ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 6, pp. 1101–1108. © Pleiades Publishing, Ltd., 2019.
Russian Text © The Author(s), 2019, published in Sibirskii Matematicheskii Zhurnal, 2019, Vol. 60, No. 6, pp. 1414–1423.

BEST LINEAR APPROXIMATION METHODS FOR SOME CLASSES OF ANALYTIC FUNCTIONS ON THE UNIT DISK

M. Sh. Shabozov and M. R. Langarshoev UDC 517.5

Abstract: Considering Banach Hardy spaces and weighted Bergman spaces, we find the sharp values of the Bernstein, Kolmogorov, Gelfand, and linear n-widths for the classes of analytic functions on the unit disk whose moduli of continuity of the rth derivatives averaged with weight are majorized by a given function satisfying some constraints.

DOI: 10.1134/S0037446619060181

Keywords: best linear approximation methods, modulus of continuity, weighted Bergman space, majorant, n-widths, best coding method

1. Suppose that X is an arbitrary Banach space, with S the unit ball of X. Let $\Lambda_n \subset X$ be an arbitrary subspace of dimension n, while $\Lambda^n \subset X$ is a linear subspace of codimension n, and $\mathscr{L}(f, \Lambda_n)$ is a continuous linear operator from X to Λ_n. Let $E(f, \Lambda_n)_X$ stand for the best approximation of $f \in X$ by $\varphi \in \Lambda_n$; i.e.,

$$E(f, \Lambda_n)_X = \inf\{\|f - \varphi\|_X : \varphi \in \Lambda_n\},$$

and let

$$\mathscr{E}(f, \mathscr{L}(f, \Lambda_n))_X = \|f - \mathscr{L}(f, \Lambda_n)\|_X$$

stand for the deviation of $f \in X$ from $\mathscr{L}(f, \Lambda_n)$ in the metric of X. Given a centrally symmetric set $\mathscr{M} \subset X$, we put

$$E(\mathscr{M}, \Lambda_n)_X \stackrel{\text{def}}{=} \sup\{E(f, \Lambda_n)_X : f \in \mathscr{M}\},$$

$$\mathscr{E}(\mathscr{M}, \mathscr{L}, \Lambda_n)_X \stackrel{\text{def}}{=} \sup\{\mathscr{E}(f, \mathscr{L}(f, \Lambda_n))_X : f \in \mathscr{M}\}.$$

The values

$$b_n(\mathscr{M}; X) = \sup\{\sup\{\varepsilon > 0 : (\varepsilon S \cap \Lambda_{n+1}) \subset \mathscr{M}\} : \Lambda_{n+1} \subset X\}, \tag{1}$$

$$d^n(\mathscr{M}; X) = \inf\{\sup\{\|f\|_X : f \in \mathscr{M} \cap \Lambda^n\} : \Lambda^n \subset X\}, \tag{2}$$

$$d_n(\mathscr{M}; X) = \inf\{E(\mathscr{M}, \Lambda_n)_X : \Lambda_n \in X\}, \tag{3}$$

and

$$\delta_n(\mathscr{M}; X) = \inf\{\inf\{\mathscr{E}(\mathscr{M}, \mathscr{L}, \Lambda_n)_X : \mathscr{L} : X \to \Lambda_n\} : \Lambda_n \subset X\} \tag{4}$$

are called the *Bernstein, Gelfand, Kolmogorov*, and *linear n-widths* respectively. Note the following relations [1, 2]:

$$b_n(\mathscr{M}; X) \le \begin{matrix} d_n(\mathscr{M}; X) \\ d^n(\mathscr{M}; X) \end{matrix} \le \delta_n(\mathscr{M}; X). \tag{5}$$

2. In the Hardy spaces H_q, $q \ge 1$, and Bergman spaces $\mathscr{B}_{q,\gamma}$, $q \ge 1$, with the weight $\gamma \ge 0$, the questions of calculation of the exact values of various n-widths for some classes of analytic functions on the unit disk and construction of the best linear approximation methods were considered, for example, in the monographs [1, 2] and articles [3–20]. We continue the study in this direction and calculate the exact values for all above-listed n-widths of the classes $W_a^{(r)} X(\Phi, \mu)$, $r \in \mathbb{N}$, $\mu \ge 1$, of analytic functions on the unit disk (where X is H_q or $\mathscr{B}_{q,\gamma}$) whose moduli of continuity of the rth derivatives averaged with weight are majorized by a given function satisfying some natural constraints.

Original article submitted January 9, 2017; revised May 31, 2019; accepted July 24, 2019.

