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ABOUT MULTIPOINT DISTORTION THEOREMS FOR RATIONAL FUNCTIONS

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Abstract: We prove some two- and three-point distortion theorems for rational functions that generalize some recent results on Bernstein-type inequalities for polynomials and rational functions. The rational functions under study have either majorants or restrictions on location of their zeros. The proofs are based on the new version of the Schwarz Lemma and univalence condition for regular functions which was suggested by Dubinin.

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1. Introduction and Auxiliary Results

The inequalities for rational functions generalizing the classical polynomial inequalities, as well as the latter themselves, occurred originally in approximation theory for proving the inverse theorems [1, 2]. At present, these inequalities are of interest in their own right which is witnessed by many recent articles (see, for instance, [3–9]). Below we will exhibit several classical and new results for polynomials and rational functions. Let us start with introducing the notations of use in this article. We consider the rational function

$$R(z) = \frac{P(z)}{\prod\limits_{k=1}^{n}(z-a_k)},$$

where $P(z) = b_m z^m + b_{m-1}z^{m-1} + \dots + b_0$ is a polynomial of degree m. The Blaschke product

$$B(z) = B_a(z) \equiv \prod_{k=1}^{n} \frac{1-\bar{a}_k z}{z-a_k}$$

is defined in the general case for every collection of poles $a = (a_1, \dots, a_n)$, $|a_k| \neq 1$, $k = 1, \dots, n$. In the problems under consideration, $R(z)$ either plays the role of the extremal function or takes part in the explicit representation of extremal rational functions. Observe that $|(z^{m-n}B(z))'| = d|z^{m-n}B(z)|/d|z| = m - n + |B'(z)|$.

The Bernstein-type inequalities on the circle and interval are interconnected. For instance, suppose that $a_k \in \mathbb{C} \setminus [-1, 1]$, $k = 1, \dots, n$, and unreal numbers a_k from this collection constitute complex conjugate pairs. Define c_k, $k = 1, \dots, n$, as follows:

$$a_k := \frac{c_k + 1/c_k}{2}, \quad |c_k| < 1, \ k = 1, \dots, n, \tag{1}$$

implying that

$$c_k = a_k - \sqrt{a_k^2 - 1}, \quad k = 1, \dots, n, \tag{2}$$

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with the appropriate branch of the square root. In this case the positive value

$$B_n(x) := \operatorname{Re}\sum_{k=1}^{n} \frac{\sqrt{a_k^2-1}}{a_k-x} = \sum_{k=1}^{n} \frac{1-|c_k|^2}{|\zeta-c_k|^2}, \quad x\in[-1,1] \quad (x=1/2(\zeta+1/\zeta))$$

is called the *Bernstein factor*.

In line with [6], put

$$M_n(\zeta) = \left(\prod_{k=1}^{n}(\zeta-c_k)(\zeta-\bar{c}_k)\right)^{1/2},$$

where the branch of the square root is chosen such that $\zeta^n M_n(\zeta^{-1})$ is an analytic function in some neighborhood of the closed unit disk, and let

$$f_n(\zeta) = \frac{M_n(\zeta)}{\zeta^n M(\zeta^{-1})} = \prod_{k=1}^{n} \frac{\zeta-c_k}{1-\zeta c_k}.$$

The last equality holds, since, by assumption, c_k constitute complex conjugate pairs too.

Consider the following rational analogs of the Chebyshev polynomials of the first, second, third, or fourth kind (see [6, 10]):

$$T_n^r(z) = \frac{1}{2}(f_n(\zeta)+f_n(\zeta)^{-1}), \tag{3}$$

$$U_n^r(z) = \frac{f_n(\zeta)-f_n(\zeta)^{-1}}{\zeta-\zeta^{-1}}, \tag{4}$$

$$\widetilde{U}_n^r(z) = \frac{\zeta^2 f_n(\zeta)-f_n(\zeta)^{-1}}{\zeta^2-1}, \tag{5}$$

$$V_n^r(z) = \frac{\zeta f_n(\zeta)+f_n(\zeta)^{-1}}{\zeta+1}, \tag{6}$$

$$W_n^r(z) = \frac{\zeta f_n(\zeta)-f_n(\zeta)^{-1}}{\zeta-1}. \tag{7}$$

If all a_k in (3)–(7) tend to infinity, then we obtain the Chebyshev polynomials of the first (T_n), second (U_{n-1} and U_n), third (V_n), and fourth (W_n) kind respectively. Moreover, observe that

$$2(z^2-1)U_n^r(z)^2+1 = \frac{1}{2}(f_n(\zeta)^2+f_n(\zeta)^{-2}), \tag{8}$$

$$2(z^2-1)\widetilde{U}_n^r(z)^2+1 = \frac{1}{2}((\zeta f_n(\zeta))^2+(\zeta f_n(\zeta))^{-2}), \tag{9}$$

$$(z+1)V_n^r(z)^2-1 = \frac{1}{2}(\zeta f_n(\zeta)^2+(\zeta f_n(\zeta)^2)^{-1}), \tag{10}$$

$$(z-1)W_n^r(z)^2+1 = \frac{1}{2}(\zeta f_n(\zeta)^2+(\zeta f_n(\zeta)^2)^{-1}). \tag{11}$$

