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EMBEDDING OF JORDAN SUPERALGEBRAS INTO THE SUPERALGEBRAS OF JORDAN BRACKETS

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Abstract: We show that the Jordan bracket on an associative commutative superalgebra is extendable to the superalgebra of fractions. In particular, we prove that a unital simple abelian Jordan superalgebra is embedded into a simple superalgebra of a Jordan bracket. We also study the unital simple Jordan superalgebras whose even part is a field. We demonstrate that each of these superalgebras is either a superalgebra of a nondegenerate bilinear form, or a four-dimensional simple Jordan superalgebra, or a superalgebra of a Jordan bracket, or a superalgebra whose odd part is an irreducible module over a field.

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In the theory of Jordan superalgebras, the superalgebras of Jordan brackets play an important role. The main properties of these superalgebras were studied in [1–3]. The question of speciality of the superalgebras of Jordan brackets was under consideration in [4–10]. It was shown in [11, 12] that the commutator with respect to the Novikov product on the associative commutative part of a Novikov–Poisson algebra defines a Jordan bracket. If the associative commutative part of the Novikov–Poisson algebra is unital then the obtained bracket is a bracket of vector type. Therefore, the superalgebra, constructed by this bracket, is special; and the speciality of the superalgebra was proved in the general case in [13].

In [5], some examples of prime degenerate Jordan algebras were constructed using the superalgebras of Jordan brackets of vector type. In [6], some simpler construction of prime degenerate Jordan algebras was given employing the superalgebras of vector type. The superalgebras of Jordan brackets of vector type (see [14]) turned out to be an effect tool for the investigation of prime degenerate Jordan algebras.

Some examples of Jordan superalgebras may be obtained by the Kantor doubling process from an associative commutative superalgebra equipped with a Jordan bracket (see [1, 2]). If a Jordan bracket is given on an associative commutative algebra then the even part of the so-obtained Jordan superalgebra is associative, and the odd part is an associative module over the even part. Following [15], we call these superalgebras *abelian*.

Simple Jordan superalgebras with an associative even part were studied in [16–21]. In particular, it was shown in [17] that a unital simple special Jordan superalgebra that is not isomorphic to a superalgebra of a bilinear form and has an associative even part is embedded into a simple superalgebra of a Jordan bracket of vector type. In [20], under some restrictions, it was formulated that a unital simple (nonspecial) Jordan superalgebra with an associative even part is embedded into a simple superalgebra of a Jordan bracket.

In this article, we study the question of extension of a Jordan bracket from an associative commutative superalgebra to its algebra of fractions with respect to some multiplicatively closed set. As it turned out, such extension exists for multiplicatively closed sets of zero nondivisors. Using this fact we give another proof of the fact that a simple Jordan superalgebra that is an associative module over the even part is embedded into a simple superalgebra of a Jordan bracket. We also study the unital simple Jordan superalgebras whose even part is a field. We show that each of these superalgebras is either a superalgebra

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of a nondegenerate bilinear form, or a superalgebra of Jordan bracket, or a four-dimensional simple Jordan superalgebra $J(\mathscr{C}, v)$, or a superalgebra whose odd part is an irreducible module over a field.

The investigation of the nonassociative superalgebras with an associative even part is of great interest. For example, the infinite-dimensional right-alternative abelian superalgebras whose even part is a field were described in [22].

§ 1. The Main Notions and Examples

Let F be a field of characteristic not 2, and let $A = A_0 + A_1$ be an arbitrary $\mathbb{Z}_2$-graded algebra; i.e., $A_0 \cap A_1 = 0$, $A_0^2 \subseteq A_0$, $A_1^2 \subseteq A_0$, $A_0 A_1 \subseteq A_1$, and $A_1 A_0 \subseteq A_1$. We call A a *superalgebra*. The vector space A_0 (A_1) is the *even* (*odd*) *part* of the $\mathbb{Z}_2$-graded algebra A. The elements in $A_0 \cup A_1$ are *homogeneous*. The expression $|x|$, where $x \in A_0 \cup A_1$, denotes the parity index of a homogeneous element x; i.e.,

$$|x| = \begin{cases} 0 & \text{if } x \in A_0 \text{ and } x \text{ is even,} \\ 1 & \text{if } x \in A_1 \text{ and } x \text{ is odd.} \end{cases}$$

Let G be the Grassmann algebra over F, i.e., G is an associative algebra given by the generators 1, $e_1, e_2, \dots$ and the defining relations

$$e_i^2 = 0, \quad e_i e_j = -e_j e_i.$$

The products $1, e_{i_1} \dots e_{i_k}$ with $i_1 < i_2 < \dots < i_k$ form a basis for G. Let G_0 and G_1 be the vector subspaces spanned respectively by the products of even and odd lengthes. Then $G = G_0 + G_1$ is a $\mathbb{Z}_2$-graded algebra.

Let $A = A_0 + A_1$ be an arbitrary $\mathbb{Z}_2$-graded algebra. Then $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ is a subalgebra of $G \otimes A$ (the tensor product over F) which is called the *Grassmann envelope* of A.

An associative superalgebra $A = A_0 + A_1$ is an *associative commutative superalgebra* provided that its Grassmann envelope $G(A)$ is an associative commutative algebra.

A superalgebra $J = J_0 + J_1$ is a *Jordan superalgebra* if its Grassmann envelope $G(J)$ is a Jordan algebra; i.e., the identities

$$xy = yx, \quad (x^2 y)x = x^2(yx)$$

hold in $G(J)$. We denote the even part of a superalgebra J by A and the odd part, by M. The following identities hold for the homogeneous elements in a Jordan superalgebra J:

$$ab = (-1)^{|a||b|} ba, \tag{1}$$

$$[(ab)c]d + (-1)^{|b||c|+|b||d|+|c||d|}[(ad)c]b + (-1)^{|a|(|b|+|c|)+(|a|+|b|+|c|)|d|}[(db)c]a$$
$$= (ab)(cd) + (-1)^{|b||c|}(ac)(bd) + (-1)^{|b||d|+|c||d|}(ad)(bc), \tag{2}$$

$$[(ab)c]d + (-1)^{|b||c|+|b||d|+|c||d|}[(ad)c]b + (-1)^{|a|(|b|+|c|)+(|a|+|b|+|c|)|d|}[(db)c]a$$
$$= [a(bc)]d + (-1)^{|c||d|}[a(bd)]c + (-1)^{|b||d|+|c||d|}[a(dc)]b, \tag{3}$$

$$(ab, c, d) + (-1)^{|b||c|+|c||d|+|d||b|}(ad, c, b) + (-1)^{|a|(|b|+|c|+|d|)+|d||c|}(bd, c, a) = 0, \tag{4}$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator of x, y, and z. From here, we have

$$(a, bc, d) = (-1)^{|a||b|} b(a, c, d) + (-1)^{|c||d|}(a, b, d)c. \tag{5}$$

Also,

$$(a, b, c) + (-1)^{|a||b|+|a||c|}(b, c, a) + (-1)^{|a||c|+|b||c|}(c, a, b) = 0, \tag{6}$$

$$(a, b, c) = -(-1)^{|a||b|+|a||c|+|b||c|}(c, b, a). \tag{7}$$

The identity

$$a(b, c, d) - (ab, c, d) - (a, b, cd) + (a, bc, d) + (a, b, c)d = 0 \tag{8}$$

holds in every algebra.

Given an arbitrary algebra A and some subsets X, Y, and Z of A, we denote by (X, Y, Z) the vector space spanned by the associators (x, y, z), where $x \in X$, $y \in Y$, and $z \in Z$.

The superalgebra of a bilinear form. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space over F with a supersymmetric bilinear form $f(x, y)$ (i.e., f is symmetric on V_0, skew-symmetric on V_1, and $f(V_0, V_1) = 0$). Consider the direct sum of the vector spaces $J = F \cdot 1 + V$. Define the product on J by putting $1 \cdot v = v \cdot 1 = v$, and $v_1 \cdot v_2 = f(v_1, v_2) \cdot 1$. Then J is a Jordan superalgebra with even part $A = F \cdot 1 + V_0$ and odd part $M = V_1$. If f is nondegenerate then J is a simple superalgebra, except for the case that $V_1 = 0$, $V_0 = F \cdot e$, and $f(e, e) = \alpha^2$.

