, $m \geq 2$.

Remark 7. Let $H_{1,3} = H_1 + H_3$. Then $\dim(\Gamma_{2m+1} \cap H_{1,3}) = 4m^2 - 1$, $m \geq 3$.

Remark 8. Let $H_{2,4} = H_2 + H_4$. Then $\dim(\Gamma_{2m} \cap H_{2,4}) = 4m^2 - 4m$, $m \geq 3$.

§ 2. The T-Ideal H' of the Weak Hall Elements in $F^{(5)}$

2.1. Preliminary lemmas. A triple of elements a, b, and c is *J-associative* provided that all Jordan associators in a, b, and c are zero. A triple of sets A, B, and C is *J-associative* provided that all triples of the shape a, b, and c, where $a \in A$, $b \in B$, and $c \in C$, are J-associative.

Lemma 2.1. *The triples A, $V^{(2)}$, $V^{(2)}$ and A, $V^{(2)}, V^{(2)} \circ V^{(2)}$ are J-associative in $A = F^{(5)}$.*

Proof. The first assertion follows from Lemma 1.2(b). Using the identity $(a, b, c)^+ = [b, [a, c]]$, we get

$$(A, V^{(2)}, V^{(2)} \circ V^{(2)})^+ = [V^{(2)}, [A, V^{(2)} \circ V^{(2)}]] \subseteq [V^{(2)}, V^{(3)} \circ V^{(2)}]$$
$$\subseteq [V^{(2)}, V^{(3)}] \circ V^{(2)} + V^{(3)} \circ [V^{(2)}, V^{(2)}] = 0;$$
$$(V^{(2)}, A, V^{(2)} \circ V^{(2)})^+ = [A, [V^{(2)}, V^{(2)} \circ V^{(2)}]] \subseteq [A, V^{(4)} \circ V^{(2)}]$$
$$\subseteq V^{(4)} \circ [A, V^{(2)}] + [A, V^{(4)}] \circ V^{(2)} = 0.$$

The next two lemmas hold in every associative algebra.

Lemma 2.2. $[[\bar{a}p][\bar{b}q], \bar{c}] = [[\bar{p}a][\bar{q}b], \bar{c}] + [[\bar{p}a][\bar{b}c], \bar{q}] + [[\bar{a}b][\bar{p}c], \bar{q}]$ *in F.*

PROOF. Develop every summand, marking with the same indices the equal summands:

$$[[\bar{a}p][\bar{b}q],\bar{c}] = [[ap][bq],c]_1 + [[bp][cq],a]_2 + [[cp][aq],b]_3$$
$$+[[bp][aq],c]_4 + [[cp][bq],a]_5 + [[ap][cq],b]_6,$$
$$[[\bar{p}a][\bar{q}b],\bar{c}] = [[pa][qb],c]_1 + [[qa][cb],p]_7 + [[ca][pb],q]_8$$
$$+[[qa][pb],c]_4 + [[ca][qb],p]_9 + [[pa][cb],q]_{10},$$
$$[[\bar{p}a][\bar{b}c],\bar{q}] = [[pa][bc],q]_{10} + [[ba][qc],p]_{11} + [[qa][pc],b]_3$$
$$+[[ba][pc],q]_{12} + [[qa][bc],p]_7 + [[pa][qc],b]_6,$$
$$[[\bar{a}b][\bar{p}c],\bar{q}] = [[ab][pc],q]_{12} + [[pb][qc],a]_2 + [[qb][ac],p]_9$$
$$+[[pb][ac],q]_8 + [[qb][pc],a]_5 + [[ab][qc],p]_{11}.$$

Lemma 2.3. *Let $f(p,q,r|a,b,c) = [\bar{p}a]\circ[\bar{q}b]\circ[\bar{r}c]$. Then in F*
(a) *$f(p,q,r|a,b,c)$ is symmetric in each of the sets $\{p,q,r\}$ and $\{a,b,c\}$;*
(b) *$f(p,q,r|a,b,c) + f(a,b,c|p,q,r) = 0$;*
(c) *$f(p,b,c|a,q,r) + f(p,c,a|b,q,r) + f(p,a,b|c,q,r) + f(a,b,c|p,q,r) = 0$.*

PROOF. Write the left-hand side of (c), and mark the opposite summands by the corresponding indices from 1 to 12 among the 24 summands:

$$f(p,b,c|a,q,r) + f(p,c,a|b,q,r) + f(p,a,b|c,q,r) + f(a,b,c|p,q,r)$$
$$= [\bar{p}a]\circ[\bar{b}q]\circ[\bar{c}r] + [\bar{p}b]\circ[\bar{a}q]\circ[\bar{c}r] + [\bar{p}c]\circ[\bar{a}q]\circ[\bar{b}r] + [\bar{a}p]\circ[\bar{b}q]\circ[\bar{c}r]$$
$$= [pa]\circ[bq]\circ[cr]_1 + [ba]\circ[cq]\circ[pr]_2 + [ca]\circ[pq]\circ[br]_3$$
$$+[ba]\circ[pq]\circ[cr]_4 + [ca]\circ[bq]\circ[pr]_5 + [pa]\circ[cq]\circ[br]_6$$
$$+[pb]\circ[aq]\circ[cr]_7 + [ab]\circ[cq]\circ[pr]_2 + [cb]\circ[pq]\circ[ar]_8$$
$$+[ab]\circ[pq]\circ[cr]_4 + [cb]\circ[aq]\circ[pr]_9 + [pb]\circ[cq]\circ[ar]_{10}$$
$$+[pc]\circ[aq]\circ[br]_{11} + [ac]\circ[bq]\circ[pr]_5 + [bc]\circ[pq]\circ[ar]_8$$
$$+[ac]\circ[pq]\circ[br]_3 + [bc]\circ[aq]\circ[pr]_9 + [pc]\circ[bq]\circ[ar]_{12}$$
$$+[ap]\circ[bq]\circ[cr]_1 + [bp]\circ[cq]\circ[ar]_{10} + [cp]\circ[aq]\circ[br]_{11}$$
$$+[bp]\circ[aq]\circ[cr]_7 + [cp]\circ[bq]\circ[ar]_{12} + [ap]\circ[cq]\circ[br]_6 = 0.$$