If there are no additional constraints on the poles, except for the fact that they do not lie on the considered compact sets (the circle or the interval), then the Bernstein-type inequalities [3, Theorem 7.1.7 and Corollary 7.1.9] are valid. Namely, if the prescribed poles $a_1,\dots,a_n$ of R do not lie on the unit circle, then at points of the circle $|z|=1$ the following inequality holds:

$$|R'(z)| \le \max\left(\sum_{|a_k|>1} \frac{|a_k|^2-1}{|a_k-z|^2}, \sum_{|a_k|<1} \frac{1-|a_k|^2}{|a_k-z|^2}\right) \max_{|z|=1}|R(z)|.$$

If $a_1, \dots, a_n$ do not belong to $[-1,1]$, then at points of this interval the estimate is valid:

$$|R'(x)| \le \max\left(\sum_{k=1}^{n} \frac{1-|c_k|^2}{|c_k - z|^2}, \sum_{k=1}^{n} \frac{|c_k|^{-2}-1}{|c_k^{-1} - z|^2}\right) \frac{\max\limits_{[-1,1]} |R(x)|}{\sqrt{1-x^2}},$$

where a_k and c_k, $k = 1, \dots, n$, satisfy (1) and (2). Equality is attained in the first case for the Blaschke product with the poles located strictly beyond or inside the circle, while in the second case it is attained for the function T_n^r defined above.

In [4, Theorem 2] Dubinin obtained the elaboration of the above. If, given R, we define

$$L = L(R) := \min_{|z|=1} \operatorname{Re} R(z), \quad H = H(R) := \max_{|z|=1} \operatorname{Re} R(z)$$

and put

$$\Lambda(R, a_k) := \lim_{z \to a_k} |z^{n-m} R(z)/B(z)|, \quad k = 1, \dots, n;$$

then

$$\frac{|\operatorname{Im}(zR'(z))|}{\sqrt{(\operatorname{Re} R(z) - L)(H - \operatorname{Re} R(z))}} \le |(z^{m-n}B(z))'| + \min_{1 \le l \le n} \frac{|a_l|^2 - 1}{|1 - \bar{a}_l z|^2}\left[\sqrt{\frac{2\Lambda(R, a_l)}{H-L}} - 1\right]$$

at every point on the circle $|z| = 1$ in which $\operatorname{Re} R(z)$ differs from $L(R)$ and $H(R)$ for $m \ge n$ and $|a_k| \ge 1$, $k = 1, \dots, n$. For instance, if $n = 1$, $a_1 = a > 1$, and $|R(x)| \le 1$, $-1 \le x \le 1$, then we have the Vidensky inequality

$$|R'(x)|\sqrt{1-x^2} \le m - 1 + \frac{\sqrt{a^2-1}}{a-x}, \quad -1 < x < 1.$$

The next two inequalities complement the Turan and Lax inequalities [7, Lemma 4 and Theorem 4] (also see, for instance, [4]). If the rational function R with prescribed poles $a_1, \dots, a_n$ has exactly m zeros belonging to the disk $|z| \le 1$, then

$$\operatorname{Re} \frac{zR'(z)}{R(z)} \ge \frac{1}{2}\left(m - n + \frac{zB'(z)}{B(z)}\right)$$

at the points on the circle $|z| = 1$ different from zeros. If all zeros lie in the complement to the unit disk, then the reverse inequality is valid:

$$\operatorname{Re} \frac{zR'(z)}{R(z)} \le \frac{1}{2}\left(m - n + \frac{zB'(z)}{B(z)}\right).$$

Note also the result of [11, Theorem 5] as an example of the Bernstein-type inequality for polynomials with a curved majorant on the interval. If a real polynomial P of degree m satisfies the condition

$$|P(x)|\sqrt{1-x^2} \le 1, \quad x \in [-1,1];$$

then

$$|xP(x) - (1-x^2)P'(x)| \le (m + \sqrt{2^{-m}|b_m|})\sqrt{1-(1-x^2)P^2(x)}$$

at the points $x \in [-1,1]$; equality is attained in the case of the polynomials U_m of the second kind. Other inequalities for polynomials and rational functions with majorants on the interval can be found, for instance, in [10, 12, 13].