The superalgebras of type $J(\mathscr{C}, v)$. Let $A = Fe_1 + Fe_2$ be the direct sum of two fields, and let $M = Fx + Fy$ be a two-dimensional space over F. Equip the vector space $A + M$ with the product by putting

$$e_1 e_2 = 0, \quad e_i^2 = e_i, \quad e_i x = x e_i = \frac{1}{2} x, \quad e_i y = y e_i = \frac{1}{2} y, \quad i = 1, 2,$$

$$x^2 = y^2 = 0, \quad xy = -yx = e_1 + t e_2,$$

where $t \in F$. Denote the so-obtained algebra by D_t. Then D_t is a Jordan superalgebra. The superalgebra D_t is simple if and only if $t \neq 0$.

Let $A = \mathscr{C} = F + Fv$ be a two-dimensional composition algebra over F, $v^2 \in F$, and $v^2 \neq 0$. Equip $A + M$ with the product by putting $a \cdot b = ab$ for $a, b \in A$, where ab is the product of a and b in A,

$$v \cdot x = x \cdot v = y \cdot v = v \cdot y = 0, \quad x \cdot y = -y \cdot x = \alpha + v\beta,$$

where α and β belong to F and are nonzero simultaneously. Denote the so-obtained algebra by $J(\mathscr{C}, v)$ (see [23]). A basis for $J(\mathscr{C}, v)$ may be chosen so that either $\alpha = 1, \beta = 0$, or $\alpha = 0, \beta = 1$, or $\alpha = 1, \beta = 1$.

Let $\overline{F}$ be an algebraic closure of F. Consider the tensor product $J(\mathscr{C}, v) \otimes \overline{F}$. Identify $v \otimes 1$, $x \otimes 1$, and $y \otimes 1$ with v, x, and y. Then

$$\mathscr{C} \otimes \overline{F} = \overline{F} + v\overline{F} = \overline{F} e_1 + \overline{F} e_2,$$

$v = \gamma s$, where $\gamma \in \overline{F}$, $s = e_2 - e_1$, and $e_i^2 = e_i$, $i = 1, 2$.

If $\alpha = 1$ and $\beta = 0$ then the superalgebra $J(\mathscr{C}, v) \otimes \overline{F}$ is isomorphic to D_1. If $\alpha = 0$ and $\beta = 1$ then $J(\mathscr{C}, v) \otimes \overline{F}$ is isomorphic to D_{-1}. It follows that $J(\mathscr{C}, v)$ is a simple Jordan superalgebra in these two cases.

If $\alpha = 1$ and $\beta = 1$ then

$$xy = 1 + \gamma s = (1 - \gamma) e_1 + (1 + \gamma) e_2$$

in $J(\mathscr{C}, v) \otimes \overline{F}$. Hence, $J(\mathscr{C}, v) \otimes \overline{F}$ is a simple Jordan superalgebra when $\gamma \neq \pm 1$. Therefore, $J(\mathscr{C}, v)$ is a simple Jordan superalgebra if $v^2 \neq 1$.

Let us exhibit one of the principal examples of a superalgebra of a Jordan bracket.

The Kantor double $J(\Gamma, \{\,,\})$. Let $\Gamma = \Gamma_0 + \Gamma_1$ be a unital associative commutative superalgebra over a field F, and let $\{\,,\} : \Gamma \times \Gamma \mapsto \Gamma$ be a skew-symmetric bilinear mapping on Γ which is called a *bracket*. Given Γ and $\{\,,\}$, we can construct the superalgebra $J(\Gamma, \{\,,\})$. Consider the direct sum of the vector spaces $J(\Gamma, \{\,,\}) = \Gamma \oplus \Gamma x$, where Γx is an isomorphic copy of Γ. Let a and b be some elements of Γ. Then the product $\cdot$ on $J(\Gamma, \{\,,\})$ is defined by the formulas

$$a \cdot b = ab, \quad a \cdot bx = (ab)x, \quad ax \cdot b = (-1)^{|b|}(ab)x, \quad ax \cdot bx = (-1)^{|b|}\{a, b\},$$

where $a, b \in \Gamma_0 \cup \Gamma_1$ and ab is the product of a and b in Γ. Put $A = \Gamma_0 + \Gamma_1 x$ and $M = \Gamma_0 x + \Gamma_1$. Then $J(\Gamma, \{\,,\}) = A + M$ is a $\mathbb{Z}_2$-graded algebra.

A bracket $\{\,,\}$ is a *Jordan bracket* provided that $J(\Gamma, \{\,,\})$ is a Jordan superalgebra.

A bracket $\{\,,\}$ is Jordan (see [1, 2]) if and only if the following hold:

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|}b\{a, c\} - \{a, 1\}bc; \tag{9}$$

$$\begin{aligned}\{a, b\}\{c, 1\} + (-1)^{|a||b|+|a||c|}\{b, c\}\{a, 1\} + (-1)^{|a||c|+|b||c|}\{c, a\}\{b, 1\}\\ = \{a, \{b, c\}\} + (-1)^{|a||b|+|a||c|}\{b, \{c, a\}\} + (-1)^{|a||c|+|b||c|}\{c, \{a, b\}\},\end{aligned} \tag{10}$$

$$\{\{d, d\}, d\} = -\{d, d\}\{d, 1\}, \tag{11}$$

where $a, b, c \in \Gamma_0 \cup \Gamma_1$ and $d \in \Gamma_1$. Clearly, $D : a \mapsto \{a, 1\}$ is a derivation of A. Then (9) is equivalent to

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|}b\{a, c\} - D(a)bc. \tag{12}$$

A Jordan bracket is a *bracket of vector type* provided that $\{a, b\} = D(a)b - aD(b)$ for all $a, b \in \Gamma$. A Jordan bracket is a *Poisson bracket* if $D(a) = 0$ for every $a \in \Gamma$.

§ 2. Extension of a Jordan Bracket to the Algebra of Fractions

Let A be a unital associative commutative algebra, and let S be a multiplicatively closed subset of zero nondivisors of A. In what follows, we assume that $1 \in S$. Consider the algebra of fractions $S^{-1}A$ of A with respect to S. Since the elements in S are zero nondivisors; therefore, A is embedded into $S^{-1}A$.

Let $\{\,,\}$ be a Jordan bracket on A. Since $D : a \mapsto \{a, 1\}$ is a derivation of A; therefore, D may be extended to a derivation of $S^{-1}A$. Extend $\{\,,\}$ to $S^{-1}A$ by putting

$$\begin{aligned}\{as^{-1}, bt^{-1}\} = -a\{s, b\}s^{-2}t^{-1} + \{a, b\}s^{-1}t^{-1} - \{a, t\}bs^{-1}t^{-2} + \{s, t\}abs^{-2}t^{-2}\\ +abD(t)s^{-1}t^{-2} - abD(s)s^{-2}t^{-1} - aD(b)s^{-1}t^{-1} + D(a)bt^{-1}s^{-1}.\end{aligned} \tag{13}$$

Then

$$\{a, bc\} = \{a, b\}c + b\{a, c\} - D(a)bc$$

for all $a, b, c \in S^{-1}A$, which follows from (13) by direct computation.