2.2. An additive basis for the T-ideal H' generated by a weak Hall element.

Proposition 2.1. *The space of the proper polynomials that belong to the ideal H' of $A = F^{(5)}$ has an additive basis from the following right elements f and g:*
(a) *if $X_5 = \{a,b,c,p,q\}$ then there are 7 elements $g(a,b,c,p,q) = [[\bar{a}p][\bar{b}q],\bar{c}]$, where $\{p,q\} \not\subset \{x_1,x_2,x_3\}$;*
(b) *if $X_6 = \{a,b,c,p,q,r\}$ then there are 5 elements*

$$f(a,b,c,p,q,r) = [\bar{a}p]\circ[\bar{b}q]\circ[\bar{c}r],$$

where $r = x_6$, $\{p,q\} \not\subset \{x_1,x_2,x_3\}$, $\{p,q\} \neq \{x_1,x_5\}$, $\{p,q\} \neq \{x_2,x_5\}$.

PROOF. We show firstly that the full linearization of the polynomial $[a,b]^3$ is a derivation in all variables. Consider $f(x,y,z) = [a,x]\circ[a,y]\circ[a,z]$. This element is symmetric in x, y, and z by Lemma 1.2(b); furthermore, $f(V^{(2)},y,z) = 0$ by Volichenko's Lemma, whence f is a Jordan derivation in all variables. Linearize this element by a:

$$f(x,y,z,b) = [b,x]\circ[a,y]\circ[a,z] + [a,x]\circ[b,y]\circ[a,z] + [a,x]\circ[a,y]\circ[b,z].$$

Clearly, $f(x,y,z,V^{(2)}) = 0$. Now, $f(x,y,z,b^2) = f(x,y,z,b)\circ b$ by Lemma 2.1.

Thus, we proved that the full linearization of $[a,b]^3$, i.e. the element

$$f(x_1,x_2,x_3,y_1,y_2,y_3)=\sum_{\sigma\in S_3}[x_{1\sigma},y_1]\circ[x_{2\sigma},y_2]\circ[x_{3\sigma},y_3],$$

is a derivation.

Show that the linearization $h'\Delta$ (see [12]) of h' is a derivation in all variables. Put $g(y)=[[a,y]\circ[a,b],a]$. We have

$$g(y^2)-g(y)\circ y=[[a,y^2]\circ[a,b],a]-g(y)\circ y$$
$$=[[a,y]\circ[a,b]\circ y,a]-g(y)\circ y=[a,y]\circ[a,b]\circ[y,a]=-2[a,y]^2\circ[a,b].$$

Consider the element

$$[[y^2,b]\circ[z,b],t]-y\circ[[y,b]\circ[z,b],t]$$
$$=[y\circ[y,b]\circ[z,b],t]-y\circ[[y,b]\circ[z,b],t]=[y,t]\circ[y,b]\circ[z,b].$$

Hence, we have to verify that $[yt]\circ[yb]\circ[zb]=0$. Let $v=[yt]$ and $g'(b)=v\circ([yb]\circ[zc]+[yc]\circ[zb])$. By Lemma 2.1 we have

$$g'(b^2)-g'(b)\circ b=v\circ([y,b^2]\circ[z,c]+[y,c]\circ[z,b^2])-g'(b)\circ b=0.$$

Then

$$g(x_1,x_2,x_3,y_1,y_2)=\sum_{\sigma\in S_3,\tau\in S_2}[[x_{1\sigma},y_{1\tau}]\circ[x_{2\sigma},y_{2\tau}],x_{3\sigma}]$$

is a Jordan derivation.

If $y_1\in V^{(2)}$ then $g(x_1,x_2,x_3,y_1,y_2)=0$ by Latyshev's Lemma. Assume that $x_1\in V^{(2)}$. By analogy,

$$[[x_1c]\circ[bc],b]+[[bc]^2,x_1]=[[x_1c]\circ[bc],b]=[[x_1c]b]\circ[bc]=0,$$

whence $g(x_1,x_2,x_3,y_1,y_2)$ is zero on the commutators. Therefore, the ideal H' of A is generated by the elements of the shape

$$g(x_1,x_2,x_3,p,q)=[[\overline{x_1}p][\overline{x_2}q],\overline{x_3}].$$

By Lemma 2.2, H' is generated by the elements of the shape

$$g(a,b,c,p,q)=[[\bar{a}p][\bar{b}q],\bar{c}],$$

where the pair $\{p,q\}$ satisfies the conditions $\{p,q\}\neq\{x_1,x_2\},\{x_2,x_3\},\{x_3,x_1\}$.

Prove that these elements are linearly independent. Assume that

$$\alpha_1[[\bar{q}p][\bar{b}a],\bar{c}]+\alpha_2[[\bar{a}p][\bar{q}b],\bar{c}]+\alpha_3[[\bar{a}p][\bar{b}c],\bar{q}]$$
$$+\beta_1[[\bar{p}q][\bar{b}a],\bar{c}]+\beta_2[[\bar{a}q][\bar{p}b],\bar{c}]+\beta_3[[\bar{a}q][\bar{b}c],\bar{p}]+\gamma[[\bar{a}p][\bar{b}q],\bar{c}]=0.$$

By the Poincaré–Birkhoff–Witt Theorem (shortly, the PBW Theorem) [14] the following equality should hold between the elements that contain the commutator $[ap]$:

$$\alpha_2[[\bar{a}p][\bar{q}b],\bar{c}]+\alpha_3[[\bar{a}p][\bar{b}c],\bar{q}]+\beta_1[[\bar{p}q][\bar{b}a],\bar{c}]+\gamma[[\bar{a}p][\bar{b}q],\bar{c}]=0.$$

Canceling by this commutator in the free associative algebra, we get

$$\alpha_2[\bar{q}b\bar{c}]+\alpha_3[\bar{b}c\bar{q}]-\beta_1[\bar{b}q\bar{c}]+\gamma[\bar{b}q\bar{c}]=0.$$

Combining the similar terms, we have

$$(-\alpha_2+\alpha_3)[bcq]+(\alpha_2+\beta_1-\gamma)[qbc]+(-\alpha_3-\beta_1+\gamma)[cqb]=0.$$