In the recent articles [14, 15], where some approaches to obtaining the polynomial inequalities are developed on using the methods and results of the geometric theory of functions of complex variables, the attention was focused on multipoint distortion theorems for algebraic polynomials. This topic was covered partially in [16]. The goal of the present article is to obtain similar inequalities for rational functions with constraints on zeros which are normed on the circle or interval. The key role in the proofs is played by the following

Lemma 1 [15, Lemma 1]. *Let f be a regular function in the unit disk $|z| < 1$ satisfying the condition $|f| < 1$. Suppose that f and its derivative are also defined at the different boundary points z_k such that $w_k = f(z_k)$, $k = 1, 2, 3$, are located on the unit circle. Then*

$$\left|\prod_{k=1}^{3} f'(z_k)\right| \geq \left|\frac{(w_1 - w_2)(w_2 - w_3)(w_3 - w_1)}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}\right|.$$

Equality is attained for linear-fractional automorphisms f of the unit disk and arbitrary different points z_k on the circle $|z| = 1$, $k = 1, 2, 3$.

Lemma 2 [14, Theorem A]. *Let $w = f(z)$ be a regular function on the open set $\mathcal{B} \subset \mathbb{C}_z$ and all limit values of $|f(z)|$ be greater than or equal to unity as z tends to the boundary of $\mathcal{B}$. Suppose that $w = f(z)$ sends a unique point z_0 of $\mathcal{B}$ into the origin of coordinates; moreover, $f'(z_0) \neq 0$. Then $w = f(z)$ conformally and univalently sends the connected component of $\widetilde{\mathcal{B}} := \{z \in \mathcal{B} : |f(z)| \neq 1\}$ containing z_0 onto the disk $|w| < 1$ and $f(\mathcal{D}) \subset \{w : |w| > 1\}$ for each other connected component $\mathcal{D}$ of $\widetilde{\mathcal{B}}$.*

Denote by $\mathscr{R}$ the class of functions $w = f(z)$ for each of which there exists an open set $G = G(f)$, $0 \in G \subset \mathbb{C}$, on which $w = f(z)$ is regular; moreover,

$$f(z) = \sum_{k=1}^{\infty} d_k z^k, \quad d_1 \neq 0,$$

in some sufficiently small neighborhood of $z = 0$, while

$$\varlimsup_{\substack{z \to \zeta \\ z \in G}} |f(z)| \leq |\zeta|^2$$

for every point ζ of the boundary of G. Let $\mathscr{R}_1$ be the subclass of functions from $\mathscr{R}$ for which $G(f)$ belongs to the disk $|z| < 1$. Put

$$G_0(f) = \left\{z \in G(f) : \left|\frac{z^2}{f(z)}\right| < 1\right\}, \quad f \in \mathscr{R}_1.$$

Lemma 3 [14, Theorem 7]. *Let $w = f(z)$ belong to $\mathscr{R}_1$ and let some different points z_k, $|z_k| = 1$, $k = 1, 2$, be the supports of the boundary elements of $G_0(f)$ whose neighborhoods are valid:*

$$f(z) = w_k + b_{1k}(z - z_k) + o(z - z_k) \quad \text{as } z \to z_k,\ z \in G_0(f),$$

where $|w_k| = 1$, $k = 1, 2$. Then

$$\left|\frac{2w_1}{z_1} - b_{11}\right|\left|\frac{2w_2}{z_2} - b_{12}\right| \leq \left|\frac{w_1 z_2^2 - w_2 z_1^2}{z_2 - z_1}\right|^2, \tag{12}$$

$$\left|\frac{2w_1}{z_1} - b_{11}\right|^{t_1^2}\left|\frac{2w_2}{z_2} - b_{12}\right|^{t_2^2} \leq \sqrt{|d_1|}\left|\frac{z_2 - z_1}{w_1 z_2^2 - w_2 z_1^2}\right|^{2t_1t_2} \tag{13}$$

hold for arbitrary reals t_1 and t_2 such that $t_1 + t_2 = 1$. Equality in (12) *and* (13) *is attained for $f(z) \equiv d_1 z$, $|d_1| = 1$.*

2. The Main Results

Theorem 1. *Let a rational function R with prescribed poles $a_1, \dots, a_n$ such that $|a_k| \geq 1$ have exactly m zeros, all of them lying in the disk $|z| \leq 1$. Then, for arbitrary points z_k, $k = 1, 2, 3$, from the unit circle $|z| = 1$, different from zeros and poles of R, the following inequality is valid:*

$$\prod_{k=1}^{3}\left(\operatorname{Re}\frac{z_k R'(z_k)}{R(z)} - \frac{1}{2}\left(m - n + \frac{z_k B'(z_k)}{B(z_k)}\right)\right) \geq \frac{1}{24\sqrt{3}} \prod_{k=1}^{3}\left|\frac{R(z_k)}{\overline{R(z_k)}}\frac{z_k^{n-m}}{B(z_k)} - \frac{R(z_{k+1})}{\overline{R(z_{k+1})}}\frac{z_{k+1}^{n-m}}{B(z_{k+1})}\right|, \tag{14}$$

where $z_4 = z_1$. Equality in (14) *is attained when the zeros of R belong to the circle $|z| = 1$.*