Let $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{C}$ be the sets of naturals, positive reals, and complexes respectively, let $U_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ be the disk of radius ρ $(0 < \rho \le 1)$, $U_1 = U$, and let $A(U_\rho)$ be the set of analytic functions on U_ρ.

Given $f \in A(U_\rho)$, put

$$M_q(f;\rho) \overset{\text{def}}{=} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})|^q \, dt\right)^{1/q}, \quad 1 \le q \le \infty,$$

where the integral is understood in the Lebesgue sense. If $q = \infty$, then assume that $f(z)$ is continuous on the closed disk $\overline{U}_\rho := \{z \in \mathbb{C} : |z| \le \rho\}$. By H_q, $1 \le q \le \infty$, we denote the Hardy Banach space consisting of $f \in A(U)$ for which the following norm is finite:

$$\|f\|_q := \|f\|_{H_q} = \lim_{\rho \to 1-0} M_q(f;\rho). \tag{6}$$

It is well known that the norm (6) is attained at the angular boundary values $f(t) := f(e^{it})$ of $f \in H_q$. By $H_{q,\rho}$, $1 \le q \le \infty$, $0 < \rho \le 1$, $H_{q,1} \equiv H_q$, we understand the Hardy space of $f \in A(U_\rho)$ for which $\|f(z)\|_{q,\rho} \overset{\text{def}}{=} \|f(\rho z)\|_q < \infty$. If $r \in \mathbb{N}$ then $f_a^{(r)}(z)$ is the derivative of the rth order of $f \in A(U)$ with respect to the argument of the complex variable $z = \rho \exp(it)$. Moreover,

$$f_a^{(1)}(z) := \frac{\partial f(z)}{\partial t} = \frac{df(z)}{dz} \cdot \frac{\partial z}{\partial t} = f'(z)zi \quad \text{and} \quad f_a^{(r)}(z) = \left\{f_a^{(r-1)}(z)\right\}_a^{(1)}, \quad r \ge 2.$$

We denote by $H_{q,a}^{(r)}$ the class of $f \in A(U)$ for which $f_a^{(r)} \in H_q$, $q \ge 1$.

The Banach space of complex-valued functions f on the disk U with the finite norm

$$\|f\|_{l_q} = \left(\frac{1}{2\pi}\iint_{(U)} |f(z)|^q \, dxdy\right)^{1/q} = \left(\frac{1}{2\pi}\int_0^1 \int_0^{2\pi} \rho |f(\rho e^{it})|^q \, d\rho dt\right)^{1/q}$$

is denoted by $l_q \overset{\text{def}}{=} l_q(U)$, $1 \le q < \infty$, where the integral is understood in the Lebesgue sense.

Let $\gamma(|z|) \ge 0$ be some measurable and summable function not equivalent to the zero function on U. The set of the complex-valued functions f for which $\gamma^{1/q} f \in l_q(U)$, $\|f\|_{l_{q,\gamma}} = \|\gamma^{1/q} f\|_{l_q}$, is denoted by $l_{q,\gamma} \overset{\text{def}}{=} l_q(U,\gamma)$, $1 \le q < \infty$, while $\mathscr{B}_{q,\gamma} \overset{\text{def}}{=} \mathscr{B}_q(U,\gamma)$, $1 \le q < \infty$, is the Banach space of $f \in A(U)$ such that $f \in l_{q,\gamma}$. Moreover,

$$\|f\|_{\mathscr{B}_{q,\gamma}} = \left(\int_0^1 \rho\gamma(\rho) M_q^q(f,\rho) \, d\rho\right)^{1/q}.$$

In the particular case when $\gamma \equiv 1$, $\mathscr{B}_q := \mathscr{B}_{q,1}$ is a usual Bergman space. By $\mathscr{B}_{q,\gamma,\rho}$, $1 \le q \le \infty$, $0 < \rho \le 1$, $\mathscr{B}_{q,\gamma,1} \equiv \mathscr{B}_{q,\gamma}$, we understand the space of $f \in A(U_\rho)$ for which

$$\|f(z)\|_{\mathscr{B}_{q,\gamma,\rho}} \overset{\text{def}}{=} \|f(\rho z)\|_{\mathscr{B}_{q,\gamma}} < \infty,$$

