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COMPRESSED ZERO-DIVISOR GRAPHS OF FINITE ASSOCIATIVE RINGS

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Abstract: We study the compressed zero-divisor graph of a finite associative ring R. In particular, we describe commutative finite rings with compressed zero-divisor graphs of order 2. Moreover, we find all graphs of order 3 that are the compressed zero-divisor graphs of some finite rings.

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1. Introduction

All rings under consideration in the article are associative. Throughout $F = GF(q)$ is a finite ring, $q = p^r$, with p a prime, $\mathbb{Z}_n$ is the residue ring modulo n, and $|M|$ is the size of a finite set M.

Let R be an associative finite ring. Given $x \in R$, put $l(x) = \{a \in R;\ ax = 0\}$ and $r(x) = \{a \in R;\ xa = 0\}$. Let $D(R)$ be the set of (one- and two-sided) zero divisors of R and let $D(R)^* = D(R) \setminus \{0\}$. Put $\mathrm{Ann}(R) = \{a \in R;\ aR = Ra = (0)\}$ and $r(R) = \{a \in R;\ Ra = (0)\}$, $l(R) = \{a \in R;\ aR = (0)\}$. By a *local ring* we mean a finite unital ring R such that the quotient ring $R/J(R)$ is a field. The ring of all $n \times n$-matrices with entries in R will be denoted by $M_n(R)$.

An element $e \in R$ is called an *idempotent* of R if $e = e^2$. A system of nonzero idempotents $e_1, \dots, e_k$ $(k \geq 2)$ of a ring R is called *orthogonal* if $e_ie_j = e_je_i = 0$ for every pair of distinct numbers $i, j \in \{1, 2, \dots, k\}$. Furthermore, let R be an arbitrary ring (possibly, without unity) and let e be a nontrivial idempotent of R, i.e. an idempotent different from the unity (if it exists) and zero. Put

$$eRe = \{ere;\ r \in R\}, \quad eR(1-e) = \{er - ere;\ r \in R\},$$

$$(1-e)Re = \{re - ere;\ r \in R\}, \quad (1-e)R(1-e) = \{r - re - er + ere;\ r \in R\}.$$

Then the additive ring of R admits the following expansion, called the *two-sided Pierce decomposition* (see [1]):

$$R = eRe \dotplus eR(1-e) \dotplus (1-e)Re \dotplus (1-e)R(1-e).$$

Assume that the additive group of a ring R decomposes into the direct sum of its additive subgroups A_i, where $i = 1, \dots, n$ and $n \geq 2$, i.e., $R = A_1 \dotplus \dots \dotplus A_n$. If all subgroups A_i are two-sided ideals of R then the ring R is called *decomposable* and we write $R = A_1 \oplus \dots \oplus A_n$.

A graph G is called *connected* if each pair of its distinct points is joined by a simple chain. A graph is called *finite* if its vertices and the edge sets are finite. The *order* of a finite graph is the number of its vertices. A *complete n-vertex graph* K_n is a graph (without edges and multiple edges) with n vertices in which each vertex is adjacent to any other vertex in this graph. A *null-graph* E_m is a graph consisting of m isolated vertices. A *bipartite graph* G is a graph whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 so that each edge in G joins vertices from different subsets. If a bipartite graph G contains all edges joining each vertex of a set V_1 to every vertex in V_2 then this graph is called *complete bipartite*. Complete bipartite graphs are denoted by $K_{n,m}$, where $n = |V_1|$ and $m = |V_2|$.

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The *zero-divisor graph* $\Gamma(R)$ of a ring R is the graph whose vertices are nonzero zero divisors of the ring (one- and two-sided), and two different vertices x and y are joined by an edge if and only if $xy = 0$ or $yx = 0$.

These graphs were defined by Anderson and Livingston in [2]. To describe the rings whose zero-divisor graph satisfies a certain condition it has become one of the directions of investigations in this area. For example, full description was obtained of rings with planar zero-divisor graphs [3–6], rings having zero-divisor Euler graphs [7], and finite rings with complete bipartite graphs [8]. In [9, 10], description was obtained of the varieties of rings in which finite rings with isomorphic zero-divisor graphs are isomorphic to each other.

The geometric depiction of the zero-divisor graph is rather complicated even for rings of small order. Therefore, it is necessary to partition the vertex set of the graph into cosets so that the impression of the structure of the graph as a whole be preserved. In [11, 12], Bloomfield and Wickham proposed some method for solving this problem for commutative rings. In this article, we extend their approach by generalizing it to the noncommutative case.

Introduce the equivalence on $D(R)^*$ as follows:

$$\forall\, x, y \in D(R)^* \quad x \sim y \Leftrightarrow l(x) \cup r(x) = l(y) \cup r(y).$$

Let $[x]$ be a coset of $x \in D(R)^*$. If $a \in [x]$, $b \in [y]$, and $x, y \in D(R)^*$; then, obviously, $ab = 0$ or $ba = 0$ if and only if $xy = 0$ or $yx = 0$.