Lemma 1. *The bracket is correctly defined on $S^{-1}A$.*

Proof. Let $\alpha \in S^{-1}A$ and $as^{-1} = bt^{-1}$, where $s, t \in S$. Then $at = bs$, whence

$$\{\alpha, at\} = a\{\alpha, t\} + \{\alpha, a\}t - D(\alpha)at,$$

$$\{\alpha, bs\} = b\{\alpha, s\} + \{\alpha, b\}s - D(\alpha)bs.$$

Therefore,

$$\{\alpha, a\}t - b\{\alpha, s\} = \{\alpha, b\}s - a\{\alpha, t\}.$$

Multiplying the both sides of this equality by st, we get

$$\{\alpha, a\}st^2 - bst\{\alpha, s\} = \{\alpha, b\}s^2t - ast\{\alpha, t\}.$$

Since $at = bs$,

$$\{\alpha, a\}st^2 - at^2\{\alpha, s\} = \{\alpha, b\}s^2t - bs^2\{\alpha, t\}.$$

Hence,

$$\{\alpha, a\}s^{-1} - a\{\alpha, s\}s^{-2} = \{\alpha, b\}t^{-1} - b\{\alpha, t\}t^{-2}$$

in $S^{-1}A$. Then

$$\begin{aligned}\{\alpha, as^{-1}\} &= \{\alpha, a\}s^{-1} + a\{\alpha, s^{-1}\} - D(\alpha)as^{-1}\\ &= \{\alpha, a\}s^{-1} + 2aD(\alpha)s^{-1} - a\{\alpha, s\}s^{-2} - D(\alpha)as^{-1}\\ &= \{\alpha, a\}s^{-1} - a\{\alpha, s\}s^{-2} + D(\alpha)as^{-1} = \{\alpha, b\}t^{-1} - b\{\alpha, t\}t^{-2} + D(\alpha)bt^{-1}\\ &= \{\alpha, bt^{-1}\}.\end{aligned}$$

Thus, the value $\{\alpha, as^{-1}\}$ of the bracket does not depend on the choice of representation of as^{-1} in $S^{-1}A$.

Lemma 2. *The bracket $\{\,,\}$ is Jordan on $S^{-1}A$.*

Proof. Let

$$J(a,b,c) = \{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\},$$
$$S(a,b,c) = \{a,b\}\{c,1\} + \{b,c\}\{a,1\} + \{c,a\}\{b,1\}.$$

By the definition of the bracket on $S^{-1}A$, (9) holds for all $a,b,c \in S^{-1}A$. Take $a \in A$, $s \in S$, and $b,c \in S^{-1}A$. Direct computation yields

$$J(as^{-1},b,c) = -J(s,b,c)as^{-2} + J(1,b,c)as^{-1} + J(a,b,c)s^{-1},$$
$$S(as^{-1},b,c) = -S(s,b,c)as^{-2} + S(a,b,c)s^{-1}.$$

Note that $S(1,b,c) = 0$. Therefore, it suffices to prove (10) when one of the elements a, b, and c belongs to A. If $b,c \in A$ then (10) holds.

For example, let $b \in A$. Then for $a \in A$ and $s \in S$ we have

$$J(s,b,c) = S(s,b,c), \quad J(a,b,c) = S(a,b,c), \quad J(1,b,c) = S(1,b,c),$$

since $a,b,s \in A$. Hence, $J(as^{-1},b,c) = S(as^{-1},b,c)$.

Thus, (10) holds, i.e., $\{\,,\}$ is a Jordan bracket on $S^{-1}A$.

Let $\Gamma = \Gamma_0 + \Gamma_1$ be a unital associative commutative superalgebra, and let S be a multiplicatively closed subset of Γ_0 of zero nondivisors of Γ with $1 \in S$. Consider the algebra of fractions $S^{-1}\Gamma$ for Γ with respect to S. Since the elements in S are zero nondivisors, Γ is embedded into $S^{-1}\Gamma$.

Lemma 3. *Let $\{\,,\}$ be a Jordan bracket on $\Gamma = \Gamma_0 + \Gamma_1$. Then $\{\,,\}$ may be extended to a Jordan bracket on the superalgebra $S^{-1}\Gamma$.*

Proof. Define the bracket on the Grassmann envelope $G(\Gamma)$ of Γ by putting

$$\langle a \otimes g_1, b \otimes g_2\rangle = \{a,b\} \otimes g_1g_2.$$

Then $\langle\,,\rangle$ is a Jordan bracket on the associative commutative algebra $G(\Gamma)$. Since $S\otimes 1 = \{s\otimes 1 \mid s \in S\}$ is a multiplicatively closed subset of zero nondivisors of $G(\Gamma)$; therefore, $\langle\,,\rangle$ may be extended to a Jordan bracket on $(S\otimes 1)^{-1}G(\Gamma)$. Since $(S\otimes 1)^{-1}G(\Gamma) \cong G(S^{-1}\Gamma)$, the bracket $\{\,,\}$ may be extended to $S^{-1}\Gamma$. Moreover, (9), (10), and (13) hold for this bracket as well.

Show that the bracket $\{\,,\}$ on $S^{-1}\Gamma$ satisfies (11). Take $s \in S$ and $a \in \Gamma_1$. Direct computation yields

$$\begin{aligned}\{\{as^{-1},as^{-1}\},as^{-1}\} = {} & -a(2\{\{s,a\},a\} + \{\{a,a\},s\})s^{-4} + \{\{a,a\},a\}s^{-3} \\ & +(D(\{a,a\}) - 2\{a,D(a)\})as^{-3}.\end{aligned}$$

By (10)

$$-a(2\{\{s,a\},a\} + \{\{a,a\},s\})s^{-4} = a(2\{s,a\}D(a) + \{a,a\}D(s))s^{-4}.$$

Since $D(a) \in \Gamma_1$; therefore, $D(a)D(a) = 0$ and $D(\{a,a\}) - 2\{a,D(a)\} = 0$ by (10). On the other hand,

$$\{as^{-1},as^{-1}\}D(as^{-1}) = -a(2\{s,a\} + \{a,a\}D(s))s^{-4} + \{a,a\}D(a)s^{-3}.$$

Then $\{\{as^{-1},as^{-1}\},as^{-1}\} = -\{as^{-1},as^{-1}\}\{as^{-1},1\}$ by (11).

Theorem 1. *Let $\Gamma = \Gamma_0 + \Gamma_1$ and $V = V_0 + V_1$ be $\mathbb{Z}_2$-graded vector spaces, and let $J = \Gamma + V$ be a unital Jordan superalgebra with the even part $\Gamma_0 + V_0$ and the odd part $\Gamma_1 + V_1$. Assume that the following hold*:

(1) *Γ is an associative subsuperalgebra of J, and V is an associative $\mathbb{Z}_2$-graded Γ-submodule of the Γ-module J*;

(2) $V_iV_j \subseteq \Gamma_{i+j \mod 2}$;

(3) *S is a multiplicatively closed subset of Γ_0, which contains the unity, J is S-torsion free, $S^{-1}V_1 = S^{-1}\Gamma_0 x$ and $S^{-1}V_0 = S^{-1}\Gamma_0(nx)$ as $S^{-1}\Gamma_0$-modules for some $x \in V_1$ and $n \in \Gamma_1$;*

(4) *$S^{-1}J = S^{-1}(\Gamma + \Gamma x)$ as $S^{-1}\Gamma_0$-modules and x is Γ-torsion free.*

Then J is embedded into a superalgebra of the Jordan bracket $J(S^{-1}\Gamma, \{\,,\})$. If J is a simple superalgebra then $J(S^{-1}\Gamma, \{\,,\})$ is simple as well.

Proof. By hypotheses, $\Gamma_0+\Gamma_1 x+\Gamma_1+\Gamma_0 x$ is a subsuperalgebra of J. Hence, $\Gamma+\Gamma x$ is a superalgebra of the Jordan bracket $\{a, b\} = (-1)^{|b|}(ax)(bx)$, where $a, b \in \Gamma_0\cup\Gamma_1$. By Lemma 3, the Jordan bracket $\{\,,\}$ may be extended to the superalgebra $S^{-1}\Gamma$, whence $S^{-1}J = J(S^{-1}\Gamma, \{\,,\})$ is a superalgebra of a Jordan bracket. Since J is S-torsion free, the Γ-module J is embedded into the $S^{-1}\Gamma$-module $S^{-1}J$. Hence, we may assume that $J \subseteq S^{-1}J$.

Show that J is embedded into the superalgebra $S^{-1}J$. The product of elements from V in J will be denoted by $u \cdot v$.