and $\mathscr{B}_{q,\gamma,a}^{(r)}$ is the space of $f \in A(U)$ such that $f_a^{(r)} \in \mathscr{B}_{q,\gamma}$, $1 \le q \le \infty$. It is proven in [16] that $\mathscr{B}_{q,\gamma}$ enables us to consider $f \in A(U)$ with constraints less stringent in comparison with $\mathscr{B}_q$ on the behavior of f near the boundary circle $\Gamma := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. It is obvious that $H_q \subset \mathscr{B}_q \subset \mathscr{B}_{q,\gamma}$, $1 \le q < \infty$. We denote by $X := X(U)$ any of the above Banach spaces H_q and $\mathscr{B}_{q,\gamma}$, while $X_\rho := X_\rho(U)$ means $H_{q,\rho}$ or $\mathscr{B}_{q,\gamma,\rho}$. Similarly, $X_a^{(r)} := X_a^{(r)}(U)$ is either $H_{q,a}^{(r)}$ or $\mathscr{B}_{q,\gamma,a}^{(r)}$, and $X_{\rho,a}^{(r)}$ is either $H_{q,\rho,a}^{(r)}$ or $\mathscr{B}_{q,\gamma,\rho,a}^{(r)}$.

Given $f \in X(U)$, consider the modulus of continuity

$$\omega(f;2t)_X = \sup\{\|f(ze^{ih}) - f(ze^{-ih})\|_X : |h| \le t\}.$$

We denote by $\mathscr{P}_n$ the set of the complex algebraic polynomials

$$p_n(z) = \sum_{k=0}^n a_k z^k \quad (n \in \mathbb{N},\ a_k \in \mathbb{C})$$

of degree n. The quantity

$$E_n(f)_X = E(f,\mathscr{P}_n)_X \overset{\text{def}}{=} \inf\{\|f - p_n\|_X : p_n \in \mathscr{P}_n\}$$

is the best approximation of $f \in X(U)$ by $\mathscr{P}_n$.

Let $\Phi(u)$ be a nondecreasing positive function defined for $u \geq 0$ such that $\lim\{\Phi(u) : u \to 0\} = \Phi(0) = 0$. Using Φ as a majorant and given $\mu \geq 1$ and $r \in \mathbb{N}$, we introduce the class of the functions

$$W_a^{(r)} X(\Phi, \mu) = \left\{ f \in X_a^{(r)} : \frac{1}{h} \int\limits_0^h \omega\big(f_a^{(r)}; 2t\big)_X \left[1 + (\mu^2 - 1)\sin\frac{\pi t}{2h}\right] dt \leq \Phi(h),\ h \in (0, \pi] \right\}.$$

In [18, 19] for $X = H_q$ and $X = \mathscr{B}_{q,\gamma}$ respectively, it is proven that if the majorant Φ satisfies the condition

$$\frac{\Phi(h)}{\Phi(\pi/(2\mu n))} \geq \frac{\pi}{2\mu} \int\limits_0^1 (\sin nht)_* \left[1 + (\mu^2 - 1)\sin\frac{\pi t}{2}\right] dt \tag{7}$$

for $\mu \geq 1$ and all $h \in (0, \pi]$, $n \in \mathbb{N}$, where

$$(\sin u)_* = \begin{cases} \sin u & \text{if } 0 < u \leq \pi/2, \\ 1 & \text{if } u > \pi/2; \end{cases}$$

then

$$\begin{aligned} b_n\big(W_a^{(r)} X(\Phi; \mu); X(U)\big) &= d_n\big(W_a^{(r)} X(\Phi; \mu); X(U)\big) \\ &= E_{n-1}\big(W_a^{(r)} X(\Phi; \mu)\big)_{X(U)} = \frac{\pi}{4\mu n^r} \Phi\left(\frac{\pi}{2\mu n}\right) \end{aligned} \tag{8}$$

for all $n, r \in \mathbb{N}$. Condition (7) holds, for example, for $\Phi_*(h) = h^\alpha$, where

$$\alpha := \alpha(\mu) = \left(\frac{\pi}{2\mu}\right)^2 \int\limits_0^1 t \cos\left(\frac{\pi t}{2\mu}\right) \left[1 + (\mu^2 - 1)\sin\left(\frac{\pi t}{2}\right)\right] dt. \tag{9}$$

It follows from (9) that $\alpha(1) = (\pi/2) - 1$, $\lim\{\alpha(\mu) : \mu \to \infty\} = 1$, and $(\pi/2) - 1 \leq \alpha(\mu) \leq 1$ for all $\mu \in [1, \infty)$.

Since $X \subset X_\rho$ $(0 < \rho \leq 1)$, it is of indubitable interest to extend (8) from the above-listed n-widths (1)–(4) to a more general space X_ρ $(0 < \rho \leq 1)$:

$$\begin{aligned} \lambda_n\big(W_a^{(r)} X(\Phi; \mu) : X_\rho(U)\big) &= E_{n-1}\big(W_a^{(r)} X(\Phi; \mu)\big)_{X_\rho(U)} \\ &= \mathscr{E}\big(W_a^{(r)} X(\Phi; \mu), \mathscr{L}_{\rho,r-1}, \mathscr{P}_{n-1}\big)_{X_\rho(U)} = \frac{\pi\rho^n}{4\mu n^r} \Phi\left(\frac{\pi}{2\mu n}\right), \end{aligned} \tag{10}$$

where $\lambda_n(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d^n(\cdot)$, $d_n(\cdot)$, or $\delta_n(\cdot)$.