Proposition 1. *Let R be an arbitrary ring and let $x \in D(R)^*$. Then*
(1) *if $x^2 = 0$ then $yz = 0$ or $zy = 0$ for any $y, z \in [x]$;*
(2) *if $x^2 \neq 0$ then $yz \neq 0$ and $zy \neq 0$ for $y, z \in [x]$.*

Proof. Let $x^2 = 0$. Then $x \in l(x) \cup r(x)$. If $y \in [x]$ then $x \in l(x) \cup r(x) = l(y) \cup r(y)$, and so $xy = 0$ or $yx = 0$. Therefore,

$$y \in l(x) \cup r(x) = l(y) \cup r(y),$$

and so $y^2 = 0$. Further, let $y, z \in [x]$. Then

$$l(y) \cup r(y) = l(z) \cup r(z),$$

and since $y^2 = 0$, we have $y \in l(z) \cup r(z)$. Consequently, $yz = 0$ or $zy = 0$.

Assume that $x^2 \neq 0$ and $y, z \in [x]$. Suppose the contrary: $yz = 0$ or $zy = 0$. Then

$$y \in l(z) \cup r(z) = l(y) \cup r(y)$$

and $y^2 = 0$. Since $y^2 = 0$; therefore, $y \in l(y) \cup r(y) = l(x) \cup r(x)$, and so $xy = 0$ or $yx = 0$. Hence, $x \in l(y) \cup r(y) = l(x) \cup r(x)$, and so $x^2 = 0$; a contradiction. $\square$

Given a ring R, denote by $\Gamma_\sim(R)$ the graph with vertex set $\{[x];\ x \in D(R)^*\}$ whose every two (not necessarily distinct) vertices $[x]$ and $[y]$ are joined by an edge if and only if $xy = 0$ or $yx = 0$. We will refer to $\Gamma_\sim(R)$ as the *compressed zero-divisor graph* of R.

Proposition 1 implies that all vertices in $\Gamma_\sim(R)$ fall into two types. If $x^2 = 0$ then $[x]$ is a vertex with a loop; if $x^2 \neq 0$ then $[x]$ is a vertex without any loop. Knowing the size of each coset $[x]$, it is always possible to pass from the compressed zero-divisor graph to the conventional zero-divisor graph. Therefore, in studying the properties of the compressed zero-divisor graph $\Gamma_\sim(R)$, we can use the properties of Γ_R. Moreover, the nilpotent elements of nilpotency index 2 are distinguished by a loop in the compressed graph. In contrast to $\Gamma(R)$, the compressed graph $\Gamma_\sim(R)$ is depicted more compactly and transparently. For example, we can estimate the maximal size number of elements in any system of pairwise orthogonal idempotents of a ring R from its compressed zero-divisor graph since, obviously, idempotents generate cosets without any loop. Moreover, the pairwise distinct idempotents belong to different cosets pairwise adjacent to each other. The form of the compressed zero-divisor graph makes it easy to determine whether the annihilator of the ring is zero because each nonzero element of the annihilator generates a coset adjacent to all remaining vertices in the compressed zero-divisor graph of this ring.

The present article deals with associative finite rings whose compressed zero-divisor graphs have at most three vertices. Namely, we find the graphs containing at most three vertices that can be realized as the compressed zero-divisor graphs of some finite associative ring. We also describe the finite commutative rings whose compressed zero-divisor graph consists of two vertices one of which is with a loop and the other is without any loop.

2. Rings with Compressed Zero-Divisor Graphs of Order 1 or 2

It was proved in [2, 13] that the zero-divisor graph of a finite associative ring is connected. Consequently, the compressed zero-divisor graph of a finite associative ring is connected as well.

Theorem 1. *Let R be a finite ring. The graph $\Gamma_\sim(R)$ has order 1 if and only if one of the following conditions holds:*

(1) *R is a nilpotent ring with zero multiplication;*

(2) *R is a local ring and $J(R)^2 = (0)$.*

Proof. Let the graph $\Gamma_\sim(R)$ of a finite ring R consist of a single vertex $[a]$. If $[a]$ is without any loop; then, since the graph $\Gamma(R)$ is connected, a is a unique nonzero zero divisor. In this case $a^2 = 0$, and so $[a]$ is with a loop; a contradiction.

Thus, let $[a]$ be a vertex with a loop. Then the two conditions are fulfilled in R:

(1) $xy = 0$ or $yx = 0$ for all $x, y \in D(R)$;

(2) $x^2 = 0$ for every $x \in D(R)$.