Take $u, v \in V_0\cup V_1$. Then $u = as^{-1}x, v = bt^{-1}x$, where $a, b \in \Gamma_0\cup\Gamma_1$, $s, t \in S$. Thus, $su = ax, tv = bx$, and

$$su \cdot tv = s(u \cdot tv) + (s, u, tv) = s(u \cdot v)t - s(u, v, t) + (s, u, tv)$$

in J. By (5), $(u, v, t) = (-1)^{|b|}(u, x, t)bt^{-1}$, and $(s, u, tv) = a(s, x, bx)s^{-1}$ in $S^{-1}\Gamma$. Then

$$s(u, x, t) = (su, x, t) + (s, u, xt) - (s, u, x)t = (ax, x, t) + a(s, x, xt)s^{-1} - a(s, x, x)s^{-1}t$$

by (8). Hence,

$$s(u, v, t) = (-1)^{|b|}(ax, x, t)bt^{-1} + (-1)^{|b|}a(s, x, xt)bs^{-1}t^{-1} - (-1)^{|b|}a(s, x, x)bs^{-1},$$

whence

$$\begin{aligned} ax \cdot bx = su \cdot tv = (u \cdot v)st - (-1)^{|b|}(ax, x, t)bt^{-1} - (-1)^{|b|}a(s, x, xt)bs^{-1}t^{-1} \\ +(-1)^{|b|}a(s, x, x)bs^{-1} + a(s, x, bx)s^{-1}. \end{aligned}$$

Then

$$\begin{aligned} u \cdot v = (ax \cdot bx)(st)^{-1} + (-1)^{|b|}(ax, x, t)bs^{-1}t^{-2} + (-1)^{|b|}a(s, x, xt)bs^{-2}t^{-2} \\ -(-1)^{|b|}a(s, x, x)bs^{-2}t^{-1} - a(s, x, bx)s^{-2}t^{-1} = (-1)^{|b|}\{a, b\}(st)^{-1} \\ +(-1)^{|b|}\{a, 1\}bs^{-1}t^{-1} - (-1)^{|b|}\{a, t\}bs^{-1}t^{-2} + (-1)^{|b|}\{s, t\}abs^{-2}t^{-2} \\ +(-1)^{|b|}\{t, 1\}abs^{-1}t^{-2} - (-1)^{|b|}\{s, 1\}abs^{-2}t^{-1} - (-1)^{|b|}a\{s, b\}s^{-2}t^{-1} \\ -(-1)^{|b|}a\{b, 1\}s^{-1}t^{-1} = (-1)^{|b|}\{as^{-1}, bt^{-1}\}. \end{aligned}$$

Thus, J is a subsuperalgebra of $S^{-1}J = J(S^{-1}\Gamma, \{\,,\})$.

Let J be a simple superalgebra. Show that $S^{-1}J$ is simple.

Let I be a nonzero ideal of $S^{-1}J$, and let u be a nonzero element in I. Then $u = ys^{-1}$, where $s \in S$ and $y \in J$, whence $y = us \in I$. Therefore, $J \cap I \neq 0$, and $1 \in J \cap I$, whence $I = S^{-1}J$. Thus, $S^{-1}J$ is a simple superalgebra.

Corollary 1. *Let $J = A + M$ be a simple abelian Jordan superalgebra, which is not isomorphic to a superalgebra of a bilinear form. Then J is embedded into a simple superalgebra of a Jordan bracket.*

Proof. By [20], J is unital, and A is a simple differential algebra. Moreover, M is a finitely generated projective A-module of rank 1. By Lemma 4 of [20], we may assume that the characteristic of the main field is 0. Then A does not have zero divisors, and M is A-torsion free.

Let P be a prime ideal of A, and put $S = A \setminus P$. Then J is S-torsion free, and $S^{-1}M$ is a finitely generated projective $S^{-1}A$-module. Since $S^{-1}A$ is a local algebra and M is an A-module of rank 1; therefore, $S^{-1}M = S^{-1}Ax$. We may assume that $x \in M$. Since $S^{-1}J = S^{-1}(A + Ax)$; therefore, $S^{-1}J = J(S^{-1}A, \{\,,\})$ is a simple superalgebra of a Jordan bracket by Theorem 1, and J is embedded into $S^{-1}J$.

Lemma 4. *Let $J = A + M$ be a simple abelian Jordan superalgebra over a field of characteristic 0, and let S be a multiplicatively closed subset of A. Assume that $S^{-1}M = S^{-1}Ax$, where $x \in M$ and $(a, x, bx) = (a, x, x)b$ for all $a, b \in A$. Then $(a, u, vb) = (a, u, v)b$ for all $a, b \in A$ and $u, v \in M$.*

Proof. By [20], A is unital and does not contain zero divisors, and M is A-torsion free. Take $u \in M$. Then $su = cx$, where $s \in S$ and $c \in A$. By (5)

$$s(a, u, xb) = (a, su, bx) = (a, cx, bx) = c(a, x, bx)$$
$$= c(a, x, x)b = (a, xc, x)b = s(a, u, x)b$$

for all $a, b \in A$, whence $(a, u, bx) = (a, u, x)b$ for all $a, b \in A$, $u \in M$. By (8)

$$a(u, x, b) = (au, x, b) - (a, ux, b) + (a, u, bx) - (a, u, x)b = (au, x, b)$$

for all $a, b \in A$ and $u \in M$. Take $v \in M$. Then $sv = cx$, where $s \in S$, $c \in A$, and

$$s(au, v, b) = (au, cx, b) = c(au, x, b) = ca(u, x, b) = a(u, cx, b) = sa(u, v, b).$$

Hence, $(au, v, b) = a(u, v, b)$ for all $a, b \in A$ and $u, v \in M$. By (7), $(a, u, vb) = (a, u, v)b$ for all $a, b \in A$ and $u, v \in M$.

In [20], it was proved

Theorem 2. *Let $J = A + M$ be a unital simple Jordan superalgebra with an associative nil-semisimple even part. Assume that $N = (A, M, A) \neq 0$. Then the following hold:*

(1) *The even part $A = A_0 + A_1$ is a $\mathbb{Z}_2$-graded algebra, and A_1 is a faithful finitely generated projective A_0-module of rank 1.*

(2) *The odd part $M = M_0 \oplus N$ is the direct sum of associative A_0-modules, and $N \otimes_{A_0} M_0 \cong A_1$ is an A_0-module isomorphism. Moreover, M_0 and N are faithful finitely generated projective A_0-modules of rank 1.*

(3) *$A_0 + M_0$ is a unital simple abelian subsuperalgebra of J which is not isomorphic to a superalgebra of a bilinear form. Moreover, A_0 is a simple differential algebra.*

(4) *If the characteristic of the main field is 0 then A_0 has no zero divisors; moreover, the A_0-modules A_1, M_0, and N are A_0-torsion free.*

(5) *$(A, N, A) = 0$, $NN = 0$, $A_1 = NM_0$, $A_1M_0 \subseteq N$, $A_1N = 0$, $\Gamma = A_0 + N$ is an associative commutative $\mathbb{Z}_2$-graded algebra, and $V = A_1 + M_0$ is an associative commutative Γ-module.*

Lemma 5. *Let $J = A+M$ be a unital simple Jordan superalgebra over a field of characteristic 0 with an associative nil-semisimple even part, and $N = (A, M, A) \neq 0$. Let A_0 and M_0 be such as in Theorem 2, and let S be a multiplicatively closed subset of A_0. Assume that $S^{-1}M_0 = S^{-1}A_0x$, where $x \in M_0$. Then $(Nx, x, A_0) \neq 0$.*

Proof. Assume that $(Nx, x, A_0) = 0$. Then by (8)

$$a(ux, x, b) = (a(ux), x, b) - (a, (ux)x, b) + (a, ux, xb) - (a, ux, x)b$$

for all $a, b \in A_0$ and $u \in N$. By Theorem 2(5) and (5)

$$0 = a(ux, x, b) = (a, ux, xb) - (a, ux, x)b = u((a, x, xb) - (a, x, x)b)$$

for all $a, b \in A_0$ and $u \in N$. By Theorem 2(4) we infer that $(a, x, xb) = (a, x, x)b$. Then $(a, u, vb) = (a, u, v)b$ for all $a, b \in A_0$ and $u, v \in M_0$ by Lemma 4.