Indeed, in the case when X is $\mathscr{B}_{q,\gamma}$, (10) follows for all n-widths from the results of Theorems 2.1 and 3.1 in [18] and can be derived almost similarly in the case $X = H_q$. The best linear method realizing the sharp value of the linear n-width has the form

$$\mathscr{L}_{\rho,r-1}(f, \mathscr{P}_n; z) = c_0(f) + \sum_{k=1}^{n-1} \mu_{k,\rho,r-1} c_k(f) z^k, \tag{11}$$

where $c_k(f)$ is the Taylor coefficients of f, while

$$\mu_{k,\rho,r-1} \overset{\text{def}}{=} 1 - \rho^{2(n-k)} \left(\frac{k}{2n-k}\right)^{r-1} \left\{1 - \gamma_{k,n}\left(1 - \frac{k^2}{(2n-k)^2}\right)\right\},$$

$$\gamma_{k,n} \overset{\text{def}}{=} n\mu \int\limits_0^{\pi/(2n)} \cos kx \cos nx\, dx, \quad k = 1, \dots, n-1.$$

In line with [13], by $\widetilde{\mathscr{P}}_{n-1}$ we denote the n-dimensional subspace spanned by the basis

$$\widetilde{\varphi}_k(z) = \left\{1 - \left(\frac{k}{2n-1}\right)^{r-1}\left[1 - \gamma_{k,n}\left(1 - \left(\frac{k}{2n-1}\right)^2\right)\right]|z|^{2(n-k)}\right\}z^k,$$
$$k = 0, 1, \dots, n-1, \quad r \in \mathbb{N}.$$

Given $f \in X(U)$, put

$$\widetilde{\mathscr{L}}_{r-1}(f, \widetilde{\mathscr{P}}_{n-1}; z) = \sum_{k=0}^{n-1} c_k(f)\widetilde{\varphi}_k(z).$$

Theorem 1. *If $\mu \geq 1$ and $r, n \in \mathbb{N}$ then*

$$\lambda_n\big(W_a^{(r)}X(\Phi;\mu); l_{q,\gamma}\big) = \bar{\lambda}_n\big(W_a^{(r)}X(\Phi;\mu); \mathscr{B}_{q,\gamma}\big) = \mathscr{E}\big(W_a^{(r)}X(\Phi;\mu), \widetilde{\mathscr{L}}_{r-1}, \widetilde{\mathscr{P}}_{n-1}\big)_{\mathscr{B}_{q,\gamma}}$$
$$= \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\right)^{1/q}, \quad 1 \leq q \leq \infty, \tag{12}$$

where $\lambda_n(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d^n(\cdot)$, $d_n(\cdot)$, or $\delta_n(\cdot)$, while $\bar{\lambda}_n(\cdot)$ is one of the n-widths $d^n(\cdot)$ or $b_n(\cdot)$.

Proof. Following the arguments of the proof of Theorem 2 in [13], we verify that

$$M_q(f - \widetilde{\mathscr{L}}_{r-1}(f, \widetilde{\mathscr{P}}_{n-1}); \rho) \leq \frac{\pi\rho^n}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right) \tag{13}$$

for all $f \in W^{(r)}H_q(\Phi;\mu)$. Taking the power q ($1 \leq q \leq \infty$) of both sides of (13), multiplying the result by $\rho\gamma(\rho)$ and integrating over ρ in the range from 0 to 1, and using the definition of $\mathscr{B}_{q,\gamma}$, we derive that

$$\mathscr{E}\big(W_a^{(r)}X(\Phi;\mu), \widetilde{\mathscr{L}}_{r-1}, \widetilde{\mathscr{P}}_{n-1}\big)_{\mathscr{B}_{q,\gamma}} \leq \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\right)^{1/q}. \tag{14}$$

Hence, by the definition of linear n-width, we obtain

$$\delta_n\big(W_a^{(r)}X(\Phi;\mu); \mathscr{B}_{q,\gamma}\big) \leq \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\right)^{1/q}. \tag{15}$$