Proof. Consider the function

$$f(z) = \frac{R(z)}{\overline{R(1/\bar{z})}}\frac{z^{n-m}}{B(z)}$$

regular in the disk $|z| \leq 1$ and satisfying the conditions of Lemma 1 for arbitrary different points z_k, $k = 1, 2, 3$, on the circle $|z| = 1$. Direct calculations yield

$$f'(z) = \frac{R'(z)\overline{R(1/\bar{z})} + R(z)\overline{R'(1/\bar{z})}/z^2}{(\overline{R(1/\bar{z})})^2}\frac{z^{n-m}}{B(z)} + \frac{R(z)}{\overline{R(1/\bar{z})}}\left(\frac{z^{n-m}}{B(z)}\right)',$$

whence

$$\begin{aligned}
|f'(z_k)| &= \left|\frac{R'(z_k)\overline{R(z_k)} + R(z_k)\overline{R'(z_k)}/z_k^2}{(\overline{R(z_k)})^2}\frac{z_k^{n-m}}{B(z_k)} + \frac{R(z_k)}{\overline{R(z_k)}}\left(\frac{z_k^{n-m}}{B(z_k)}\right)'\right| \\
&= \left|\frac{R(z_k)z_k^{n-m-1}}{\overline{R(z_k)}B(z_k)}\right|\left|\frac{z_k R'(z_k)}{R(z_k)} + \overline{\frac{z_k R'(z_k)}{R(z_k)}} - \left(m - n + \frac{z_k B'(z_k)}{B(z_k)}\right)\right| \\
&= \left|2\operatorname{Re}\frac{z_k R'(z_k)}{R(z_k)} - \left(m - n + \frac{z_k B'(z_k)}{B(z_k)}\right)\right| = 2\operatorname{Re}\frac{z_k R'(z_k)}{R(z_k)} - \left(m - n + \frac{z_k B'(z_k)}{B(z_k)}\right).
\end{aligned}$$

The last equality is valid by, for instance, [4, Lemma 3]. Now,

$$|f(z_k) - f(z_{k+1})| = \left|\frac{R(z_k)}{\overline{R(z_k)}}\frac{z_k^{n-m}}{B(z_k)} - \frac{R(z_{k+1})}{\overline{R(z_{k+1})}}\frac{z_{k+1}^{n-m}}{B(z_{k+1})}\right|, \quad k = 1, 2, 3. \tag{15}$$

Application of Lemma 1 and the Schur inequality

$$|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)| \leq 3\sqrt{3} \tag{16}$$

completes the proof of (14). The case of equality can be checked directly. □

Theorem 2. *Let a rational function R with prescribed poles $a_1, \dots, a_n$ such that $|a_k| \geq 1$ have exactly m zeros, all of them lying in the disk $|z| \leq 1$. Then, for arbitrary points z_k, $k = 1, 2$, of the unit circle $|z| = 1$, different from zeros and poles of R, either the inequality*

$$\operatorname{Re}\frac{z_k R'(z_k)}{R(z_k)} \geq \frac{1}{2}\left(m - n + 1 + \frac{z_k B'(z_k)}{B(z_k)}\right) \tag{17}$$

holds in at least one of them, or the following inequalities are valid:

$$\begin{aligned}
&\left(m - n + 1 + \frac{z_1 B'(z_1)}{B(z_1)} - 2\operatorname{Re}\frac{z_1 R'(z_1)}{R(z_1)}\right)\left(m - n + 1 + \frac{z_2 B'(z_2)}{B(z_2)} - 2\operatorname{Re}\frac{z_2 R'(z_2)}{R(z_2)}\right) \\
&\qquad \leq \left|\frac{z_1^{n-m-1}R(z_1)\overline{R(z_2)}B(z_2) - z_2^{n-m-1}R(z_2)\overline{R(z_1)}B(z_1)}{R(z_1)R(z_2)B(z_1)B(z_2)(z_1 - z_2)}\right|,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\left(m - n + 1 + \frac{z_1 B'(z_1)}{B(z_1)} - 2\operatorname{Re}\frac{z_1 R'(z_1)}{R(z_1)}\right)^{t_1^2}\left(m - n + 1 + \frac{z_2 B'(z_2)}{B(z_2)} - 2\operatorname{Re}\frac{z_2 R'(z_2)}{R(z_2)}\right)^{t_2^2} \\
&\qquad \leq \sqrt{\left|\frac{b_0}{b_m}\right|}\left|\frac{R(z_1)R(z_2)B(z_1)B(z_2)(z_1 - z_2)}{z_1^{n-m-1}R(z_1)\overline{R(z_2)}B(z_2) - z_2^{n-m-1}R(z_2)\overline{R(z_1)}B(z_1)}\right|^{2t_1t_2},
\end{aligned} \tag{19}$$

where t_1 and t_2 are arbitrary reals such that $t_1 + t_2 = 1$. Equality in (18) *and* (19) *is attained when the zeros of R belong to $|z| = 1$.*