Take $b \in A_0$, $u \in M_0$, and $v \in N$. Then $su = ax$ for some $s \in S$ and $a \in A_0$. By (8)

$$s(vu, x, b) = (s(vu), x, b) - (s, (vu)x, b) + (s, vu, xb) - (s, vu, x)b.$$

By Theorem 2(5)

$$(s(vu), x, b) = (v(su), x, b) = (v(ax), x, b) = ((av)x, x, b) = 0$$

and $(s,(vu)x,b)=0$, whence by (5)

$$s(vu,x,b)=v(a,u,xb)-v(a,u,x)b=v((a,u,xb)-(a,u,x)b)=0.$$

Since A_0+M_0 is a subsuperalgebra of J and M is A_0-torsion free, $(vu,x,b)=0$. Since M_0 is a finitely generated A_0-module, $sM_0\subseteq A_0x$ for some $s\in S$. Then by (5)

$$s(vu,M_0,b)\subseteq(vu,sM_0,b)\subseteq(vu,A_0x,b)\subseteq A_0(vu,x,b)=0.$$

Since M is A_0-torsion free, $(vu,M_0,b)=0$. Hence, $(NM_0,M_0,A_0)=0$. From here and by Theorem 2(5) we infer that $(A,M,A)=0$; a contradiction.

Thus, $(Nx,x,A_0)\neq 0$.

Corollary 2. *Let $J=A+M$ be a unital simple Jordan superalgebra with an associative nil-semisimple even part. Assume that $(A,M,A)\neq 0$. Then J is embedded into a simple superalgebra $J(A_0+N,\{\,,\})$ of a Jordan bracket, where A_0+N is a split null extension of the unital associative commutative algebra A_0. Furthermore, $\{n,a\}\neq a\{n,1\}$ for some $a\in A_0$, $n\in N$.*

Proof. Let A_1, N, and M_0 be as in Theorem 2. Put $\Gamma_0=A_0$, $\Gamma_1=N$, $V_0=A_1$, and $V_1=M_0$. Then $\Gamma=A_0+N=\Gamma_0+\Gamma_1$ is an associative commutative $\mathbb{Z}_2$-graded algebra, and $V=V_0+V_1$ is an associative commutative $\mathbb{Z}_2$-graded Γ-module. Moreover, $J=\Gamma+V$.

Assume that A_1, N, and M_0 are cyclic A_0-modules. Then $A_1=A_0v$, $N=A_0n$, and $M_0=A_0x$. By Theorem 2(2) we may assume that $v=nx$. Consequently,

$$J=A_0+A_1+M_0+N=A_0+A_0n+(A_0n)x+A_0x=\Gamma+\Gamma x,$$

whence the product of elements in Γx defines a Jordan bracket $\{\,,\}$ on Γ. Namely, $\{a,b\}=(-1)^{|b|}(ax)(bx)$, where $a,b\in A_0\cup N$.

Let the characteristic p of the main field be greater than 2. Then $a^pA=A$ for every $a\neq 0$ by Theorem 2(3). Hence, A is a field, while A_1, N, and M_0 are cyclic A_0-modules by Theorem 2(2). Thus, J is a superalgebra of a Jordan bracket.

Since $M_0=A_0x$, $(NM_0,M_0,A_0)\subseteq(Nx,M_0,A_0)$ by Theorem 2(5). By (5) $(NM_0,M_0,A_0)\subseteq A_0(Nx,x,A_0)$. Since $(A,M,A)\subseteq(NM_0,M_0,A_0)\neq 0$; therefore, $(Nx,x,A_0)\neq 0$. Hence, $\{n,a\}\neq a\{n,1\}$ for some $a\in A_0$ and $n\in N$.

Let the characteristic of the main field be 0, let P be a prime ideal of Γ_0, and let $S=\Gamma_0\setminus P$. Then S is a multiplicatively closed subset of Γ, J is S-torsion free by Items (2), (4), and (5) of Theorem 2, and $S^{-1}V_1=S^{-1}\Gamma_0x$, where $x\in V_1=M_0$.

Analogously, $S^{-1}V_0=S^{-1}A_1=S^{-1}\Gamma_0v$, where $v\in A_1=NM_0$. Hence, $v=\sum_{i=1}^k n_im_i$, where $n_i\in N$ and $m_i\in M_0$. Since $S^{-1}V_1=S^{-1}\Gamma_0x$; therefore, $s_im_i=a_ix$ for some $s_i\in S$, $a_i\in\Gamma_0$, $i=1,\dots,k$. Let $s=\prod_{i=1}^k s_i$. Then $sm_i=b_ix$ for some $b_i\in\Gamma_0$, $i=1,\dots,k$. Put $n=\sum_{i=1}^k b_in_i$. So $n\in N$, and by Theorem 2(5)

$$sv=s\sum_{i=1}^k n_im_i=\sum_{i=1}^k n_i(sm_i)=\sum_{i=1}^k n_i(b_ix)=\sum_{i=1}^k(b_in_i)x=nx.$$

Thus, $S^{-1}V_0=S^{-1}\Gamma_0(nx)$, where $n\in N=\Gamma_1$. By Theorem 2(5), Nx is a Γ_0-submodule of the Γ_0-module NM_0. Then $S^{-1}V_0=S^{-1}(Nx)$. Hence, $S^{-1}J=S^{-1}(\Gamma+\Gamma x)$. Then J is embedded into a simple superalgebra $J(S^{-1}\Gamma,\{\,,\})$ of a Jordan bracket by Theorem 1.

Since $(A,M,A)\neq 0$, $(NM_0,M_0,A_0)\neq 0$ by Theorem 2(5). Then $(Nx,x,A_0)\neq 0$ by Lemma 5. Therefore, $\{n,a\}\neq a\{n,1\}$ for some $a\in A_0$.

§ 3. Simple Jordan Superalgebras Whose Even Part Is a Field

In this section we study the unital simple Jordan superalgebras whose even part is a field.

Let $J = A + M$ be a unital simple Jordan superalgebra such that $(A, J, A) = 0$ and $(M, A, M) \neq 0$. Recall that the unital simple Jordan superalgebras with the associative nil-semisimple even part satisfying $(A, M, A) \neq 0$ were studied in [20]. The unital simple Jordan superalgebras with the associative even part satisfying $(A, M, A) = 0$ and $(M, A, M) = 0$ were described in [21].

Given $a \in J$, we denote by $R_a : x \mapsto xa$ the operator of right multiplication by a. Let $R^M(A)$ be the subalgebra of $\mathrm{End}_F(M)$ which is generated by the operators $R_a : m \in M \mapsto ma \in M$, $a \in A$. Then $R^M(A)$ is an associative commutative algebra, and M is a faithful associative $R^M(A)$-module. In what follows, the operator $\phi \in R^M(A)$ will be written to the right of its argument. Take $x, y \in M$. Then $R_x \circ R_y = R_x R_y + R_y R_x$ is a derivation of J by (5).

Note that by (4) and $(A, J, A) = 0$,

$$(xa, y, z) = -(-1)^{|x||y|+|x||z|+|y||z|}(za, y, x) \tag{14}$$

holds in $J = A + M$ for all $a \in A$, $x, z \in M$, and $y \in A \cup M$.

Lemma 6. *Let N be an A-submodule of M. Then $(A(N, A, N))M \subseteq N$.*

PROOF. By (6) and (7), $(N, A, N) \subseteq (A, N, N)$. By (8)

$$A(N, A, N) \subseteq (AN, A, N) + (A, NA, N) + (A, N, AN) + (A, N, A)N \subseteq (A, N, N).$$

By (5)

$$(A, N, N)M \subseteq (A, NM, N) + N(A, M, N) \subseteq N.$$

Consequently, $(A(N, A, N))M \subseteq N$.