Since $\mathscr{B}_{q,\gamma}$ is isomorphic to and isometrically embedded in $l_{q,\gamma}$, from the definitions and properties of the Bernstein and Gelfand n-widths [2, Chapter II, § 3, Proposition 3.2] we find

$$\begin{gathered} d^n\big(W_a^{(r)}H_q(\Phi;\mu); \mathscr{B}_{q,\gamma}\big) = d^n\big(W_a^{(r)}H_q(\Phi;\mu); l_{q,\gamma}\big), \\ b_n\big(W_a^{(r)}H_q(\Phi;\mu); \mathscr{B}_{q,\gamma}\big) = b_n\big(W_a^{(r)}H_q(\Phi;\mu); l_{q,\gamma}\big). \end{gathered} \tag{16}$$

From (5), (15), and (16) we obtain the upper estimate for all n-widths under consideration. To obtain the lower estimates for these n-widths, we introduce the $(n+1)$-dimensional ball of the polynomials

$$\widetilde{S}_{n+1} := \left\{p_n \in \mathscr{P}_n : \|p_n\|_{\mathscr{B}_{q,\gamma}} \leq \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\left(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\right)^{1/q}\right\}$$

and prove the embedding $\widetilde{S}_{n+1} \subset W_a^{(r)}H_q(\Phi;\mu)$. Observe that

$$\omega\big((p_n)_a^{(r)}; 2t\big)_X \leq 2n^r(\sin nt)_*\|p_n\|_X \tag{17}$$

for an arbitrary polynomial $p_n \in \mathscr{P}_n$, which is proved for $X = H_q$ in [4] and for $X = \mathscr{B}_{q,\gamma}$ in [18]. It is proven in [16] that

$$\big\|(p_n)_a^{(r)}\big\|_{H_q} \le n^r \bigg(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\bigg)^{-1/q} \|p_n\|_{\mathscr{B}_{q,\gamma}} \tag{18}$$

for all $p_n \in \mathscr{P}_n$, $1 \le q \le \infty$, and $r, n \in \mathbb{N}$. Using (17) and (18), we see that

$$\omega\big((p_n)_a^{(r)}; 2t\big)_{H_q} \le 2n^r (\sin nt)_* \bigg(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\bigg)^{-1/q} \|p_n\|_{\mathscr{B}_{q,\gamma}}. \tag{19}$$

By (19) and (7), we have

$$\frac{1}{h}\int_0^h \omega\big((p_n)_a^{(r)}; 2t\big)_{H_q} \left[1 + (\mu^2 - 1)\sin\frac{\pi t}{2h}\right] dt$$

$$\le 2n^r \bigg(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\bigg)^{-1/q} \|p_n\|_{\mathscr{B}_{q,\gamma}} \cdot \frac{1}{h}\int_0^h (\sin nt)_* \left[1 + (\mu^2 - 1)\sin\frac{\pi t}{2h}\right] dt$$

$$\le \frac{\pi}{2\mu}\Phi\left(\frac{\pi}{2\mu n}\right)\int_0^1 (\sin nht)_* \left[1 + (\mu^2 - 1)\sin\frac{\pi t}{2}\right] dt \le \Phi(h)$$

for all $p_n \in \widetilde{S}_{n+1}$; whence the embedding $\widetilde{S}_{n+1} \subset W_a^{(r)} H_q(\Phi; \mu)$ is immediate. In view of the proven embedding and the definition of Bernstein n-width, we conclude that

$$b_n\big(W_a^{(r)} H_q(\Phi; \mu); \mathscr{B}_{q,\gamma}\big) \ge b_n(\widetilde{S}_{n+1}; \mathscr{B}_{q,\gamma}) \ge \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\bigg(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\bigg)^{1/q}. \tag{20}$$

The sought equality (12) follows from (15), (16), and (20). Theorem 1 is proven.

In solving the extremal problems of approximation theory for the analytic functions on the disk is of interest, the computation of the sharp upper bounds for the moduli of the Taylor coefficients (see, for example, [13, 16]) on various classes of analytic functions. We present a solution to this problem for the classes of functions under consideration.

Theorem 2. *Put $L_n(\mathscr{M}) := \sup\{|c_n(f)| : f \in \mathscr{M}\}$. Then*

$$L_n\big(W_a^{(r)} H_q(\Phi; \mu)\big) = \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right), \tag{21}$$

$$L_n\big(W_a^{(r)} \mathscr{B}_{q,\gamma}(\Phi; \mu)\big) = \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\bigg(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\bigg)^{-1/q} \tag{22}$$

for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, and $q \in [1, \infty]$.