PROOF. Consider the function

$$f(z) = \frac{R(z)}{\overline{R(1/\bar{z})}} \frac{z^{n-m+1}}{B(z)}$$

and put $g(z) = z^2/f(z)$. If at some point z_k, $k = 1, 2$, the derivative of $g(z)$ exists and this point does not belong to the boundary of the set $G_0(f)$, then by Lemma 2

$$0 \geq \frac{d|g|}{d|z|}(z_k) = 2 - |f'(z_k)|.$$

After calculations similar to those in the proof of Theorem 1, we arrive at (17). If at both points the derivative of $g(z)$ exists and the inequality reverse to (17) holds, then $\zeta_k \in \partial G_0(f)$, $k = 1, 2$; and while (12) and (13) imply (18) and (19) respectively. The points where $g(z)$ is not differentiable constitute a finite set, and the sought inequalities in these points are obtained by passage to the limit. The case of equality is checked by direct calculations. □

Theorem 3. *Let a rational function R with prescribed poles $a_1, \dots, a_n$ such that $|a_k| \geq 1$ have exactly m zeros, all of them lying in $|z| \geq 1$. Then, for arbitrary points z_k, $k = 1, 2, 3$, of the unit circle $|z| = 1$, different from zeros and poles of R, the following inequality is valid:*

$$\prod_{k=1}^{3} \left(\frac{1}{2} \left(m - n + \frac{z_k B'(z_k)}{B(z_k)} \right) - \operatorname{Re} \frac{z_k R'(z_k)}{R(z)} \right) \geq \frac{1}{24\sqrt{3}} \prod_{k=1}^{3} \left| \frac{\overline{R(z_k)}}{R(z_k)} \frac{B(z_k)}{z_k^{n-m}} - \frac{\overline{R(z_{k+1})}}{R(z_{k+1})} \frac{B(z_{k+1})}{z_{k+1}^{n-m}} \right|, \tag{20}$$

where $z_4 = z_1$. Equality in (20) *is attained when the zeros of R belong to $|z| = 1$.*

PROOF. Consider the function

$$f(z) = \frac{\overline{R(1/\bar{z})}}{R(z)} z^{m-n} B(z)$$

regular in the disk $|z| \leq 1$. It is easy to see that $f(z)$ satisfies the conditions of Lemma 1 for arbitrary different points z_k, $k = 1, 2, 3$, on the circle $|z| = 1$. Direct calculations yield

$$f'(z) = -\frac{R(z)\overline{R'(1/\bar{z})}/z^2 + R'(z)\overline{R(1/\bar{z})}}{R^2(z)} z^{m-n} B(z) + \frac{\overline{R(1/\bar{z})}}{R(z)} (z^{m-n} B(z))',$$

whence

$$\begin{aligned}
|f'(z_k)| &= \left| \frac{\overline{R(1/\overline{z_k})}}{R(z_k)} \left(z_k^{m-n} B(z_k) \right)' - \frac{R(z_k)\overline{R'(1/\overline{z_k})}/z_k^2 + R'(z_k)\overline{R(1/\overline{z_k})}}{R^2(z_k)} z_k^{m-n} B(z_k) \right| \\
&= \left| \frac{\overline{R(z_k)}}{R(z_k)} B(z_k) z_k^{m-n-1} \right| \left| \left(m - n + \frac{z_k B'(z_k)}{B(z_k)} \right) - \left(\frac{z_k R'(z_k)}{R(z_k)} + \frac{\overline{z_k R'(z_k)}}{\overline{R(z_k)}} \right) \right| \\
&= \left| \left(m - n + \frac{z_k B'(z_k)}{B(z_k)} \right) - 2 \operatorname{Re} \frac{z_k R'(z_k)}{R(z_k)} \right| = \left(m - n + \frac{z_k B'(z_k)}{B(z_k)} \right) - 2 \operatorname{Re} \frac{z_k R'(z_k)}{R(z_k)}.
\end{aligned}$$

The latter is valid by [4, Lemma 3] for instance. Application of (15), Lemma 1, and Schur's inequality (16) completes the proof of (20). The case of equality can be checked directly. □

REMARK. Observe that an analog of Theorem 2 also holds in the case when the zeros of the rational function lie in the complement to the unit disk. It is sufficient to repeat the arguments of the proof of Theorem 2 for the function

$$f(z) = \frac{\overline{R(1/\bar{z})}}{R(z)} B(z) z^{m-n+1}.$$

Henceforth

$$\Psi(\omega) = \omega + \sqrt{\omega^2 - 1}, \quad \omega \in \overline{\mathbb{C}} \setminus [-1,1],$$

is one of the branches of the inverse Joukowsky function, $\Phi(\infty) = \infty$. Speaking of the values of Ψ on $[-1,1]$, for definiteness we mean the values on the upper face of the cut.