Now, assume that the even part A of J is a field.

Lemma 7. *The module M is one-generated over $R^M(A)$. Let I be a proper ideal of $R^M(A)$. Then*
(1) $(MI, A, MI) = 0$;
(2) $I^2 = 0$, $(MI, M, MI) = 0$, and $R^M(A)$ is a local ring.

PROOF. (1): Since $(M, A, M) \neq 0$, $(x, A, x) \neq 0$ for some $x \in M$. Let $N = xR^M(A)$. By Lemma 6 $(A(x, A, x))M \subseteq N$. Since $A(x, A, x)$ is a nonzero ideal of A; therefore, $A(x, A, x) = A$, whence $M = N = xR^M(A)$.

Let I be a proper ideal of $R^M(A)$, and $N = MI$. Then $(A(N, A, N))M \subseteq N$ by Lemma 6. If $(N, A, N) \neq 0$ then $M = MI$, whence there is nonzero ϕ in $R^M(A)$ such that $M\phi = 0$. Since M is a faithful $R^M(A)$-module, we arrive to a contradiction. Hence, $(N, A, N) = 0$. Item (1) is proved.

(2): Take $\phi, \psi, \tau \in I$. Since $(N, A, N) = 0$,

$$x\phi\psi \cdot x\tau = x\phi \cdot x\tau\psi = x\phi \cdot x\psi\tau = x\phi\tau \cdot x\psi = x\tau \cdot x\psi\phi = x\tau \cdot x\phi\psi.$$

On the other hand, $x\phi\psi \cdot x\tau = -x\tau \cdot x\phi\psi$. Therefore, $x\phi\psi \cdot x\tau = 0$ and $MI^2 \cdot MI = 0$. By (14) we get $(MI^2, A, M) \subseteq (MI, A, MI) = 0$ and $(MI^2)A \cdot M \subseteq MI^2 \cdot MA$, whence $MI^3 \cdot M \subseteq MI^2 \cdot MI = 0$, i.e., MI^3 is an ideal of J. Hence, $I^3 = 0$, and by (14) we obtain $(MI^2, J, MI) \subseteq (MI^3, J, M) = 0$.

Let $MI^2 \cdot M \neq 0$. Then $(A, MI^2, M) + MI^2 \cdot M$ is a nonzero ideal of A by (5), whence $A = (A, MI^2, M) + MI^2 \cdot M$. By (5)

$$(A, MI^2, M) \cdot MI \subseteq (A, MI, M) \cdot MI^2 \subseteq MI^2.$$

Since $(MI^2, J, MI) = 0$; therefore,

$$(MI^2 \cdot M) \cdot MI \subseteq MI^2 \cdot (M \cdot MI) \subseteq MI^2,$$

whence

$$MI = A(MI) \subseteq (A, MI^2, M) \cdot MI + (MI^2 \cdot M) \cdot MI \subseteq MI^2.$$

Consequently, $MI^2 = 0$, and $I^2 = 0$. Then $(MI, J, MI) = 0$ by (14).

Let K_1 and K_2 be nonzero maximal ideals of $R^M(A)$. Then $R^M(A) = K_1 + K_2$. By the above $K_1^2 = K_2^2 = 0$, whence $R^M(A)$ is a nilpotent algebra. Therefore, we infer that $R^M(A)$ is a local ring. Item (2) is proved.

Lemma 8. *Let I be an ideal of $R^M(A)$. Then*

(1) $(A, MI, MI)M \subseteq MI$, $(A, MI, MI) \cdot MI = 0$.

If $MI \cdot MI \neq 0$ then

(2) $(MI \cdot MI, J, MI) = 0$, $A = MI \cdot MI + (A, MI, MI)$ is a $\mathbb{Z}_2$-graded algebra whose even part is $MI \cdot MI$ and odd part is (A, MI, MI);

(3) $MI \cdot MI$ is a field and $(A, MI, MI) = v(MI \cdot MI)$, where $v \in (A, MI, MI)$.

Proof. (1): Let I be an ideal of $R^M(A)$. Then by (5)

$$(A, MI, MI)M \subseteq (A, MI \cdot M, MI) + (A, M, MI)MI \subseteq MI.$$

Therefore, $(A, MI, MI) \cdot MI \subseteq MI^2 = 0$. Then $(A, MI \cdot MI, MI) = 0$ by (5). Item (1) is proved.

(2): Let $MI \cdot MI \neq 0$. Then $(A, MI, MI) + MI \cdot MI$ is an ideal of A by (5). Hence, $A = (A, MI, MI) + MI \cdot MI$. By Lemma 1 from [21] $M = (A, A, M)$, whence $MI = (A, A, M)I = (A, A, MI)$, since $(A, M, A) = 0$. Then

$$MI \cdot MI = (A, A, MI) \cdot MI = (A, (A, MI, MI) + MI \cdot MI, MI) \cdot MI$$
$$= (A, (A, MI, MI), MI) \cdot MI \overset{\text{by (5)}}{=} (A, (A, MI, MI) \cdot MI, MI)$$
$$+ (A, MI, MI)(A, MI, MI) = (A, MI, MI)(A, MI, MI).$$

It follows from here that $(A, MI, MI) \neq 0$. By (4)

$$(MI \cdot MI, J, MI) = ((A, MI, MI)^2, J, MI) \subseteq ((A, MI, MI) \cdot MI, J, (A, MI, MI)) = 0.$$

By (5)

$$(MI \cdot MI)(A, MI, MI) \subseteq (A, (MI \cdot MI) \cdot MI, MI) + (A, MI \cdot MI, MI) \cdot MI$$
$$\subseteq (A, MI, MI).$$

Let $K = (A, MI, MI) \cap MI \cdot MI$. Then K is an ideal of A. Since $MI \cdot K = 0$, we get $K = 0$. Hence, $A = MI \cdot MI + (A, MI, MI)$ is a $\mathbb{Z}_2$-graded algebra with the even part $MI \cdot MI$ and the odd part (A, MI, MI). Item (2) is proved.

Show that $MI \cdot MI$ is a field. Let $\alpha \in MI \cdot MI$ and $\alpha \neq 0$. By (8)

$$\alpha(\alpha^{-1}, MI, MI) \subseteq (\alpha, MI, MI) + (\alpha, \alpha^{-1}, MI \cdot MI) + (\alpha, \alpha^{-1}, MI) \cdot MI = 0,$$

whence $(\alpha^{-1}, MI, MI) = 0$. Let $\alpha^{-1} = \beta + v$, where $\beta \in MI \cdot MI$ and $v \in (A, MI, MI)$. Then

$$v(MI \cdot MI) = (v, MI, MI) = (\alpha^{-1} - \beta, MI, MI) = 0,$$

and $\alpha^{-1} \in MI \cdot MI$.

Let $v \in (A, MI, MI)$ and $v \neq 0$. Then $v^2 \in MI \cdot MI$. Since $MI \cdot MI$ is a field, $v^{-1} = v^{-2}v \in (A, MI, MI)$. If $u \in (A, MI, MI)$ then $u = (v^{-1}u)v$, whence $(A, MI, MI) = v(MI \cdot MI)$, where $v \in (A, MI, MI)$.

Lemma 9. *If I is the largest ideal of $R^M(A)$ then $MI \cdot MI = 0$.*

Proof. Assume that $MI \cdot MI \neq 0$. Let $A_0 = MI \cdot MI$ and $A_1 = (A, MI, MI)$. By Lemma 8, $A = A_0 + A_1$ is a $\mathbb{Z}_2$-graded algebra, the even part A_0 is a field, $A_1 = vA_0$, and $v^2 \in A_0$. By Lemma 8, $MA_1 \subset MI$ and $MI \cdot A_1 = 0$. By Lemma 7 and (2), the odd part M is equal to $xA + (xA)A$. By (4)

$$(A_0, J, MI) = \left(A_1^2, J, MI\right) \subseteq (A_1 \cdot MI, J, A_1) = 0.$$

Analogously,

$$(A_0, A_0, M) \subseteq (A_1M, A_0, A_1) \subseteq (A_1M, A_1, A_0).$$

Since $MA_1 \subseteq MI$; therefore, $(A_0, A_0, M) \subseteq (MI, A_1, A_0) = 0$. Thus, M is a vector space over A_0. Then $M = xA_0 + (x(vA_0))A_0$.