Applying the Jacobi identity, we obtain

$$2\alpha_2 - \alpha_3 + \beta_1 - \gamma = 0, \quad \alpha_2 - 2\alpha_3 - \beta_1 + \gamma = 0.$$

Arguing analogously, distinguish the terms that contain $[aq]$ as a factor:

$$-2\alpha_1 - \beta_2 - \beta_3 + \gamma = 0, \quad -\alpha_1 - 2\beta_2 + \beta_3 + \gamma = 0.$$

Similarly, choose the terms that contain $[ab]$:

$$\alpha_1 - \alpha_2 - 2\beta_1 + 2\beta_2 = 0, \quad -\alpha_1 + \alpha_2 - \beta_1 + \beta_2 = 0.$$

Finally, distinguish the terms that contain $[ac]$:

$$\alpha_1 - \alpha_3 - 2\beta_1 + 2\beta_3 = 0, \quad -\alpha_1 + \alpha_3 - \beta_1 + \beta_3 = 0.$$

The system of the first seven equations has nonzero determinant; therefore, it possesses only the zero solution. Hence, $\dim_K(\Gamma_5(A) \cap H') = 7$.

Which elements of the form $f(X')$ over $X' = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ are linearly independent? There are only $\binom{6}{3} = 20$ of these elements. Since

$$f(x_1, x_2, x_3, y_1, y_2, y_3) = -f(y_1, y_2, y_3, x_1, x_2, x_3),$$

we may assume that y_3 is the greatest variable. By Lemma 2.3, $f(x_1, x_2, x_3, y_1, y_2, y_3)$ is linearly expressible by the five elements $f(p, q, a, b, c, r)$ such that

$$\{b, c\} \not\subset \{x_1, x_2, x_3\}, \quad \{b, c\} \neq \{x_1, y_2\}, \quad \{b, c\} \neq \{y_1, y_2\}.$$

Call such elements *f-right*.

Thus, assume that there exists some relation

$$\alpha_2 f(,,,b,q,r) + \alpha_3 f(,,,c,q,r) + \beta_1 f(,,,a,p,r) + \beta_2 f(,,,b,p,r) + \beta_3 f(,,,c,p,r) = 0$$

among the f-right elements. This means that in $A = F^{(5)}$ we have

$$\alpha_1[\bar{p}b] \circ [\bar{a}q] \circ [\bar{c}r] + \alpha_2[\bar{p}c] \circ [\bar{a}q] \circ [\bar{b}r]$$
$$+\beta_1[\bar{q}a] \circ [\bar{b}p] \circ [\bar{c}r] + \beta_2[\bar{q}b] \circ [\bar{a}p] \circ [\bar{c}r] + \beta_3[\bar{q}c] \circ [\bar{a}p] \circ [\bar{b}r] = 0. \tag{1}$$

Note firstly that the commutator $[x_1, \ldots, x_i, a]$ is a linear combination of the commutators of the form $[y_1, a, \ldots, y_i]$, where $(y_1, \ldots, y_i)$ is a permutation of $(x_1, \ldots, x_i)$. Further, if $w = [ab]$ then in the free associative algebra F we get

$$[a, b^2, x, y, z] = [w \circ b, x, y, z] \in [V^{(2)} \circ F, x, y, z] \subseteq [V^{(3)} \circ F + V^{(2)} \circ V^{(2)}, y, z]$$
$$\subseteq [V^{(4)} \circ F + V^{(3)} \circ V^{(2)}, z] \subseteq V^{(5)} \circ F + V^{(4)} \circ V^{(2)} + V^{(3)} \circ V^{(3)}.$$

It follows from here that each proper polynomial of degree 6, which is contained in $T^{(5)}$, may be represented as a linear combination of u_6, u_4u_2, and u_3v_3, where u_i and v_i are the commutators of degree i. Using the PBW Theorem, select in (1) the terms that contain $[ap]$, and, replacing $x \circ y$ by $2xy - [xy]$, we obtain

$$\beta_2[\bar{q}b][\bar{c}r][ap] + \beta_3[\bar{q}c][\bar{b}r][ap] = 0.$$

Canceling $[ap]$, we get $\beta_2[\bar{q}b][\bar{c}r] + \beta_3[\bar{q}c][\bar{b}r] = 0$. Combine the similar terms to obtain

$$(-\beta_2)[bq][cr] + (-\beta_2 + \beta_3)[bc][qr] + (-\beta_3)[cq][br] = 0.$$

Since the mentioned products of commutators are linearly independent, $\beta_2 = \beta_3 = 0$. Arguing analogously with the commutators $[bp]$ and $[cp]$, we obtain $\alpha_2 = 0$ and $\beta_1 = 0$. Hence, $\alpha_1 = \alpha_2 = 0$, and $\beta_1 = \beta_2 = \beta_3 = 0$. Thus, $\dim_K(\Gamma_6(A) \cap H') = 5$.

REMARK 9. $\dim_K(\Gamma_5(A) \cap H') = 7$ and $\dim_K(\Gamma_6(A) \cap H') = 5$ in $A = F^{(5)}$.

§ 3. The Sequence of Codimensions of $T^{(5)}$

Denote by $c_n^{(l)}$ the codimension of the ideal $T^{(l)}$. Recall that

$$c_n^{(3)} = 2^{n-1}, \quad c_n^{(4)} = 2^{n-1} + 2\binom{n}{4} + 2\binom{n}{3}.$$

Using the constructed additive basis, we can compute the exact value of $c_n^{(5)}$ as a function of n. Since the exact value is not needed, prove the validity of the asymptotic estimation $c_n^{(5)} \approx n^2 \cdot 2^{n-2}$.

Lemma 3.1. *If* $A = \mathfrak{H}[X]$ *then* $\dim_K(T^{(4)} \cap P_n(A)) = 6\binom{n}{4} + 18\binom{n}{5}$.

Lemma 3.1 is immediate from Remarks 1 and 2.

Lemma 3.2. *Let* $A = \mathfrak{E}[X]$ *and* $d_n = \dim_\Phi H(A) \cap P_n(A)$. *Then* $d_n \approx n^2 \cdot 2^{n-2}$.