Proof. Indeed, if $f \in A(U)$ then the Taylor coefficient $c_n(f)$ is represented as

$$c_n(f) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} f(\zeta)\zeta^{-n-1}\,d\zeta = \frac{1}{2\pi\rho^n}\int_0^{2\pi} [f(\rho e^{it}) - \mathscr{L}_{\rho,r-1}(f, \mathscr{P}_n; \rho e^{it})] e^{-int}\,dt, \tag{23}$$

where $\mathscr{L}_{\rho,n-1}(f,\mathscr{P}_n;\rho e^{it})$ is defined by (11). By Hölder's inequality and (10), from (21) we obtain

$$|c_n(f)| \le \rho^{-n}\mathscr{E}(f;\mathscr{L}_{\rho,r-1}(f,\mathscr{P}_n))_{H_{q,\rho}} \le \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)$$

for all $f \in W_a^{(r)}H_q(\Phi;\mu)$, whence the upper estimate

$$L_n\big(W_a^{(r)}H_q(\Phi;\mu)\big) \le \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)$$

is immediate. On the other hand, writing the coefficient $c_n(f)$ as

$$c_n(f) = \frac{1}{2\pi(\rho R)^n}\int_0^{2\pi}[f(\rho Re^{i\tau}) - \mathscr{L}_{\rho,r-1}(f,\mathscr{P}_n;\rho Re^{i\tau})]e^{-in\tau}\,d\tau$$

and using Hölder's inequality, we obtain

$$R^n|c_n(f)| \le \rho^{-n}M_q(f-\mathscr{L}_{\rho,r-1}(f,\mathscr{P}_n);\rho R)$$

for all $\rho, R \in (0,1)$, whence, by the definition of the norm on $X(U)$, we derive

$$|c_n(f)| \le \rho^{-n}\mathscr{E}(f;\mathscr{L}_{\rho,r-1}(f,\mathscr{P}_n);X_\rho)\bigg(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\bigg)^{-1/q}$$

for $f \in W_a^{(r)}X(\Phi;\mu)$. This inequality together with (10) yields the upper estimate

$$L_n\big(W_a^{(r)}X(\Phi;\mu)\big) \le \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\begin{cases} 1 & \text{if } X(U) = H_q, \\ \big(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\big)^{-1} & \text{if } X(U) = \mathscr{B}_{q,\gamma}. \end{cases} \tag{24}$$

To obtain the lower estimate, consider

$$f_0(z) = \frac{1}{(in)^r}\cdot\frac{\pi}{4\mu}\Phi\left(\frac{\pi}{2\mu n}\right)\frac{z^n}{\|z^n\|_X} \in W_a^{(r)}X(\Phi;\mu).$$

Using the definition of $L_n(\cdot)$ for this function, we write down the lower estimate

$$\mathscr{L}_n\big(W_a^{(r)}X(\Phi;\mu)\big) \ge |c_n(f_0)| = \frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\begin{cases} 1 & \text{if } X(U) = H_q, \\ \big(\int_0^1 \rho^{nq+1}\gamma(\rho)\,d\rho\big)^{-1} & \text{if } X(U) = \mathscr{B}_{q,\gamma}. \end{cases} \tag{25}$$

Equalities (21) and (22) are obtained by comparing of the upper and lower estimates (24) and (25). Theorem 2 is proven.

3. In Sections 1 and 2 we calculated the exact values of the various widths from the above-indicated classes of analytic functions on the unit disk. We can relate the problems of optimal recovery and coding of the functions as interpreted by Korneichuk [21, Chapter 8, § 8.3; 22] to the widths from these function classes.

Let us give the necessary notions and definitions. Suppose that the collection $M_n := \{\mu_1,\mu_2,\dots,\mu_n\}$ of some functionals μ_k, $k = 1,\dots,n$, is given on the normed function space X. The set M_n can be considered as the coding method sending $f \in X$ to $T(f,M_n) = \{\mu_1(f),\dots,\mu_n(f)\}$. The problem of recovery of f from T is solved by relating

$$A(f,M_n;G_n,\Gamma_n:z) = \sum_{k=1}^n \gamma_k\mu_k(f)g_k(z)$$

to $T(f,M_n)$, where $G_n=\{g_k(z)\}_{k=1}^n$ and $\Gamma_n=\{\gamma_k\}_{k=1}^n\in\mathrm{Im}$ are respectively a system of linearly independent functions from X and a collection of numerical coefficients giving the best representation of the elements of the class $\mathscr{M}\subset X$, while $\mathrm{Im}=\{\Gamma_n\}$ is the vector of numerical coefficients. We assume the error of recovery on $\mathscr{M}$ is equal to

$$\mathscr{R}_n(\mathscr{M};M_n,G_n)=\inf\{\sup\{\|f-A(f;M_n,G_n,\Gamma_n)\|_{B_{q,\gamma}}:f\in\mathscr{M}\}:\Gamma_n\subset C^n\} \tag{26}$$

and put $\mathscr{R}_n(\mathscr{M},X)=\inf\{\mathscr{R}(\mathscr{M};M_n,G_n):M_n,G_n\}$.