Since $A_1M \subseteq MI$; therefore, $R_{A_1} \subseteq I$ and $R_{A_1}^2 = 0$ by Lemma 7, whence $R^M(A) = R_{A_0} + R_{A_1}R_{A_0}$ and $I = R_{A_1}R_{A_0}$ by Lemma 8.

Prove that $(M, A_1, M) \subseteq A_1$ and $(M, A_0, M) \subseteq A_0$.

Let $x, y \in M$ and $a \in A$. Then $R_{(x,a,y)} = [R_a, R_x \circ R_y]$ by (3), where $[u, w] = uw - wu$ is the commutator of u and w. Since $I^2 = 0$, we have

$$2[u, R_x \circ R_y]^2 = [[u^2, R_x \circ R_y], R_x \circ R_y] - 2[[u, R_x \circ R_y], R_x \circ R_y]u \in I$$

for $u \in I$. Consequently, $[I, R_x \circ R_y] \subseteq I$, since I is the largest ideal. Hence, $R_{(M,A_1,M)} \subseteq I$. Then $(M, A_1, M) \subseteq A_1$. By (5)

$$(M, A_0, M) \subseteq \left(M, A_1^2, M\right) \subseteq (M, A_1, M)A_1 \subseteq A_1^2 = A_0.$$

Thus, $(M, A_1, M) \subseteq A_1$ and $(M, A_0, M) \subseteq A_0$.

Prove that $(MI, A_0, M) = 0$. By (4)

$$\begin{aligned}(M, MI, A_0) &\subseteq \left(M, MI, A_1^2\right) \subseteq (A_1, MI, MA_1)\\ &\subseteq (A_1, MI, MI) \subseteq A_1(MI \cdot MI) \subseteq A_1.\end{aligned}$$

By (7) and $(A_0, M, MI) = 0$ we get

$$(M, MI, A_0) \subseteq (MI, A_0, M) + (A_0, M, MI) = (MI, A_0, M) \subseteq A_0.$$

Hence,

$$(MI, A_0, M) = (M, MI, A_0) \subseteq A_0 \cap A_1 = 0.$$

Prove that $MI \cdot MI = 0$. Let $\alpha, \beta, \gamma, \delta \in A_0$. By $(MI, A_0, MI) = 0$ and $(MI, J, A_0) = 0$

$$\begin{aligned}(x(v\alpha))\beta \cdot (x(v\gamma))\delta &= ((x(v\alpha)), \beta, (x(v\gamma))\delta) + x(v\alpha)(\beta \cdot (x(v\gamma))\delta)\\ = x(v\alpha)((x(v\gamma))(\beta\delta)) &= -(x(v\alpha), x(v\gamma), \beta\delta) + (x(v\alpha) \cdot x(v\gamma))(\beta\delta)\\ &= (x(v\alpha) \cdot x(v\gamma))(\beta\delta).\end{aligned}$$

Show that $x(v\alpha) \cdot x(v\gamma) = 0$. Indeed, since $A_1 \cdot MI = 0$ and $(MI, A_0, M) = 0$; therefore,

$$\begin{aligned}x(v\alpha) \cdot x(v\gamma) = (x, v\alpha, x(v\gamma)) &\overset{\text{by (5)}}{=} v(x, \alpha, x(v\gamma)) + (x, v, x(v\gamma))\alpha = (x, v, x(v\gamma))\alpha\\ = (xv \cdot x(v\gamma))\alpha = -(xv, v\gamma, x)\alpha &\overset{\text{by (5)}}{=} -(xv, v, x)(\gamma\alpha) = (xv \cdot xv)(\gamma\alpha) = 0,\end{aligned}$$

whence $MI \cdot MI = 0$.

Lemma 10. *Let I be the largest ideal of $R^M(A)$, and let $N = MI$. Then $R_{(NM)} \subseteq I$; i.e., $(NM)M \subseteq N$ and $(NM)N = 0$.*

Proof. Take $x, y, z \in M$ and $w \in I$. Then $(MI, M, MI) = 0$ by Lemma 7, and $(x, yw, zw) = -(x, zw, yw)$ by (6) and (7), whence

$$\begin{aligned}(x, yw, zw) &\overset{\text{by (14)}}{=} (z, yw, xw) = -(z, xw, yw) \overset{\text{by (14)}}{=} -(y, xw, zw)\\ &= (y, zw, xw) \overset{\text{by (14)}}{=} (x, zw, yw) = -(x, yw, zw),\end{aligned}$$

and $(x, yw, zw) = 0$. Then $(x \cdot yw)(zw) = 0$ by Lemma 9. Hence, $R_{x \cdot yw}$ is not invertible in $R^M(A)$. By Lemma 7, $R^M(A)$ is a local algebra. Then $R_{x \cdot yw} \in I$, whence $(M \cdot MI)M \subseteq MI$ and $(NM)N \subseteq MI^2 = 0$ by Lemma 7.

Lemma 11. *Let I be the largest ideal of $R^M(A)$ and $N = MI$. The following hold:*
(1) $((NM)^2, M, M) \subseteq (NM)^2$ and

$$((NM)^2, (NM)^2, M) = ((NM)^2, N, M) = ((NM)^2, M, N) = 0.$$

Assume that $NM \neq 0$. Then
(2) $A = NM + (NM)^2$, $NM \cap (NM)^2 = 0$, and A is a $\mathbb{Z}_2$-graded algebra with the even part $A_0 = (NM)^2$ and the odd part $A_1 = NM$. Furthermore, A_0 is a field, and $A_1 = vA_0$ for every nonzero $v \in A_1$.
(3) $M = A_0x + A_0n$ is a two-dimensional vector space over the field A_0, $N = A_0n$, and $A_1 = (nx)A_0$.
(4) $A_0 + A_0x$ is an abelian subsuperalgebra of J.

Proof. (1): By identity (4) and Lemmas 10 and 9

$$((NM)^2, M, M) \subseteq ((NM)M, M, NM) \subseteq (N, M, NM) \subseteq (NM)^2.$$

Show that $((NM)^2, (NM)^2, M) = 0$. By (4) and Lemma 10

$$((NM)^2, (NM)^2, M) \subseteq ((NM)M, (NM)^2, NM) \subseteq (N, (NM)^2, NM).$$

Since

$$(N, (NM)^2, NM) \subseteq (N, NM, NM)(NM) \subseteq N(NM)$$

by (5); therefore, $((NM)^2, (NM)^2, M) = 0$ by Lemma 10. Analogously, by Lemma 9 $((NM)^2, N, M) = ((NM)^2, M, N) = 0$. Thus, Item (1) is proved.

(2): Note that $(NM)^3 \subseteq NM$. Indeed, by identity (4) and Lemma 10

$$\begin{aligned}(NM)(NM)(NM) &\subseteq (N, M, (NM)^2) + N(M(NM)^2)\\ &\subseteq (NM, M, N(NM)) + NM = NM.\end{aligned}$$

Since A is a field, $A = (NM)A$ and $A = NM + (N, M, A)$. By (4)

$$\begin{aligned}(N, M, A) &\subseteq (N, M, NM) + (N, M, (NM)A)\\ &\subseteq (N, M, NM) + (A, M, N(NM)) + (NM, M, AN).\end{aligned}$$

Then $(N, M, A) \subseteq (NM)^2$ by Lemma 10. Hence, $A = NM + (NM)^2$.