Proof. Firstly, we compute the number γ_m of the basis proper multilinear polynomials of degree m which lie in the ideal $H(A)$; i.e., $\gamma_m = \dim_\Phi H(A) \cap \Gamma_m(A)$. Taking into account Remarks 3, 7, and 8, we have $\gamma_5 = 5$, $\gamma_{2k} = 4k^2 - 4k$, and $\gamma_{2k+1} = 4k^2 - 1$ when $k \geq 3$.

We get $d_n = \sum_{m\geq 5} \gamma_m \binom{n}{m}$. Give the further computation as a sequence of items, using the following combinatorial formulas (see [13]):

$$\sum_k k^2 \binom{n}{2k} = n(n+1)2^{n-5} \approx n^2 \cdot 2^{n-5},$$

$$\sum_k k^2 \binom{n}{2k+1} = (n^2 - 3n + 4)2^{n-5} \approx n^2 \cdot 2^{n-5}.$$

1. $\sum_{k\geq 3} \gamma_{2k} \binom{n}{2k} \approx n^2 \cdot 2^{n-3}$. Since $\sum_{k\geq 3} k^2 \binom{n}{2k} \approx \sum_k k^2 \binom{n}{2k}$; therefore,

$$\sum_{k\geq 3} \gamma_{2k} \binom{n}{2k} = \sum_{k\geq 3} (4k^2 - 4k)\binom{n}{2k} \approx 4\sum_{k\geq 3} k^2 \binom{n}{2k} \approx n^2 \cdot 2^{n-3}.$$

2. $\sum_{k\geq 3} \gamma_{2k+1} \binom{n}{2k+1} \approx n^2 \cdot 2^{n-3}$. Since $\sum_{k\geq 3} k^2 \binom{n}{2k+1} \approx \sum_k k^2 \binom{n}{2k+1} \approx n^2 \cdot 2^{n-5}$; therefore,

$$\sum_{k\geq 3} \gamma_{2k+1} \binom{n}{2k+1} \approx 4\sum_{k\geq 3} k^2 \binom{n}{2k+1} \approx n^2 \cdot 2^{n-3}.$$

3. We have

$$d_n = \gamma_5 \, \mathrm{C}_n^5 + \sum_{k\geq 3} \gamma_{2k} \binom{n}{2k} + \sum_{k\geq 3} \gamma_{2k+1} \binom{n}{2k+1} \approx n^2 \cdot 2^{n-2}.$$

Theorem 3.1. *The asymptotic estimation* $c_n^{(5)} \approx n^2 \cdot 2^{n-2}$ *holds.*

Proof. By Remark 9, $\dim_K(H' \cap P_n(A)) = 7\binom{n}{5} + 5\binom{n}{6}$ with $n \geq 5$. Hence, applying Lemmas 3.1 and 3.2, we get $\dim_K(T^{(4)} \cap P_n(A)) \approx n^2 \cdot 2^{n-2}$. Thus, $c_n^{(5)} = c_n^{(4)} + \dim_K(T^{(4)} \cap P_n(A)) \approx n^2 \cdot 2^{n-2}$ by [5].

§ 4. The Multilinear Components of the Centers of $F^{(3)}$ and $F^{(4)}$

In this section we give a description for the centers of the relatively free algebras $F^{(n)}$ with the Lie nilpotency identity $\mathrm{LN}(n)$ of degree $n = 3, 4$. The description of the proper central polynomials of these algebras is well known (it is immediate from the articles by Latyshev [2] and Volichenko [5]). The proper central polynomials of $F^{(5)}$ and $F^{(6)}$ were described in [8]. Concerning the descriptions of the centers $Z(F^{(l)})$ with $l = 3, 4$, the equalities hold:

$$[Z(F^{(3)}) = ([xy])^V, \quad Z(F^{(4)}) = ([xy][zt])^V + ([xyz])^T,$$

where $(f)^T$ and $(f)^V$ is a T-subspace and T-ideal generated by f. These results belong to Grishin, and they are presented in [15, 16]. Find an additive basis for the centers and compute the dimensions of multilinear components $Z(F^{(l)}) \cap P_n$ when $l = 3, 4$.

Denote by $\mathrm{vr}(f)$ the set of variables of a homogeneous polynomial f; i.e., $\mathrm{vr}(f)$ is the set of variables of positive degree in f.

Let $A = F^{(3)}$. Construct an additive basis for $Z(A) \cap P_n(A)$.

A *right commutator word* is as usual an element of the shape $[a_1, b_1] \dots [a_k, b_k]$, where $a_i, b_i \in X$ and $a_1 < b_1 < \dots < a_k < b_k$. A *right monomial* is a word $y_1 y_2 \dots y_k$, where $y_i \in X$ and $y_1 < \dots < y_k$. A *right commutator* over X_n is an element of the form $[x_1 y_1 \dots y_l, t]$ $(l \geq 1)$, where $y_i, t \in X_n$ and $y_1 < \dots < y_l$.

Lemma 4.1 [15]. $Z(A) = [A, A]$.

Proof. It is known [2] and easily to verify that $P_n(A)$ is spanned by the *right elements* of the shape uv, where $u = [a_1, b_1] \dots [a_k, b_k]$ is a right commutator word, $v = y_1 y_2 \dots y_l$ is a right monomial, and $\{a_1, b_1, \dots, a_k, b_k, y_1, \dots, y_l\} = X_n$.

Since $a x_1 b \equiv x_1 b a \pmod{[A, A]}$ and $[x_1 y_1 \dots y_l, t] \in [A, A]$; therefore,

$$[x_1, t] y_1 \dots y_l + \sum_{i=1}^{l} x_1 y_1 \dots [y_i, t] \dots y_l \in [A, A].$$

Hence, an arbitrary element p in $P_n(A)$ is represented modulo $[A, A]$ as a linear combination of the elements of the form

$$[a_1, b_1] \dots [a_k, b_k] x_1 y_1 y_2 \dots y_l.$$

Thus, $p \equiv x_1 p_1$ modulo $[A, A]$, where $p_1 = \sum_i \alpha_i w_i v_i$ and $w_i v_i$ are some right multilinear elements in $x_2, \dots, x_n$. Assume that $p \in Z(A)$. Then $p_1 \in Z(A)$ (it suffices to apply $\frac{\partial}{\partial x_1}$); therefore, $p_1 \in Z^*(A)$. Since $Z^*(A) = 0$ (A is T-prime, i.e., the product of nonzero T-ideals of A is nonzero), $p_1 = 0$ and $p \in [A, A]$. The lemma is proved.