Since all elements in a finite ring without unity are zero divisors (see [1]), R either has a unity or is a nilpotent ring. If R has a unity then R contains no orthogonal idempotents since the squares of all zero divisors in R are zero. Consequently, in this case, R is a local ring and $D(R) = J(R)$ [1]. If R is nilpotent then $R = D(R) = J(R)$. Prove that both in the first and second cases, $xy = yx = 0$ for all $x, y \in D(R)^*$. Suppose that $xy = 0$ and $yx \neq 0$ for some $x, y \in D(R)$. Then $x + y \in D(R)$ since $D(R) = J(R)$, and so $(x+y)^2 = 0$. This gives $0 = (x+y)^2 = x^2+xy+yx+y^2 = yx \neq 0$; a contradiction. Hence, $D(R)^2 = (0)$. $\square$

Proposition 2. *Suppose that R is a finite ring and the graph $\Gamma_\sim(R)$ consists of two adjacent vertices one of which is with a loop and the other is without any loop. Then all nilpotent elements of R have nilpotency index at most 3.*

Proof. Assume that $[a]$ is a vertex of the graph $\Gamma_\sim(R)$ without any loop, and $[b]$ is a vertex with a loop, $x \in R$ is a nilpotent element. If $x^{2n} = 0$ and $x^{2n-1} \neq 0$ for some $n \geq 1$ then $x^n \in [b]$. Consequently, $x \cdot x^n = x^{n+1} = 0$, and so $n+1 \geq 2n$, $n = 1$, $x^2 = 0$. If $x^{2n+1} = 0$, $x^{2n} \neq 0$ for some $n \geq 1$ then $x^{n+1} \in [b]$. Consequently, $x \cdot x^{n+1} = x^{n+2} = 0$; hence, $n + 2 \geq 2n + 1$, $n = 1$, $x^3 = 0$. $\square$

The following theorem gives full description of the finite commutative rings whose compressed zero-divisor graphs have order 2.

Theorem 2. *Let R be a finite commutative ring. The graph $\Gamma_\sim(R)$ has order 2 if and only if one of the conditions holds:*

(1) *$R \cong GF(q_1) \oplus GF(q_2)$, where $q_i = p_i^{s_i}$, with p_i a prime, and $s_i \geq 1$, $i = 1, 2$;*

(2) *$R \cong GF(q) \oplus B$, where $B^2 = (0)$, $q = p^s$, with p a prime, and $s \geq 1$;*

(3) *R is a nilpotent ring, and $R = A \oplus B$, where A is an indecomposable ring, $|A| = p^\alpha$, $\alpha \geq 1$, $A^3 = (0)$, $B^2 = (0)$, $|B| = m$, with p a prime, and $m \geq 0$, while $(\forall x, y \in A \setminus \operatorname{Ann}(A))(xy \neq 0)$, where either $pR^2 = (0)$ or $(\forall x \in R)(px = 0 \to x^2 = 0)$;*

(4) *R is a local ring such that $J(R)$ satisfies (3).*

Proof. Let R be an arbitrary finite commutative ring whose graph $\Gamma_\sim(R)$ has order 2. Consider the variants of the geometric pictures of $\Gamma_\sim(R)$ presented in Figs. 1–3.

Note that the variant in Fig. 1 is impossible since, in this case, $a \sim b$ and so $[a] = [b]$.

Consider the variant in Fig. 2. Then $\Gamma(R) = K_{n,m}$ is a complete bipartite graph. Since both vertices of the compressed zero-divisor graph $\Gamma_\sim(R)$ are without any loop, R has no nilpotent elements. Hence, R is isomorphic to a direct sum of finite fields [1]. But $\Gamma(R)$ is a bipartite graph. This means that R is isomorphic to a direct sum of two finite fields; i.e., we obtain a ring of the first type from the statement of the theorem.

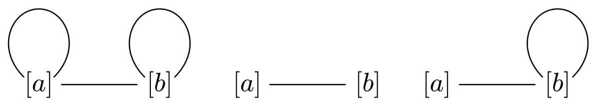

Fig. 1 **Fig. 2** **Fig. 3**

Consider the variant in Fig. 3. Note that, in this case, R contains no nonzero orthogonal idempotents. Indeed, if e_1 and e_2 are nonzero orthogonal idempotents of R then $e_1, e_2 \in [a]$. But then $e_1e_2 \neq 0$, which contradicts the orthogonality of e_1 and e_2.

If R is a unital ring then R is a local ring (see [1]); i.e., $R/J(R) \cong GF(q)$ for some $q = p^s$, with p a prime, and $s \geq 1$. Then $\Gamma_\sim(R) = \Gamma_\sim(J(R))$.

If R is a ring without unity and $R \neq J(R)$ then R contains a principal nonzero idempotent e and $R = eR \oplus (1-e)R$. Since $e^2 = e \neq 0$ and $e \cdot er = er \neq 0$ for every nonzero element $er \in eR$; therefore, $e \in [a]$ and all nonzero elements of eR belong to $[a]$. Therefore, the subring eR has no nonzero zero divisors and $eR \cong GF(q)$ for some $q = p^s$, with p a prime, and $s \geq 1$. Furthermore, since $e \cdot (1-e)R = (0)$, i.e., the vertex e is adjacent to the nonzero elements of the ideal $(1-e)R$, the nonzero elements of $(1-e)R$ belong to $[b]$ and $((1-e)R)^2 = (0)$. Thus, $R \cong GF(q) \oplus B$, where $B \trianglelefteq R$ and $B^2 = (0)$.