Let M_n' be a collection of the bounded linear functionals on X. Consider the characteristic

$$\mathscr{R}_n'(\mathscr{M},X)=\inf\{\mathscr{R}(\mathscr{M};M_n',G_n):M_n',G_n\}.$$

The method of recovery $(\overset{\circ}{M}_n,\overset{\circ}{G}_n,\overset{\circ}{\Gamma}_n)$ $\{\overset{\circ}{M}{}_n',\overset{\circ}{G}_n,\overset{\circ}{\Gamma}_n\}$ satisfying

$$\mathscr{R}_n(\mathscr{M},X)=\sup\{\|f-A(f,\overset{\circ}{M}_n,\overset{\circ}{G}_n,\overset{\circ}{\Gamma}_n)\|_X:f\in\mathscr{M}\},$$

$$\{\mathscr{R}_n'(\mathscr{M},X)=\sup\{\|f-A(f,\overset{\circ}{M}{}_n',\overset{\circ}{G}_n,\overset{\circ}{\Gamma}_n)\|_X:f\in\mathscr{M}\}\}$$

is referred to as the *optimal* (*optimal linear*) *recovery method* for the functions of $\mathscr{M}$. The following are valid [21]:

$$\mathscr{R}_n'(\mathscr{M},X)=\lambda_n(\mathscr{M},X),\quad \mathscr{R}_n(\mathscr{M},X)\ge d_n(\mathscr{M},X). \tag{27}$$

If $\mathscr{M}=\widetilde{\mathscr{M}}\otimes L$, where $\widetilde{\mathscr{M}}$ is a compact set and L is a finite-dimensional subspace, then the equalities hold in (27).

Alongside (26), consider the quantity

$$\mathscr{K}(\mathscr{M},M_n)=\sup\{\|f_1-f_2\|_X:f_1,f_2\in\mathscr{M},T(f_1,M_n)=T(f_2,M_n)\}$$

which can be interpreted as the error of the coding method on $\mathscr{M}$ by means of M_n.

Putting

$$\nu^n(\mathscr{M},X)=\inf\{\mathscr{K}(\mathscr{M},M_n):M_n\},$$

where the infimum is taken over all collections M_n on the dual space X^*, we obtain $\nu^n(\mathscr{M},X)\le 2\mathscr{R}_n'(\mathscr{M},X)$; and if $\mathscr{M}$ is a centrally symmetric convex set, then $\nu^n(\mathscr{M},X)=2d^n(\mathscr{M},X)$.

Theorem 3. *Under condition* (7), *the collection* $\overset{\circ}{M}_n$ *of the linear functionals*

$$\overset{\circ}{\mu}_k(f)=c_k(f),\quad k=0,\dots,n-1, \tag{28}$$

provides the best coding method for the functions of $W_a^{(r)}X(\Phi;\mu)$ *in* $X_\rho(U)$. *The optimal linear recovery method for* $\overset{\circ}{M}{}_n'$, $\overset{\circ}{G}_n$, $\overset{\circ}{\Gamma}_n$ *of* $f(z)$ *from* $W_a^{(r)}X(\Phi;\mu)$ *in* $X_\rho(U)$ *is the linear method* $\mathscr{L}_{\rho,r-1}(f,\mathscr{P}_{n-1};z)$ *defined by* (11). *Moreover,*

$$\nu^n\big(W_a^{(r)}X(\Phi;\mu),X_\rho(U)\big)=\mathscr{R}_n\big(W_a^{(r)}X(\Phi;\mu),X_\rho(U)\big)$$
$$=\mathscr{R}_n'\big(W_a^{(r)}X(\Phi;\mu),X_\rho(U)\big)=\frac{\pi\rho^n}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)$$

for all $n\in\mathbb{N}$.

Theorem 4. *Under condition* (7), *the optimal linear recovery method for* $f(z)\in W_a^{(r)}H_q(\Phi;\mu)$ *in* $\mathscr{L}_{q,\gamma}$ *is the linear method* $\widetilde{V}_{r-1}(f,\widetilde{\mathscr{P}}_n,z)$ *defined in Section* 2, *while the best coding method is the collection of functionals* (28). *Moreover,*

$$\frac{1}{2}\lambda^n\big(W_a^{(r)}H_q(\Phi;\mu),\mathscr{L}_{q,\gamma}\big)=\mathscr{R}_n\big(W_a^{(r)}H_q(\Phi;\mu),\mathscr{L}_{q,\gamma}\big)$$
$$=\mathscr{R}_n'\big(W_a^{(r)}H_q(\Phi;\mu),\mathscr{L}_{q,\gamma}\big)=\frac{\pi}{4\mu n^r}\Phi\left(\frac{\pi}{2\mu n}\right)\left(\int_0^1\rho^{nq+1}\gamma(\rho)\,d\rho\right)^{-1/q}$$

for all $n\in\mathbb{N}$.