Lemma 4.2. *The following elements form a basis for* $Z(A) \cap P_n(A)$*:*

$$[a_1, b_1] \dots [a_k, b_k][x_1 y_1 \dots y_l, t], \quad \text{where } a_1 < b_1 < \dots < a_k < b_k < t,\ y_1 < \dots < y_l.$$

Proof. The above elements form a *Z-basis*:

(1) We prove firstly that each element $p \in Z(A) \cap P_n(A)$ is a linear combination of some Z-basis elements. Note that every commutator $[v, v'] \in P_n$ in monomials of degree ≥ 3 may be written as $[v_i, x_i]$ $(i \geq 2)$. Indeed, if $a = a_1 a_2$ and $x_1 \in \mathrm{vr}(b)$ then

$$[a, b] = [a_1 a_2, b] = -[a_2 b, a_1] - [b a_1, a_2] = [a_1, a_2 b] + [a_2, b a_1],$$

and we get the required representation by induction on the degree of a. Now,

$$[a x_1 b, y] = [x_1 b a, y],$$

since $[[a, x_1 b], y] = 0$. Therefore, we may assume that $v_i = x_1 v_i'$.

By Lemma 4.1 the monomial v_i' may be represented as a linear combination of some generators among $x_2, \dots, x_n$. Thus, we have a representation of p as a combination of elements

$$[a_1, b_1] \dots [a_k, b_k][x_1 y_1 \dots y_l, t]$$

and the right commutator word $[x_1, x_2] \dots [x_{n-1}, x_n]$ with n even. Noting that $[a_1, b_1] \dots [a_k, b_k][v, t]$ is skew-symmetric in $a_1, b_1, \dots, a_k, b_k$, and t, we get the required assertion.

(2) Show that the Z-basis elements are linearly independent. Let a linear combination of the Z-basis elements with nonzero coefficients be equal to zero. Among the Z-basis elements choose an element $b = [a_1, b_1] \dots [a_k, b_k][x_1 y_1 \dots y_l, t]$ such that the number l is maximal and b enters into the linear combination with a coefficient β. The element b is completely determined by $\{y_1, \dots, y_l\}$. Substituting for $y_1, \dots, y_l$ the commutators $[x_N, x_{N+1}], [x_{N+1}, x_{N+2}], \dots,$ in which $N > n$, we get $\pm \beta w = 0$, where w is a right commutator word; a contradiction. The lemma is proved.

From Lemma 4.2 and Volichenko's Lemma we easily deduce

Lemma 4.3. *The center $Z(F^{(4)})$ modulo $T^{(3)}$ possesses an additive basis of the Z-basis elements with $k \geq 1$:*

$$[a_1, b_1] \dots [a_k, b_k][x_1 y_1 \dots y_l, t],$$

where $k \geq 1$, $a_1 < b_1 < \dots < a_k < b_k < t$, $y_1 < \dots < y_l$.

Proof. Let $p \in Z(F^{(4)})$. Then $p \in Z(F^{(3)}) + T^{(3)}$. By Lemma 4.2 p can be written as

$$p \equiv \sum_{Y=\{y_1,\dots,y_l\}} \alpha_Y w_Y [x_1 y_1 \dots y_l, t] \pmod{T^{(3)}},$$

where w_Y is a right commutator word, and $w_Y < t$.

Let Z_0 be the linear span of the elements mentioned in the lemma. If $k \geq 1$ then $p \in Z_0$. Hence, we may assume that

$$p \equiv \sum_{Y=\{y_1,\dots,y_l\}} \alpha_Y [x_1 y_1 \dots y_l, t] \pmod{Z_0}.$$

Inserting the unity for $y_1, \dots, y_l$ we get $\alpha_Y [x_1, t] \in Z(F^{(4)})$ which is possible only if $\alpha_Y = 0$. Hence, $k \geq 1$ and $p \in Z_0$. The lemma is proved.

Corollary [16]. $Z(F^{(4)}) = ([xy] \cdot [zt])^V + ([xyz])^T$.

Remark. If $A = F_{(2)}$ is the free metabelian algebra then $Z(A) = ([xy] \cdot [zt])^V$.

Indeed, $Z(A) \subseteq ([xy] \cdot [zt])^V + T^{(3)}$ by Lemma 4.3. It follows from [13] that $Z(A) \cap T^{(3)} = ([xy]^2)^T = ([xy]^2)^V$. Since $[xy]^2 \in \{[xy] \cdot [zt]\}^V$, the required equality holds.

Theorem 4.1. *Let $A = F^{(3)}$; $c_n = \dim_K P_n(A)$, and $\xi_n = \dim_K Z(A) \cap P_n(A)$. Then*

$$c_n = 2\xi_n, \quad n \geq 2.$$

Proof. Compute the number of Z-basis words. Let $n = 2m + 1$. Then each basis word is defined by a choice of the set $\{y_1, \dots, y_l\} \subseteq \{x_2, \dots, x_{2m+1}\}$, where l is odd and $1 \leq l \leq 2m - 1$. Then

$$\xi_{2m+1} = \sum_{k \geq 0}^{m-1} \binom{2m}{2k+1} = \binom{2m}{1} + \binom{2m}{3} + \dots + \binom{2m}{2m-1} = 2^{2m-1}.$$

Since $c_{2m+1} = 2^{2m}$; therefore, $c_{2m+1} = 2\xi_{2m+1}$.

If $n = 2m$ then every basis word is defined by $\{y_1, \dots, y_l\} \subseteq \{x_2, \dots, x_{2m}\}$, where l is even and $0 \leq l \leq 2m - 2$. Then

$$\xi_{2m} = \sum_{k \geq 0}^{m-1} \binom{2m-1}{2k} = \binom{2m-1}{0} + \binom{2m-1}{2} + \dots + \binom{2m-1}{2m-2} = 2^{2m-2}.$$

Since $c_{2m} = 2^{2m-1}$, we have $c_{2m} = 2\xi_{2m}$. The theorem is proved.