So, consider the case when R is a nilpotent ring ($R = J(R)$). By Proposition 2, $x^3 = 0$ for all $x \in R$. Note that $[b] \cup \{0\} = \operatorname{Ann}(R)$. Take $u, v \in [a]$. Then $uv \neq 0$, $u^2 \neq 0, v^2 \neq 0$. Since $u^3 = v^3 = 0$, we have $u^2, v^2 \in [b]$. Consequently, $u^2v = uv^2 = 0$. Thus, $(uv)^2 = 0$, i.e., $uv \in [b]$. Hence, $R^2 \subseteq [b] \cup \{0\}$, and so $R^3 = (0)$ and $[a] \subseteq R \backslash R^2$.

Let $|R| = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$, where $p_1, \dots, p_s$ are different primes and $\beta_1 \geq 1, \dots, \beta_s \geq 1$. Then

$$R = R_1 \oplus R_2 \oplus \dots \oplus R_s,$$

where $|R_i| = p_i^{\beta_i}$, $1 \leq i \leq s$. If $x \in R_i$ and $y \in R_j$, where $i \neq j$, then $xy = 0$. Hence, $[a] \subseteq R_k$ for some $k \in \{1, 2, \dots, s\}$. Without loss of generality, put $[a] \subseteq R_1$. In this case, the rings $R_2, \dots, R_s$ have compressed zero-divisor graphs consisting of a single vertex with a loop. By Theorem 1, R_i is a ring with zero multiplication for $i \geq 2$. Note that if R_1 in turn splits into a direct sum of rings, for example, $R_1 = S_1 \oplus \dots \oplus S_t$, where S_i are indecomposable rings, $1 \leq i \leq t$, and $t \geq 2$; then, arguing similarly, we conclude that $[a]$ is contained in one of these indecomposable rings, for example, in S_1. Then $S_2 \oplus \dots \oplus S_t \subseteq [b] \cup \{0\} = \operatorname{Ann}(R)$. These arguments show that $R = A \oplus B$, where A is an indecomposable ring, $|A| = p^\alpha$, $\alpha \geq 1$, $B^2 = (0)$, $|B| = m$, $p = p_1$ is a prime, $m \geq 1$.

Suppose that there is $u \in [a]$ such that $pu = 0$. Then $u \cdot pR = (0)$, i.e., $pR \subseteq \operatorname{Ann}(R)$. Hence $pR \cdot R = (0)$. Thus, either all elements of additive order p belong to $\operatorname{Ann}(R)$ or $pR^2 = (0)$.

The converse is obvious. $\square$

Proposition 3. *Let A be a finite-dimensional commutative indecomposable nilpotent $\mathbb{Z}_p$-algebra* (*with p a prime*). *The graph $\Gamma_\sim(A)$ consists of two adjacent vertices one of which is with a loop and the other is without any loop if and only if $\operatorname{Ann}(A) = A^2$ and A has a basis $\{a_1, \dots, a_s, b_1, \dots, b_t\}$ such that $b_i \in \operatorname{Ann}(A)$ for all i and $a_ia_j \neq 0$ for all i and j.*

Proof. Suppose that A is a finite-dimensional commutative indecomposable $\mathbb{Z}_p$-algebra, while $\Gamma_\sim(A)$ consists of two adjacent vertices one of which is with a loop and the other, without any loop (Fig. 3). By Theorem 2, $A^3 = (0)$. Complement some basis $\{b_1, \dots, b_t\}$ for the subspace A^2 to a basis

$\{b_1, \dots, b_t, d_1, \dots, d_k\}$ for the subspace $\mathrm{Ann}(A)$ and then complement the obtained basis to some basis $\{a_1, \dots, a_s, b_1, \dots, b_t, d_1, \dots, d_k\}$ for the whole algebra A. Then $a_i a_j \neq 0$ and $a_i a_j \in A^2$ for all i and j, and $a_i b_j = a_i d_j = b_i d_j = b_i b_j = d_i d_j = 0$ for all i and j. The subalgebra B generated by all elements a_i and b_j and also the subalgebra D generated by all d_i's are ideals of A. Moreover, $A = B \oplus D$. This contradicts the indecomposability of A. Hence, $D = (0)$ and $\{a_1, \dots, a_s, b_1, \dots, b_t\}$ is a desired basis for A.