Assuming that the points a_l are different from zeros of R and predominantly following the article [4], for the fixed number l, $1 \le l \le n$, on the open set $G = \{z : |z| < 1,\ R(z) + \overline{R(1/\bar{z})} \notin [-1,1],\ |F(z)| \ne 1\}$ consider the function

$$w = F(z) \equiv z^{n-m} \prod_{\substack{k=1\\ k\ne l}}^{n} \frac{z - a_k}{1 - z\bar{a}_k} [\Phi(R(z) + \overline{R(1/\bar{z})})]^{-1},$$

where $\xi = \Phi(\zeta)$ is the unique branch of the composite function

$$\xi = \Psi(\zeta_1) = \zeta_1 + \sqrt{\zeta_1^2 - 1}, \quad \zeta_1 = \frac{\zeta - H - L}{H - L},$$

conformally and univalently sending the exterior of the interval $\gamma := [2L, 2H]$ onto the exterior of the unit disk $|\xi| > 1$ so that $\Phi(\infty) = \infty$ and $\Phi(2L) = -1$. The function F is regular on G; moreover, F sends the point $1/\bar{a}_l$, and only it, to the origin of coordinates. Now, the circle $|\omega| = 1$ is sent to the circle $|w| = 1$ in the sense of boundary correspondence and all points of the boundary of G are mapped into the points within $|w| \ge 1$.

Introduce the standard notation $[x]^+ = \max\{x, 0\}$ and consider the linear-fractional mapping χ of the unit disk onto itself sending the points w_k, $k = 1,2,3$, into the points z_k, $k = 1,2,3$. It is easy to verify that

$$\frac{1}{|(w_1 - w_2)(w_2 - w_3)(w_3 - w_1)|} = \frac{\left|\prod_{k=1}^{3} \chi'(w_k)\right|}{|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|}.$$

Theorem 4. *For every rational function R with prescribed poles $a_1, \dots, a_n$, $|a_k| \ge 1$, $k = 1, \dots, n$, $m \ge n$, for arbitrary points z_k, $k = 1,2,3$, of the unit circle $|z| = 1$ in which $\operatorname{Re} R(z)$ is different from $L(R)$ and $H(R)$, and $1 \le l \le n$, the following inequality is valid:*

$$\prod_{k=1}^{3} \left[\frac{|\operatorname{Im}(z_k R'(z_k))|}{\sqrt{(\operatorname{Re} R(z_k) - L(R))(H(R) - \operatorname{Re} R(z_k))}} - \left|\left(z_k^{m-n} B(z_k)\right)'\right| + \frac{|a_l|^2 - 1}{|1 - \bar{a}_l z_k|^2} \right]^+$$
$$\le \left| \frac{(F(z_1) - F(z_2))(F(z_2) - F(z_3))(F(z_3) - F(z_1))}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)} \right|. \tag{21}$$

Equality in (21) *is attained in the case $R(z) = \alpha z^{m-n} B(z)$, where α is an arbitrary real.*

Proof. By Lemma 2 G consists of finitely many domains $\{\widetilde{G}\}$ with piecewise smooth boundaries. If $\widetilde{G}$ does not contain $1/\bar{a}_l$, then $F(\widetilde{G})$ lies outside the unit disk; and if $1/\bar{a}_l \in G$, then $F(z)$ conformally and univalently sends the domain $\widetilde{G}$ onto the unit disk $|w| < 1$.

By direct calculations, we obtain

$$\frac{\partial |F|}{\partial |z|} = \frac{\partial}{\partial |z|} \left| z^{n-m} \prod_{\substack{k=1\\ k\ne l}}^{n} \frac{z - a_k}{1 - z\bar{a}_k} [\Phi(R(z) + \overline{R(1/\bar{z})})]^{-1} \right|$$
$$= -\frac{\partial}{\partial |z|} \left| z^{m-n} \prod_{k=1}^{n} \frac{1 - \bar{a}_k z}{z - a_k} \right| - \frac{\partial}{\partial |z|} \left| \frac{z - a_l}{1 - \bar{a}_l z} \right| + \left| \frac{d\Phi(R(z) + \overline{R(1/\bar{z})})}{dz} \right|$$
$$= -|(z^{m-n} B(z))'| + \frac{|a_l|^2 - 1}{|1 - \bar{a}_l z|^2} + \frac{|\operatorname{Im}(zR'(z))|}{\sqrt{(\operatorname{Re} R(z) - L)(H - \operatorname{Re} R(z))}}.$$

If one of the points z_k is regular for $F(z)$ on the circle $|z| = 1$ and simultaneously $|F(\widetilde{G})|$ lies outside U, then

$$\frac{\partial |F|}{\partial |z|}(z_k) \le 0.$$

In this case the assertion of the theorem is obvious. Let all three points z_k, $k = 1, 2, 3$, lie on the boundary of $\widetilde{G}$ which contains $1/\bar{a}_l$. Applying Lemma 1 to the superposition $F^{-1} \circ \chi^{-1}$, we obtain the sought inequality. The case of equality can be checked directly. □

Introduce the function

$$w = F_1(z) = z^{n-m} \prod_{k=1}^{n} \frac{1 - z\bar{c}_k}{z - c_k} \left[\Psi\left(R\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)\right]^{-1}$$

on the open set $G = \left\{z : |z| < 1,\ R\left(\frac{1}{2}(z + 1/z)\right) \notin [-1, 1],\ |F_1(z)| \ne 1\right\}$.