By analogy with Theorem 4.1 we can prove for the numbers $c_n(A) = \dim_K P_n(A)$ and $\xi_n(A) = \dim_K Z(A) \cap P_n(A)$ that the following hold:

$$\frac{\xi_n(F^{(4)})}{c_n(F^{(4)})} \to \frac{1}{2}, \quad \frac{\xi_n(F^{(2)})}{c_n(F^{(2)})} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Note that the relatively free associative algebra with the identity $[xy][zt] = 0$ has zero center.

§ 5. Upon the Center of $F^{(5)}$

The aim of this section is a description of the T-generators of the center of $F^{(5)}$ and presentation of an asymptotic estimation of the dimension of $Z_n(F^{(5)})$. It turns out that the "center is about half of the algebra."

5.1. The T-generators of $Z(F^{(5)})$.

Proposition 5.1. *Let $p = \operatorname{char}(K)$. Then*

$$Z(F^{(n)}) = (x^q)^V + Z(F^{(n)}) \cap T^{(2)},$$

where q is the least number of the form p^s such that $p^s \geq n-1$.

PROOF. This literally repeats the argument of [9, Theorem 4].

In [7] the hypothesis on the center was formulated:

$Z(F^{(5)}) = (h, [x_1, \dots, x_4])^V + (h')^T$ if $\operatorname{char}(K) = 0$;

$Z(F^{(5)}) = (x^p, h, [x_1, \dots, x_4])^V + (h')^T$ if $\operatorname{char}(K) = p \geq 5$.

It was proved therein also that $[T^{(3)}, a, b] \subseteq T^{(5)}$ [7, Lemma 2]. Hence, $[T^{(3)}, a] \subseteq Z(F^{(5)})$.

Lemma 5.1. $[T^{(3)}, a] \subseteq (h, [x_1, \dots, x_4])^V$ *for every $a \in F^{(5)}$.*

PROOF. Indeed, modulo $(h, [x_1, \dots, x_4])^V$ we have

$$[[abb] \circ x, y] \equiv -[[abx] \circ b, y] = -[[a, b^2, x] + [ab] \circ [bx], y] \equiv 0,$$

which was required.

Throughout this section, A is the free algebra over a set X in the variety generated by the model algebra $E^{(2)}$. Since $(h')^T$ in $F^{(5)}$ coincides with the ideal of identities of $E^{(2)}$; therefore, it suffices to understand that every central element in $E^{(2)}$ is contained in $[T^{(3)}, A]$.

Proposition 5.2. *Let A be a relatively free algebra of an arbitrary unitarily closed variety. If $f \in Z^*(A)$ and $f = \sum_{\vec{i}} f_{\vec{i}} X^{\vec{i}}$, where $0 \neq f_{\vec{i}} \in \Gamma(A)$ and $X^{\vec{i}}$ are some right monomials, then $f_{\vec{i}} \in Z^*(A)$.*

PROOF. We may assume that f is homogeneous. If $x \notin \operatorname{vr}(f)$ then $fx \in Z(A)$. Choose a set $\vec{i}$ of naturals, which has the maximal sum of indices. Applying the partial derivation operators $\left(\frac{\partial}{\partial x_k}\right)^{i_k}$, where $\vec{i} = (i_1, i_2, \dots)$, we obtain $f_{\vec{i}} x \in Z(A)$ or $f_{\vec{i}} \in Z^*(A)$, which was required.

Thus, the kernel of the algebra is generated by the proper kernel elements. It turns out that there is no such assertion for the center. In [7, 8] the central polynomials were under study. However, it is impossible to apply these results directly to the central polynomials, since we have the following

Lemma 5.2. *Let f be a basis polynomial of type* (2) *or* (3) *in $\Gamma(A) \cap H(A)$, and $f \notin Z(A)$. Then there exist some central polynomials $g_i \in \Gamma(A) \cap H(A)$ and $t_i \in X$ such that*

$$f + \sum_i g_i t_i \in Z(A).$$

PROOF. Without loss of generality we may assume that f is of the shape $f = vu_1 \dots u_k$, where $v \in V^{(3)}$ and $u_1, \dots, u_k \in V^{(2)}$. Consider $[f, t]$, where $t \notin \operatorname{vr}(f)$:

$$\begin{aligned}[f, t] &= [vu_1 \dots u_k, t] = [v, t]u_1 \dots u_k = [v, t]u_1 \dots u_{k-1}[ab] = [v, a]u_1 \dots u_{k-1}[bt] \\ &= [([v, a]u_1 \dots u_{k-1})b, t] = \Big[\sum_i g_i b, t\Big];\end{aligned}$$

here g_i are some proper central polynomials of even degree and $a, b \in X$, as required.

Lemma 5.3. *If $g \in \Gamma(A) \cap T^{(4)}(A) \cap Z(A)$ is a polynomial of degree s and $a \in A$ then there exist some polynomials $g_i \in \Gamma(A) \cap T^{(4)}(A)$ of degree $\geq s+1$ and a suitable $a_i \in A$ such that $ga + \sum_i g_i a_i \in [T^{(3)}, A]$.*

Proof. Let g be of even degree. Then $g \in V^{(4)}$, i.e., $g = wu_1 \dots u_k$, where $w = [v, x], v \in V^{(3)}$, $u_1, \dots, u_k \in U$, $x \in X$. Consider the element

$$ga = [v, x] \cdot u_1 \dots u_k a = [v \cdot u_1 \dots u_k a, x] - v[a, x]u_1 \dots u_k = z + \sum_i g_i a_i,$$

where g_i are some proper polynomials (of odd degree), $z = [v \cdot u_1 \dots u_k a, x] \in [T^{(3)}, A]$.