The converse is obvious. $\square$

Theorem 2 is formulated for commutative rings. In the general case, we have

Proposition 4. *Let R be a finite ring. If the graph $\Gamma_\sim(R)$ has order 2 then one of the following conditions holds:*

(1) *$R \cong GF(q_1) \oplus GF(q_2)$, where $q_i = p_i^{s_i}$, with p_i a prime, $s_i \geq 1$, and $i = 1, 2$;*

(2) *R is a nilpotent ring;*

(3) *R is a local ring and $\Gamma_\sim(R) = \Gamma_\sim(J(R))$;*

(4) *R is a nonnilpotent ring without unity and $R/J(R)$ is a field; then $J^2(R) \subseteq \mathrm{Ann}(R)$ and, in particular, $J(R)^3 = (0)$.*

Proof. Suppose that R is a finite ring and $\Gamma_\sim(R)$ has order 2. The proof of Theorem 2 implies that $\Gamma_\sim(R)$ cannot have two loops. If $\Gamma_\sim(R)$ has no loops then R is isomorphic to a direct sum of two fields (the commutativity of R was not used in proving this fact). Thus, we may assume that $\Gamma_\sim(R)$ contains exactly one loop; i.e., its geometric picture is presented in Fig. 3. Let $[a]$ be a vertex without any loop and let $[b]$ be a vertex with a loop in $\Gamma_\sim(R)$. Note that R cannot contain orthogonal idempotents (see the proof of Theorem 2). Therefore, if R contains a unity then R is local.

Let R be a ring without unity and let $R \neq J(R)$. Then R contains a principal idempotent e whose image is the unity in the quotient ring $R/J(R)$; moreover, e does not split into a sum of orthogonal idempotents; i.e., $R/J(R)$ is a finite field [1]. Consider the Pierce decomposition of R:

$$R = eRe \dotplus eR(1-e) \dotplus (1-e)Re \dotplus (1-e)R(1-e).$$

Prove that $J(eRe) = (0)$. Suppose the contrary. Then there exists a nonzero $j \in J(eRe)$ such that $j^2 = 0$. Clearly, $[e] = [a]$ and $[j] = [b]$. But the vertex $[e]$ is not adjacent with $[j]$; a contradiction. Therefore, $J(eRe) = (0)$. Moreover, $eRe \cong GF(q)$ for some $q \geq 2$ since R has no orthogonal idempotents.

Prove that $[b] \cup \{0\} = l(R) \cup r(R)$. Since $d^2 = 0$ for every $d \in l(R) \cup r(R)$, we have $l(R) \cup r(R) \subseteq [b] \cup \{0\}$. Suppose that the reverse inclusion fails. Then there exists $z \in [b]$ such that $zx = 0$, $yz = 0$, $xz \neq 0$, and $zy \neq 0$ for some $x, y \in R$, i.e., $z \notin l(R) \cup r(R)$. Note also that $x + y \neq 0$. Since R has no unity, $x + y$ is a zero divisor. Hence, $zy = z(x + y) = 0$ or $xz = (x + y)z = 0$; a contradiction. Thus, $[b] \cup \{0\} = l(R) \cup r(R)$. Consequently, for each element d of the coset $[b]$, either $dR = (0)$ or $Rd = (0)$. Since the elements of $((1-e)Re)^* \dotplus (1-e)R(1-e)$, where $((1-e)Re)^* = (1-e)Re \setminus \{0\}$, belong to $r(e)$, they all lie in $[b]$. Moreover,

$$((1-e)Re)^* \dotplus (1-e)R(1-e) \subseteq r(R)$$

since $[b] = l(R) \cup r(R)$. It is proved similarly that

$$((1-e)Re)^* \subseteq r(R), \quad (eR(1-e))^* \subseteq l(R),$$
$$(eR(1-e))^* \dotplus (1-e)R(1-e) \subseteq l(R).$$

Consequently, $(1-e)R(1-e) \subseteq l(R) \cap r(R) = \mathrm{Ann}(R)$.

Prove that $\mathrm{Ann}(R) = (1-e)R(1-e)$, $r(R) = (1-e)Re \dotplus (1-e)R(1-e)$ and $l(R) = eR(1-e) \dotplus (1-e)R(1-e)$. Let $r \in \mathrm{Ann}(R)$, $r = r_1 + r_2 + r_3 + r_4$, where $r_1 \in eRe$, $r_2 \in eR(1-e)$, $r_3 \in (1-e)Re$, $r_4 \in (1-e)R(1-e)$. Since $re = 0$, we have $(r_1 + r_3)e = 0$. Multiplying the last equality by e, we conclude that $er_1 = r_1 = 0$, i.e., $r = r_2 + r_3 + r_4$. Since $er = 0$, we have $er_2 = r_2 = 0$ and $r = r_3 + r_4$. Finally, $re = r_3 e = r_3 = 0$, i.e., $r = r_4 \in (1-e)R(1-e)$. Thus, $\mathrm{Ann}(R) = (1-e)R(1-e)$. Take an arbitrary

$r = r_1 + r_2 + r_3 + r_4 \in l(R)$, where $r_1 \in eRe$, $r_2 \in eR(1-e)$, $r_3 \in (1-e)Re$, and $r_4 \in (1-e)R(1-e)$. Then $re = 0$, i.e., $r_1 + r_3 = (r_1 + r_3)e = 0$ and $r = r_2 + r_4 \in eR(1-e) \dotplus (1-e)R(1-e)$. Thus,

$$l(R) = eR(1-e) \dotplus (1-e)R(1-e).$$

It is proved similarly that $r(R) = (1-e)Re \dotplus (1-e)R(1-e)$.