Consider $K = NM \cap (NM)^2$. Then K is an ideal of A. Indeed, take $r \in K$. Then $r \in NM$, whence $r(NM) \subseteq (NM)^2$. On the other hand, $r \subseteq (NM)^2$. Thus, $r(NM) \subseteq (NM)^3 \subseteq NM$. Hence, $r(NM) \subseteq K$. Analogously, $r(NM)^2 \subseteq K$. Thus, K is an ideal of A. By Lemma 10, $KN \subseteq (NM)N = 0$, whence $K = 0$. Therefore, $A = NM + (NM)^2$ is a $\mathbb{Z}_2$-graded algebra.

Let $A_0 = (NM)^2$ and $A_1 = NM$. Show that A_0 is a field. Let $a \in A_0$ and $a \neq 0$. Then $aA_0 + aA_1$ is a nonzero ideal of A. Therefore, $aA_0 = A_0$. Hence, A_0 is a field. Let $v \in A_1$ and $v \neq 0$. Then $u = v(v^{-2}uv) \in vA_0$ for every $u \in A_1$. Thus, Item (2) is proved.

(3): By Lemma 7, $M = xR^M(A)$. Since $R^M(A)$ is a commutative algebra, $M = xA + (xA)A$. Then by Lemma 11(1),(2) $M = xA_0 + (x(vA_0))A_0$ and $x(vA_0) \subseteq N$, whence $M = xA_0 + N$. Since $M \neq MI = N$, $x \notin N$.

Let $u \in N$ and $ux = 0$. Then by Item (1) and $Nu = 0$ we get $Mu = 0$, whence $M(A_0u) = 0$. By Lemma 10 and Item (1), $A(A_0u) \subseteq A_0u$. Therefore, A_0u is an ideal of J, and $u = 0$.

Consider some nonzero u and w in N. Then ux, wx are nonzero in A_1. Since $\dim_{A_0} A_1 = 1$; therefore, $\alpha(xu) = \beta(xw)$ for some nonzero $\alpha, \beta \in A_0$, whence $x(\alpha u - \beta w) = 0$ by Item (1). Hence, $\alpha u - \beta w = 0$.

Thus, $N = A_0n$, $M = A_0x + A_0n$, and $nx \neq 0$, whence $A_1 = (nx)A_0$. Item (3) is proved.

(4): It suffices to prove that $m_1m_2 \in A_0$ for $m_1, m_2 \in A_0x$. Assume that $m_1m_2 = a + b$, where $a \in A_1$ and $b \in A_0$. Take $y, z \in N$ and $m \in M$. Then by Lemma 10

$$\begin{aligned}(ym_2)(zm) \cdot m_1 = (ym_2, zm, m_1) &\overset{\text{by (4)}}{=} (ym_1, zm, m_2) + (m_1m_2, zm, y)\\ &= (ym_1)(zm) \cdot m_2 + (m_1m_2, zm, y).\end{aligned}$$

Since $(ym_2)(zm) \cdot m_1, (ym_1)(zm) \cdot m_2 \in A_0x$ and $(m_1m_2, zm, y) \in N$; therefore, $(m_1m_2, zm, y) = 0$. Then $(a, zm, y) = 0$ by identity (5) and Item (1). Hence, $a(zm) \cdot y = 0$, since $(zm)y \in (NM)N = 0$. Therefore, $(aA_1)N = 0$. It follows from here that $aA_1 = 0$, and $a = 0$.

Thus, $(A_0x)^2 \subseteq A_0$, i.e., $A_0 + A_0x$ is a subsuperalgebra of $A + M$. By Item (1), $A_0 + A_0x$ is an abelian superalgebra.

Theorem 3. *Let $J = A + M$ be a unital simple superalgebra, and let A be a field. Assume that $(A, J, A) = 0$ and $(M, A, M) \neq 0$. Then either M is an irreducible A-module or $J = J(A_0 + A_0n, \{\,,\})$ is a superalgebra of a Jordan bracket on the one-dimensional split null extension $A_0 + A_0n$ of a field A_0, $\{an, b\} = b\{an, 1\}$ for all $a, b \in A_0$, and $\{ac, b\} \neq \{a, cb\}$ for some $a, b, c \in A_0$. Furthermore, $J(A_0, \{\,,\})$ is a simple abelian subsuperalgebra of J.*

Proof. Let M be a reducible A-module. Then $R^M(A)$ contains the largest nonzero ideal I by Lemma 7. Hence, $N = MI \neq 0$. Since J is a simple superalgebra, $NM \neq 0$ and by Lemma 11 the even part A is equal to $A_0 + vA_0$, where A_0 is a field. The odd part $M = A_0x + A_0n$ is a two-dimensional vector space over A_0. Moreover, $v = nx$.

Let $\Gamma = A_0 + A_0n, V = A_0x + A_0v$. Then

$$J = \Gamma + V = A_0 + A_0n + A_0x + A_0n \cdot x = \Gamma + \Gamma x.$$

By identity (5) and Lemma 10 $(A_0n, A_0, x) \subseteq (A_0n, A_1, x)A_1 = 0$. Hence, $(\Gamma, \Gamma, x) = 0$ by Lemmas 10 and 11. We get from here that J is a superalgebra of the Jordan bracket $\{a, b\} = (-1)^{|b|}(ax \cdot bx)$, $a, b \in A_0 \cup A_0n$. Since $(A_0(nx), x, A_0) = 0$, $\{an, b\} = b\{an, 1\}$ for all $a, b \in A_0$.

Let $\{ac, b\} = \{a, cb\}$ for all $a, b, c \in A_0$. Then $(A_0x, A_0, A_0x) = 0$. Since $v^2 \in A_0$; therefore, $v(A_0x, v, A_0x) = 0$ by (5), whence $(A_0x, v, A_0x) = 0$, and $(A_0x, vA_0, A_0x) = 0$ by (5). By Lemmas 10 and 11

$$(A_0n, A, A_0n) = (A_0x, A, A_0n) = 0,$$

whence $(M, A, M) = 0$; a contradiction.

We also get from here that $(A_0x)^2 \neq 0$. By Lemma 11(4), $A_0 + A_0x$ is an abelian subsuperalgebra of J, whence $J(A_0, \{\,,\})$ is a simple abelian subsuperalgebra of J.

Thus, the following theorem holds.

Theorem 4. *Let $J = A + M$ be a unital simple Jordan superalgebra whose even part A is a field. Then J is one of the following superalgebras:*

(1) *a superalgebra of a nondegenerate skew-symmetric bilinear form on M;*

(2) *a superalgebra $J(A, \{\,,\})$ of a Jordan bracket;*

(3) *a superalgebra $J(A_0 + A_0n, \{\,,\})$ of a Jordan bracket on the one-dimensional split null extension $A_0 + A_0n$ of a field A_0 and $\{n, a\} \neq a\{n, 1\}$ for some $a \in A_0$;*

(4) *a superalgebra $J(\mathscr{C}, v)$;*

(5) *a superalgebra $J(A_0 + A_0n, \{\,,\})$ of a Jordan bracket on the one-dimensional split null extension $A_0 + A_0n$ of a field A_0 if the A-module M is reducible; furthermore, $\{an, b\} = b\{an, 1\}$ for all $a, b \in A_0$, $\{ac, b\} \neq \{a, cb\}$ for some $a, b, c \in A_0$, and $J(A_0, \{\,,\})$ is a simple abelian subsuperalgebra of J;*

6) *the odd part M is an irreducible module over the even part A.*

Proof. If A lies in the center of J or $(A, M, A) = 0$, $(M, A, M) = 0$ and M is not a two-generated module over $R^M(A)$ then J is a superalgebra of a nondegenerate skew-symmetric bilinear form $f(x, y) = xy$ for $x, y \in M$ by Theorem 1 from [21].

If $(A, M, M) \neq 0$ and J is an abelian superalgebra then J is a superalgebra of a Jordan bracket on A by [20, Lemma 4].

If $(A, M, A) \neq 0$ then Item (3) holds by Corollary 2.

If $(A, M, A) = 0$, $(M, A, M) = 0$, and M is a two-generated module over $R^M(A)$ then Item (4) holds by Theorem 2 from [21].

If $(A, M, A) = 0$ and $(M, A, M) \neq 0$ then Item (5) holds by Theorem 3.

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