If g is of odd degree then $g = [\varphi, x]u_1 \dots u_k$, where $\varphi \in \Phi$, $u_1, \dots, u_k \in V^{(2)}$, and $x \in X$. Consider the element

$$ga = [\varphi, x]au_1 \dots u_k = [\varphi a, x]u_1 \dots u_k - \varphi[a, x]u_1 \dots u_k.$$

Since $[\varphi a, x]u_1 \dots u_k \in [T^{(3)}, A]$ and we can find some proper polynomials $g_i \in \Phi U^{k+1}$ for $\varphi[x, a]u_1 \dots u_k$ so that $ga + \sum_i g_i a_i \in [T^{(3)}, A]$; therefore, the lemma is proved.

A monomial $y_1 y_2 \dots y_l$ is *right* provided that $y_i \in X$ and $y_1 \leq y_2 \leq \dots \leq y_l$.

Introduce the notions of regular elements of first and second types.

The *regular elements of first type* are the following proper polynomials:

(a) the basis elements of types (2) and (3) (see 1.4);

(b) the basis elements of the shape φ;

(c) the basis commutators $[abc]$ of degree 3;

(d) the elements of the form a_{ij}, b_{ij}, c_{ij} referred to in Remark 2;

(e) the right commutator words $[y_1, t_1] \dots [y_k, t_k]$ $(k \geq 1)$, i.e., $y_i, t_i \in X$ and $y_1 < z_1 < \dots < y_k < z_k$.

The *regular elements of second type* are the following proper polynomials:

(a) the basis elements of types (1) and (4) (see 1.4);

(b) the basis commutators $[abcd]$ of degree 4.

Lemma 5.4. *An arbitrary polynomial f modulo $[T^{(3)}, A]$ is represented as $f \equiv \sum_i f_i a_i$, where f_i are some regular elements of first type and a_i are some right monomials* (*it is possible that one of the factors f_i or u_i is omitted*).

Proof. Write the element f as $f = \sum_i g_i b_i$, where b_i are some right monomials, and g_i are some regular elements of first or second type. If g_i is a regular element of second type then g_i is central, and by Lemma 5.3 $g_i b_i$ modulo $[T^{(3)}, A]$ is a linear combination of $f_i a_i$, where f_i are regular elements of first type, and a_i are right monomials; i.e., $f \equiv \sum_i f_i a_i \pmod{[T^{(3)}, A]}$, as required.

Theorem 5.1. $Z(A) \cap T^{(2)} = [T^{(3)}, A]$.

Proof. Let f be a central element. By Lemma 5.4 we may assume that f is written as $f = \sum_i f_i a_i$, where f_i are some regular elements of first type and a_i are some right monomials. Applying suitable partial derivations and taking into account that the spaces $[T^{(3)}, A]$ and $Z(A)$ are invariant with respect to them, we obtain linear dependence modulo $Z(A)$ of regular elements of fist type.

Show that this assertion fails. If the elements are linearly independent modulo $Z(A)$ then we call them *Z-free*.

In [10], it was proved that the basis elements of type (3) of odd degree ≥ 5 are Z-free.

The basis commutators of degree 3 are Z-free (see [7, Lemma 10]) as well.

Verify that the following elements are Z-free:

$$\varphi(a, b, x, y), \quad \varphi(a, b, y, x) \quad \text{and} \quad [a, b][x, y].$$

In $E^{(2)}$ the equalities hold:

$$\varphi(e_1, e_2, e_3, e_4) = 0, \quad [[e_1, e_2][e_3, e_4], e_5] \neq 0.$$

Hence, the following elements should be linearly dependent modulo the center:

$$\varphi(a, b, x, y), \quad \varphi(a, b, y, x),$$

but this fails by [10].

The linear independence of the elements of the shape $a_{ij}, b_{ij}, c_{ij\varphi}$ modulo the center was noted in Remark 2. Finally, a right commutator word does not belong to $Z(E^{(2)})$. The theorem is proved.

Proposition 5.1 and Theorem 5.1 imply

Corollary. *Over a field K of characteristic 0 the center of $F^{(5)}$ as a T-space is generated by the elements $[x_1, x_2, x_3, x_4]$, $[[x_1, x_2, x_3] \cdot x_4, x_5]$, and $[[x_1, x_2]^2, x_2]$.*

If $p = \mathrm{char}(K) \geq 5$ then $Z(F^{(5)})$ is generated by

$$x^p, \quad [x_1, x_2, x_3, x_4], \quad [[x_1, x_2, x_3] \cdot x_4, x_5], \quad [[x_1, x_2]^2, x_2].$$

5.2. An asymptotic estimation of ξ_n/c_n. Let $A = F^{(5)}$, $\xi_n = \dim_K Z(A) \cap P_n(A)$, and $c_n = \dim_K P_n(A)$. In this subsection we assume that a field K is of characteristic $\neq 2, 3$. Show that $\lim_{n\to\infty} \xi_n/c_n = \frac{1}{2}$.

Note firstly that by Lemma 5.4

$$[T^{(3)}, A] \subseteq [T^{(3)}, X] \subseteq \sum_{i,j} [f_i a_i, x^j],$$

where f_i are regular elements of first type, a_i are right monomials, and $x^j \in X$.

If $\deg f_i = d \leq 5$ then the number of commutators $[f_i a_i, x^j]$ does not exceed $n\gamma_d \mathrm{C}_{n-1}^d$ (a polynomial of 6th degree in n), where γ_d is the number of regular elements of degree d. Since $c_n \approx n^2 \cdot 2^{n-2}$; therefore, we may assume that

$$f = \sum_{i,j} \alpha_i [f_i a_i, x^j],$$

where f_i are some regular elements of first type of the following shape:

(1) the right φ-words contained in ΦU^m;

(2) the right η-words contained in $V^{(3)} U^m$;

(3) the right commutator words $[y_1, t_1] \ldots [y_k, t_k]$ $(k \geq 3)$.

Find an upper estimation for N_m of linear generators of the form $\left[f_i^{(m)} a_i, x^j\right]$, where $m = 1, 2, 3$.

If $m = 3$, i.e., $f_i^{(m)}$ are some right commutator words (or the elements of type (3)) then

$$N_3 \leq n \sum_k \binom{n-1}{k} \leq n \cdot 2^{n-2}.$$

If $t \in T^{(3)}$ then $[t[ab]c, a] = 0$. Hence, the elements of the form $[t[y_1, t_1] \ldots [y_k, t_k]\ c, x]$, where $t \in \Phi \cup V^{(3)}$, are skew-symmetric in $y_1, t_1 \ldots y_k, t_k$, and x.