Furthermore, $J(R) = eR(1-e) \dotplus (1-e)Re \dotplus (1-e)R(1-e)$ (see [1]). Therefore,

$$J(R)^2 = (eR(1-e) \dotplus (1-e)Re \dotplus (1-e)R(1-e))^2$$
$$= (eR(1-e) \dotplus (1-e)Re)^2 = (1-e)Re \cdot eR(1-e) \subseteq (1-e)R(1-e) = \mathrm{Ann}(R).$$

Consequently, $J(R)^2 \cdot R = R \cdot J(R)^2 = (0)$. In particular, $J(R)^3 = (0)$. □

3. Rings with Compressed Zero-Divisor Graphs of Order 3

Proposition 5. *Suppose that the graph $\Gamma_\sim(R)$ of a finite ring R is complete and also loops are possible and the number of vertices in $\Gamma_\sim(R)$ is more than 2. Then R is a noncommutative ring, $\Gamma_\sim(R)$ contains exactly one vertex with a loop, and the ring R satisfies one of the conditions:*

(1) *R is a nilpotent ring;*

(2) *R is a ring without unity and without orthogonal idempotents;*

(3) *R is a ring with unity in which any system of orthogonal idempotents contains at most two idempotents.*

Proof. Suppose that the graph $\Gamma_\sim(R)$ of a finite ring R is complete with possible loops. Note that if two vertices $[a]$ and $[b]$ have loops then $a \sim b$; a contradiction. Thus, there can be at most one loop. If there are no loops at all then R has no nilpotent elements; i.e., R is a direct sum of finite fields. Clearly, in this case the graph of R can be complete if and only if R is a direct sum of two fields and $\Gamma_\sim(R)$ contains exactly two vertices; the so-obtained contradiction proves that $\Gamma_\sim(R)$ contains exactly one vertex with a loop. Let $[b], [a_1], [a_2], \dots, [a_n]$ be all vertices in $\Gamma_\sim(R)$, where only $[b]$ has a loop and $n \geq 2$.

Prove that R is not commutative. Suppose the contrary: let R be a commutative ring. Note that $a_1 + a_2$ is a zero divisor since $(a_1 + a_2)b = 0$. But $[a_1 + a_2]$ is adjacent neither to $[a_1]$ nor $[a_2]$ since $a_i^2 \neq 0$ for all i. This is possible only in the two cases: Firstly, $a_1 + a_2$ belongs to one of the cosets $[a_i]$, $i = 1, 2$, and this coset is a singleton. But this is impossible since in this case $a_1 = 0$ or $a_2 = 0$. Secondly, if $a_1 + a_2 = 0$ then $a_1 = -a_2$. Since the vertices $[a_1]$ and $[a_2]$ are adjacent, $a_1^2 = a_2^2 = 0$; a contradiction. Thus, R is a noncommutative ring.

Let $R \neq J(R)$. Observe that all vertices $[a_1^i]$ $(i \geq 1,\ a_1^i \neq 0)$ are adjacent to the vertices $[b], [a_2], \dots, [a_n]$. If a_1 is a nilpotent element; then, starting from some number i_0, the powers of $a_1^{i_0}$ lie in $[b]$. It is possible that not all powers of a_1 are nonzero, i.e., some power of the element a_1 is an idempotent by the finiteness of R (cp. [1]), and this idempotent is different from the unity; otherwise, a_1 is invertible and is not a zero divisor of R. Thus, we may assume that each of $a_1, a_2, \dots, a_n$ is either a nilpotent element or an idempotent different from the unity.

Let $e_1, e_2, \dots, e_t$ be a system of pairwise orthogonal idempotents of R; assume also that $t \geq 3$. The vertices $[e_1 + e_2]$, $[e_1]$, and $[e_2]$ are distinct and adjacent to $[e_3]$ but $[e_1 + e_2]$ is adjacent neither to $[e_1]$ nor $[e_2]$; a contradiction. Hence, R contains at most two orthogonal idempotents.

Consider the case when R has no unity. Let e be a principal idempotent whose image is the unity of the quotient ring $R/J(R)$, where $e = e_1 + e_2$ is a decomposition of e into a sum of orthogonal idempotents (cp. [1]). Since R has no unity, all its elements are zero divisors (cp. [1]). But the vertex $[e]$ is adjacent neither to $[e_1]$ nor $[e_2]$, which is impossible. Thus, e does not split into a sum of orthogonal idempotents. □

Theorem 3. *Let R be a finite ring such that the compressed zero-divisor graph $\Gamma_\sim(R)$ has exactly three vertices. Then the compressed zero-divisor graph is isomorphic to one of the graphs with loops depicted in Figs.* 4(3) *and* 4(9).

Proof. The possible variants of the geometric depictions of $\Gamma_\sim(R)$ are given in Fig. 4.