Theorem 5. *For every real rational function R with prescribed finite poles $a_1, \dots, a_n$, $m \ge n$, satisfying the condition*

$$|R(x)| \le 1, \quad x \in [-1, 1],$$

for arbitrary points $x_k \in (-1, 1)$, $R^2(x_k) \ne 1$, $k = 1, 2, 3$, and $1 \le l \le n$, the following inequality is valid:

$$\prod_{k=1}^{3} \left[\frac{|R'(x_k)|\sqrt{1 - x_k^2}}{\sqrt{1 - R^2(x_k)}} - m + n - B_n(x_k) + \operatorname{Re} \frac{\sqrt{a_l^2 - 1}}{a_l - x_k}\right]^{+}$$

$$\le \left|\frac{(\widetilde{F}_1(\Psi(x_1)) - \widetilde{F}_1(\Psi(x_2)))(\widetilde{F}_1(\Psi(x_2)) - \widetilde{F}_1(\Psi(x_3)))(\widetilde{F}_1(\Psi(x_3)) - \widetilde{F}_1(\Psi(x_1)))}{(\Psi(x_1) - \Psi(x_2))(\Psi(x_2) - \Psi(z_3))(\Psi(x_3) - \Psi(x_1))}\right|, \tag{22}$$

where $\widetilde{F}_1(z) = F_1(z)(z - c_l)(1 - z\bar{c}_l)$. Equality in (22) is attained in the case $R(z) = T_n^r(z)$.

Proof. By Lemma 2, G consists of finitely many domains $\{\widetilde{G}\}$ with piecewise smooth boundaries. If $\widetilde{G}$ does not contain c_l, then $\widetilde{F}_1(\widetilde{G})$ lies outside the unit disk; and if $c_l \in G$, then $\widetilde{F}_1(z)$ conformally and univalently sends $\widetilde{G}$ onto the unit disk. If one of the points $z_k = \Psi(x_k)$ is regular for $\widetilde{F}_1(z)$ on the circle $|z| = 1$ and simultaneously $|\widetilde{F}_1(\widetilde{G})|$ lies outside the unit disk, then

$$\frac{\partial |\widetilde{F}_1|}{\partial |z|}(z_k) \le 0.$$

In this case the assertion of the theorem is obvious.

Let all three points $z_k = \Psi(x_k)$, $k = 1, 2, 3$, lie on the boundary of $\widetilde{G}$ which contains the origin of coordinates. Direct calculations and application of Lemma 1 to $\widetilde{F}_1^{-1} \circ \chi^{-1}$ yield the sought inequality.

The case of equality follows from the fact that if $R(z) = T_n^r(z)$, then $\widetilde{F}_1(z) \equiv (z - c_l)(1 - z\bar{c}_l)$.

As regards $zF(z)$ and $zF_1(z)$, the analogs of Theorem 2 follow from Lemma 3 for rational functions satisfying the conditions of Theorems 4 and 5. □

Now, applying Theorem 5 to the functions

$$R_1(z) = 2(1 - z^2)R^2(z) - 1, \quad R_2(z) = (z + 1)R^2(z) - 1, \quad R_3(z) = (1 - z)R^2(z) - 1$$

and, instead of F_1, taking the functions

$$F_2(z) = z^{2n-2m-2} \prod_{k=1}^{n} \left(\frac{1 - z\bar{c}_k}{z - c_k}\right)^2 \left[\Psi\left(R_1\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)\right]^{-1},$$

$$F_3(z) = z^{2n-2m-1} \prod_{k=1}^{n} \left(\frac{1 - z\bar{c}_k}{z - c_k}\right)^2 \left[\Psi\left(R_2\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)\right]^{-1},$$

$$F_4(z) = z^{2n-2m-1} \prod_{k=1}^{n} \left(\frac{1 - z\bar{c}_k}{z - c_k}\right)^2 \left[\Psi\left(R_3\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)\right]^{-1}$$

considered on the corresponding open sets G, we obtain the following corollaries. The cases of equality ensue from (8)–(11).

Corollary 1. *For every real rational function R with prescribed finite poles $a_1, \dots, a_n$, $m \ge n$, satisfying the condition*

$$|R(z)|\sqrt{1-x^2} \le 1, \quad x \in [-1,1],$$

for arbitrary points $x_k \in [-1,1]$, $R^2(x_k)\big(1-x_k^2\big) \ne 1$, $k=1,2,3$, and $1 \le l \le n$, the following inequality is valid:

$$\prod_{k=1}^{3}\left[\frac{\big|x_kR(x_k)-\big(1-x_k^2\big)R'(x_k)\big|}{\sqrt{1-\big(1-x_k^2\big)R^2(x_k)}} - m + n - 1 - B_n(x_k) + \frac{1}{2}\operatorname{Re}\frac{\sqrt{a_l^2-1}}{a_l-x_k}\right]^+$$
$$\le \frac{1}{8}\left|\frac{(\widetilde{F}_2(\Psi(x_1))-\widetilde{F}_2(\Psi(x_2)))(\widetilde{F}_2(\Psi(x_2))-\widetilde{F}_2(\Psi(x_3)))(\widetilde{F}_2(\Psi(x_3))-\widetilde{F}_2(\Psi(x_1)))}{(\Psi(x_1)-\Psi(x_2))(\Psi(x_2)-\Psi(z_3))(\Psi(x_3)-\Psi(x_1))}\right|, \tag{23}$$

where $\widetilde{F}_2(z) = F_2(z)(z-c_l)(1-z\bar{c}_l)$. Equality in (23) is attained if $R(z) = U_n^r(z)$ or $R(z) = \widetilde{U}_n^r(z)$.

Corollary 2. *For every real rational function R with prescribed finite poles $a_1, \dots, a_n$, $m \ge n$, satisfying the condition*

$$|R(x)|\sqrt{\frac{1+x}{2}} \le 1, \quad x \in [-1,1],$$

for arbitrary points $x_k \in [-1,1]$, $(1+x_k)R^2(x_k) \ne 2$, $k=1,2,3$, and $1 \le l \le n$, the following inequality is valid:

$$\prod_{k=1}^{3}\left[\frac{|R(x_k)+2(1+x_k)R'(x_k)|\sqrt{1-x_k}}{\sqrt{2-(1+x_k)R^2(x_k)}} - 2m + 2n - 1 - 2B_n(x_k) + \operatorname{Re}\frac{\sqrt{a_l^2-1}}{a_l-x_k}\right]^+$$
$$\le \left|\frac{(\widetilde{F}_3(\Psi(x_1))-\widetilde{F}_3(\Psi(x_2)))(\widetilde{F}_3(\Psi(x_2))-\widetilde{F}_3(\Psi(x_3)))(\widetilde{F}_3(\Psi(x_3))-\widetilde{F}_3(\Psi(x_1)))}{(\Psi(x_1)-\Psi(x_2))(\Psi(x_2)-\Psi(z_3))(\Psi(x_3)-\Psi(x_1))}\right|, \tag{24}$$

where $\widetilde{F}_3(z) = F_3(z)(z-c_l)(1-z\bar{c}_l)$. Equality in (24) is attained if $R(z) = V_n^r(z)$.

Corollary 3. *For a real rational function R with prescribed finite poles $a_1, \dots, a_n$, $m \ge n$, satisfying the condition*

$$|R(x)|\sqrt{\frac{1-x}{2}} \le 1, \quad x \in [-1,1],$$

for arbitrary points $x_k \in [-1,1]$, $(1-x_k)R^2(x_k) \ne 2$, $k=1,2,3$, and $1 \le l \le n$, the following inequality is valid:

$$\prod_{k=1}^{3}\left[\frac{|R(x_k)+2(1-x_k)R'(x_k)|\sqrt{1+x_k}}{\sqrt{2-(1-x_k)R^2(x_k)}} - 2m + 2n - 1 - 2B_n(x_k) + \operatorname{Re}\frac{\sqrt{a_l^2-1}}{a_l-x_k}\right]^+$$
$$\le \left|\frac{(\widetilde{F}_4(\Psi(x_1))-\widetilde{F}_4(\Psi(x_2)))(\widetilde{F}_4(\Psi(x_2))-\widetilde{F}_4(\Psi(x_3)))(\widetilde{F}_4(\Psi(x_3))-\widetilde{F}_4(\Psi(x_1)))}{(\Psi(x_1)-\Psi(x_2))(\Psi(x_2)-\Psi(z_3))(\Psi(x_3)-\Psi(x_1))}\right|, \tag{25}$$

where $\widetilde{F}_4(z) = F_4(z)(z-c_l)/(1-z\bar{c}_l)$. Equality in (25) is attained if $R(z) = W_n^r(z)$.

Remark. The assertions complementing the inequalities in Corollaries 1–3 can also be obtained by the same arguments as in the proof of Theorem 6, applied to the following functions (cf. [11, 12, 16]):

$$F_5(z) = z^{n-m-1}\prod_{k=1}^{n}\frac{1-z\bar{c}_k}{z-c_k}\left[\Psi\left(\frac{i}{2}\left(z-\frac{1}{z}\right)R\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\right)\right]^{-1},$$

$$F_6(z) = z^{n-m}\prod_{k=1}^{2n}\frac{1-z\bar{\hat{c}}_k}{z-\hat{c}_k}\left[\Psi\left(\frac{1}{2}\left(z+\frac{1}{z}\right)R\left(\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\right)\right)\right]^{-1},$$

where $\hat{c}_k = \sqrt{c_k}$ for $k = 1, \dots, n$ and $\hat{c}_k = -\hat{c}_{k-n}$ for $k = n+1, \dots, 2n$.

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