Similarly, if $f_i^{(m)}$ are the right φ-words of the shape $\varphi(x_1, x_2, x_3, x_i)v'$ or $\varphi(x_1, x_2, x_i, x_3)v'$ then the number of linear generators of the shape $\left[f_i^{(m)} a_i, x^j\right]$ is equivalent to $Cn \cdot 2^{n-2}$, where $C = \mathrm{const}$. An analogous fact holds for the right η-words of the form $\eta^+(x_2, x_1, x_i, x^j, x_3)v'$, $\eta^+(x_3, x_1, x_2, x_i, z)v'$, $\eta^-(x_2, x_1, x_3, x_i, z)v'$, $\eta^+(x_2, x_1, x_3, x_i, z)v$, where $x_i < z < v'$.

Hence, without loss of generality we may assume that $f_i^{(m)}$ are of the shape

$$\varphi^-(y_1, y_2, y_k, y_l)v' \quad \text{or} \quad \eta^+(y_2, y_1, y_i, y^j, y_3)v', \tag{2}$$

where $y_1 < y_2 < y_3 < v' < x^j$.

Lemma 5.5. *$t' = [t[x_n, b]z_1 \ldots z_k, z]$, where $t \in T^{(3)}, b, z_1, \ldots, z_k, z, x_n \in X_n$, may be represented as a linear combination of the right elements $[f_i a_i x_n, x^j]$, $[g_i b_i, x_n]$, while f_i and g_i are some elements of the form (2), $f_i < x^j$ and a_i and b_i are some right monomials.*

Proof. Since

$$[t[x_n z_1 \ldots z_k, b], z] = [t[x_n, b]z_1 \ldots z_k, z] + [t x_n [z_1 \ldots z_k, b], z],$$

by $[t[ab]c,a]=0$ and Volichenko's Lemma we infer that t' is a linear combination of the elements $[f_ia_ix_n,x^j]$ and $[g_i,b_ix_n]$. Since the second commutator is linearly expressible by $[f_ia_ix_n,x^j]$ and $[g_ib_i,x_n]$, the second assertion holds.

Prove the linear independence of the right elements. Let

$$\sum_i \alpha_i[f_ia_ix_n,x^j]+\sum_k \beta_k[g_kb_k,x_n]=0.$$

If $\alpha_{i_0}\neq 0$ or $\beta_{k_0}\neq 0$ then assume that the word $a_{i_0}\neq 0$ or $b_{k_0}\neq 0$ is of maximal degree. If a_{i_0} is of maximal degree then putting $a_i=x_n=1$ we get $\alpha_{i_0}=0$; a contradiction. If b_{k_0} is of maximal degree then we put $b_k=1$. Then $\beta_{k_0}=0$; a contradiction. The lemma is proved.

The lengthes of φ and η are of different parity; therefore, if some monomial a_i is taken then the choice of $f_i^{(m)}$ is defined by the parity of $\deg a_i$, and a_i is connected only with one of the elements of type (2).

Firstly, compute the number N_1 of choices of the right elements of the form $[f_ia_ix_n,x^j]$. A right monomial a_i of length i may be chosen by $\binom{n-1}{i}$ ways, since we need to take i variables in X_{n-1}. For definiteness, we take $[\varphi^-(y_1,y_2,y_k,y_l)v',x^j]$ from the remaining $n-i-1$ elements. Given that y_1 and y_2 are the least symbols, the number of choices of y_k and y_l is $\binom{n-3-i}{2}$. Hence,

$$N_1=\sum_i\binom{n-3-i}{2}\binom{n-1}{i}=\sum_{i\leq n-1}\binom{n-3-i}{2}\binom{n-1}{n-1-i}.$$

Now, by $\binom{n}{k}=n\binom{n-1}{k-1}$ we have

$$\begin{aligned}N_1&=\sum_i\binom{n-3-i}{2}\binom{n-1}{i}=\sum_{i\leq n-1}\binom{n-3-i}{2}\binom{n-1}{n-1-i}\\&-\frac{1}{2}\sum_{i\leq n-1}(n-3-i)(n-4-i)\binom{n-1}{n-1-i}\\&\approx\frac{1}{2}\sum_{i\leq n-1}(n-1-i)(n-2-i)\binom{n-1}{n-1-i}\\&=\frac{1}{2}(n-1)(n-2)\sum_{i\leq n-1}\binom{n-3}{n-3-i}\approx n^2\cdot 2^{n-4}.\end{aligned}$$

Find the number N_2 of choices of the right elements of type $[g_i,b_ix_n]$. A right monomial b_i of length i may be chosen by $\binom{n-1}{i}$ ways. For definiteness, we take $\eta^+(y_2,y_1,y_i,y^j,y_3)v'$ from the remaining $n-i-1$ elements. Given that y_1, y_2, and y_3 are the least symbols, the number of choices of y_k and y_l is $\binom{n-4-i}{2}$. Therefore,

$$N_2=\sum_i\binom{n-4-i}{2}\binom{n-1}{i}\approx\sum_{i\leq n-1}\binom{n-3-i}{2}\binom{n-1}{n-1-i},$$

whence $N_1+N_2\approx n^2\cdot 2^{n-3}$. Then $\xi_n/c_n\approx[(n^2\cdot 2^{n-3}):(n^2\cdot 2^{n-2})]=\frac{1}{2}$.

Question. *Is it true that $\xi_n^{(l)}/c_n^{(l)}\approx\frac{1}{2}$ for every l?*

From the above results, it is immediate that the algebra over the ring $\mathbb{Z}\left[\frac{1}{6}\right]$ is additively torsion-free; in particular, the additive group of a free Lie nilpotent ring of degree 5 lacks the elements of finite simple degree ≥ 5. The elements of additive degree 3 in $F^{(4)}$ are known (see [17]).

It is unknown if the additive group of $F^{(5)}(\mathbb{Z})$ contains some elements of order 2 and 3.

Note that Shirshov in [18] proved that the free Lie ring is torsion-free, and existence of elements of order 3 in the free alternative ring was proved in [19, 20].

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