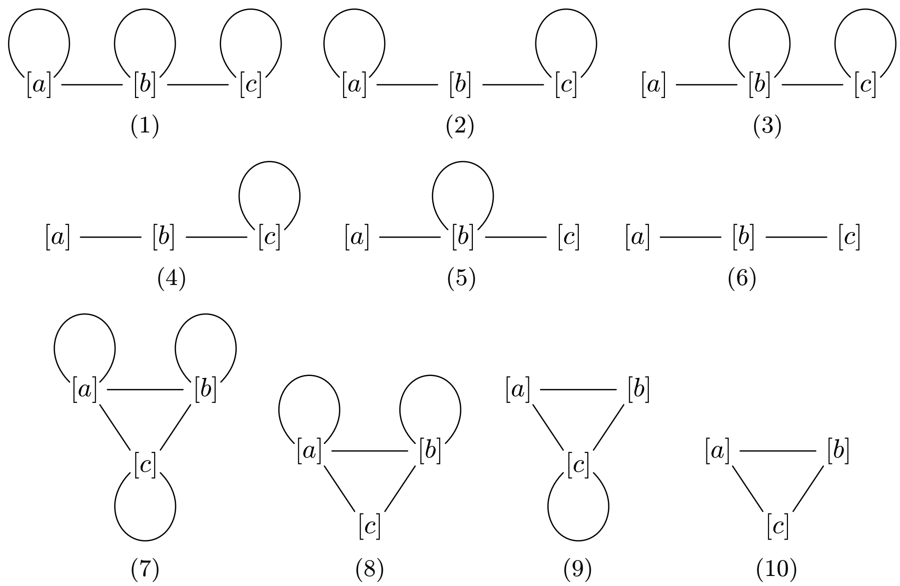

Fig. 4

Prove that the version in Fig. 4(1) is impossible. Indeed, in this case $(a+b)b = 0$ or $b(a+b) = 0$, i.e., $a + b \in D(R)$. Hence, $(a+b)^2 = ab + ba = 0$. Since $ab = 0$ or $ba = 0$, we have $ab = ba = 0$. Consequently, $b(a+c) = 0$ or $(a+c)b = 0$, i.e., $a + c \in D(R)$. But $(a+c)c = ac \neq 0$, $c(a+c) = ca \neq 0$, and $(a+c)a = ca \neq 0$, $a(a+c) = ac \neq 0$, and hence the vertex $a + c$ is not adjacent to a and c. But all vertices of the graph are adjacent either to a or c; a contradiction.

Prove that the variant in Fig. 4(2) is impossible. Let $j \in J(R)$, $j \neq 0$, $j^2 = 0$. Then $j \in [a]$ or $j \in [c]$. Assume without loss of generality that $j \in [a]$. Therefore, $jc \neq 0$ and $cj \neq 0$. Since $[jc]$ is adjacent to $[j] = [a]$ and $[c]$, we have $jc \in [jc] = [b]$.

Since $jc \in J(R)$, there exists $n \in N$ such that $(jc)^n = 0$ and $(jc)^{n-1} \neq 0$. Since $((jc)^{n-1})^2 = 0$ and $[b]$ is a vertex without any loop, $(jc)^{n-1} \notin [b]$. But $(jc)^{n-1} \cdot c = 0$ and $j \cdot (jc)^{n-1} = 0$, and so $(jc)^{n-1} \in [b]$; a contradiction.

Thus, $J(R) = (0)$. By the Wedderburn Theorem,

$$R \cong M_{n_1}(GF(q_1)) \oplus \cdots \oplus M_{n_s}(GF(q_s))$$

for some $n_1, \ldots, n_s, q_1, \ldots, q_s \in \mathbb{Z}$. Note that R contains no orthogonal idempotents, and so $R \cong GF(q_1)$. But then $\Gamma_{\sim}(R)$ is empty; a contradiction.

The variant in Fig. 4(3) is possible. Consider the ring $\mathbb{Z}_{p^4}$ for some prime p. Then $D(R) = p\mathbb{Z}_{p^4}$ and $[a] = [\bar{p}]$, $[b] = [\bar{p}^3]$, $[c] = [\bar{p}^2]$.

Prove that the variant in Fig. 4(4) is impossible. Indeed, observe that $\mathrm{Ann}(R) = (0)$ and R is a nonnilpotent ring. Consequently, R contains nonzero idempotents, and there are at most 2 of them.

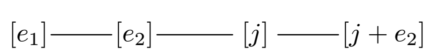

Fig. 5

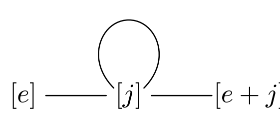

Fig. 6

Let e be a principal idempotent of R (possibly, $e = 1$) and $e = e_1 + e_2$, where e_1 and e_2 are orthogonal idempotents. If R has no unity then $e \in D(R)$ [1]. Then $[e]$, $[e_1]$, and $[e_2]$ are three different vertices without any loop in $\Gamma_\sim(R)$; a contradiction. Therefore, $e = e_1$. Let $e = 1$. We may assume without loss of generality that $[a] = [e_1]$ and $[b] = [e_2]$. If $j \in e_1Re_2$, $j \neq 0$, then $j^2 = 0$ and the vertex $[j] = [c]$ is adjacent to $[e_1]$; a contradiction. Consequently, $e_1Re_2 = (0)$. Similarly, $e_2Re_1 = (0)$. Hence,

$$R = e_1Re_1 \oplus e_2Re_2.$$

Assume that $j \in e_2Re_2$, $j \neq 0$, and $j^2 = 0$. Then the vertex $[j] = [c]$ is adjacent to $[e_1] = [a]$; a contradiction. Consequently, e_2Re_2 contains no nilpotent elements, and so it is a direct sum of fields. Since R contains at most two orthogonal idempotents, e_2Re_2 is a field. Put $e_2Re_2 = F$. Suppose that there exists a nonzero $j \in e_1Re_1$ such that $j^2 = 0$. Then $\Gamma_\sim(R)$ contains at least four vertices (Fig. 5), which is impossible. Hence, e_1Re_1 is a field and the graph $\Gamma_\sim(R)$ contains two vertices; a contradiction.

Thus, R contains a unique idempotent e. If R has a unity then R is a local ring and $\Gamma_\sim(R) = \Gamma(J(R)/\sim)$. However, $\mathrm{Ann}(J(R)) \neq (0)$; a contradiction. Therefore, R is a ring without unity.

Let e be a principal idempotent of R. Consider the Pierce decomposition

$$R = eRe \dotplus (1-e)Re \dotplus eR(1-e) \dotplus (1-e)R(1-e).$$

If $(1-e)Re \neq 0$ and $j \in (1-e)Re$, $j \neq 0$, then $j^2 = 0$ and $[j] = [c]$. Since $ej = 0$, we have $[e] = [b]$. Thus, the following inclusions hold:

(1) $(1-e)R(1-e) \setminus \{0\} \subseteq [c]$ because the vertices of $(1-e)R(1-e)$ are adjacent to e and j simultaneously;

(2) $ere + s \in [e]$, where $r \in R$, $ere \neq 0$, $s \in (1-e)Re \dotplus eR(1-e) \dotplus (1-e)R(1-e)$ since the elements of this form are not adjacent to e;

(3) $er(1-e) + (1-e)te + s \in [e]$, where $r, s \in R$, $er(1-e) \neq 0$, $(1-e)te \neq 0$, $s \in (1-e)R(1-e)$, since the elements of this form are not adjacent to e either;

(4) $r + s \subseteq [c]$, where $r \in (1-e)Re \cup eR(1-e)$, $r \neq 0$, $s \in (1-e)R(1-e)$, since the elements of this form are adjacent to e.

Consequently, all nonzero zero divisors of R are contained in $[b]$ or $[c]$; a contradiction. Therefore, $(1-e)Re = (0)$. We similarly infer that $eR(1-e) = (0)$.

Thus,

$$R = eRe \oplus (1-e)R(1-e).$$

Suppose that there exists a nonzero $j \in (1-e)R(1-e)$ such that $j^2 \neq 0$ and $j^n = 0$, $j^{n-1} \neq 0$ for some integer $n \geq 3$. Then $(j^{n-1})^2 = 0$ and $ej = 0$, and so $e \in [a]$, $j \in [b]$, and $j^{n-1} \in [c]$. But $e \cdot j^{n-1} = 0$; a contradiction. Hence, for all j in $(1-e)R(1-e)$, we have $j^2 = 0$. So, let $j \in (1-e)R(1-e)$ be a nonzero element. Then the graph $\Gamma_\sim(R)$ contains the subgraph that is depicted in Fig. 6; a contradiction.

The variant in Fig. 4(5) is impossible since $a \sim c$, and so $[a] = [c]$.

The variant in Fig. 4(6) is impossible. Indeed, R has no nilpotent elements. Therefore, $R = GF(q_1) \oplus \dots \oplus GF(q_s)$ is a direct sum of fields. But if R is a sum of three or more fields then $\Gamma_\sim(R)$ contains the cycle presented in Fig. 7. Consequently, $R = GF(q_1) \oplus GF(q_2)$ is a direct sum of two fields, but then $\Gamma_\sim(R)$ has order 2; a contradiction.

$$\begin{tikzcd}[arrows=dash]
{[(1,0,0)]} \arrow[rr] \arrow[dr] & & {[(0,1,0)]} \arrow[dl] \\
& {[(0,0,1)]} &
\end{tikzcd}$$

Fig. 7

The variants depicted in Figs. 4(7), 4(8), and 4(10) are impossible (see Proposition 4).

Consider the variant in Fig. 4(9). Such a graph is possessed by the nilpotent ring $R = \langle a \rangle \dotplus \langle b \rangle \dotplus \langle c \rangle$, where $ab = 0$, $ba = a^2 = b^2 = c$, $c^2 = ac = ca = bc = cb = 0$, and $2a = 2b = 2c = 0$. $\square$

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