The authors are grateful to the referee for the valuable remarks.

References

1. Tikhomirov V. M., “Diameters of sets in function spaces and the theory of best approximations,” Russian Math. Surveys, vol. 15, no. 3, 75–111 (1960).
2. Pinkus A., *n-Widths in Approximation Theory*, Springer-Verlag, Berlin (1985).
3. Taikov L. V., “On the best approximation in the mean of certain classes of analytic functions,” Math. Notes, vol. 1, no. 2, 104–109 (1967).
4. Taikov L. V., “Diameters of certain classes of analytic functions,” Math. Notes, vol. 22, no. 2, 650–656 (1977).
5. Ainulloev N. and Taikov L. V., “Best approximation in the sense of Kolmogorov of classes of functions analytic in the unit disc,” Math. Notes, vol. 40, no. 3, 699–705 (1986).
6. Dveirin M. Z., “On approximation of analytic functions in the unit disk,” in: *Metric Problems of the Theory of Functions and Mappings* [Russian], Naukova Dumka, Kiev, 1975, 41–54.
7. Farkov Yu. A., “Widths of Hardy classes and Bergman classes on the ball in $\mathbb{C}^n$,” Russian Math. Surveys, vol. 45, no. 5, 229–231 (1990).
8. Farkov Yu. A., “n-Widths, Faber expansion, and computation of analytic functions,” J. Complexity, vol. 2, no. 1, 58–79 (1996).
9. Fisher S. D. and Stessin M. I., “The n-widths of the unit ball of H^q,” J. Approx. Theory, vol. 67, no. 3, 347–356 (1991).
10. Vakarchuk S. B., “Best linear methods of approximation and widths of classes of analytic functions in a disk,” Math. Notes, vol. 57, no. 1, 21–27 (1995).
11. Vakarchuk S. B., “On the best linear approximation methods and the widths of certain classes of analytic functions,” Math. Notes, vol. 65, no. 2, 153–158 (1999).
12. Vakarchuk S. B., “Exact values of widths of classes of analytic functions on the disk and best linear approximation methods,” Math. Notes, vol. 72, no. 5, 615–619 (2002).
13. Vakarchuk S. B., “On some extremal problems of approximation theory in the complex plane,” Ukrainian Math. J., vol. 56, no. 9, 1371–1390 (2004).
14. Vakarchuk S. B. and Zabutnaya V. I., “Best linear approximation methods for functions of Taikov classes in the Hardy spaces $H_{q,\rho}$, $q \geq 1$, $0 < \rho \leq 1$,” Math. Notes, vol. 85, no. 3, 322–327 (2009).
15. Shabozov M. Sh. and Langarshoev M. R., “On the best approximation of some classes of functions in the weight Bergman space,” Izv. Akad. Nauk Resp. Tadzhikistan. Otd. Fiz.-Mat., Khim., Geol. i Tekh. Nauk, vol. 136, no. 3, 7–23 (2009).
16. Vakarchuk S. B. and Shabozov M. Sh., “The widths of classes of analytic functions in a disc,” Sb. Math., vol. 201, no. 8, 1091–1110 (2010).
17. Shabozov M. Sh. and Langarshoev M. R., “The best linear methods and values of widths for some classes of analytic functions in the Bergman weight space,” Dokl. Math., vol. 87, no. 3, 338–341 (2013).
18. Shabozov M. Sh. and Saidusaynov M. S., “The values of n-widths and best linear approximation methods for some classes of analytic functions in the Bergman weight space,” Izv. TulGU. Estestv. Nauki, no. 3, 40–57 (2014).
19. Shabozov M. Sh. and Yusupov G. A., “Best linear methods of approximation and widths for some classes of functions in the Hardy space,” Dokl. Akad. Nauk Resp. Tajikistan, vol. 57, no. 2, 97–102 (2014).
20. Langarshoev M. R., “On the best linear methods of approximation and the exact values of widths for some classes of analytic functions in the weighted Bergman space,” Ukrainian Math. J., vol. 67, no. 10, 1537–1551 (2016).
21. Korneichuk N. P., *Exact Constants in Approximation Theory* [Russian], Nauka, Moscow (1987).
22. Korneichuk N. P., “Widths in L_p of classes of continuous and of differentiable functions, and optimal methods of coding and recovering functions and their derivatives,” Math. USSR-Izv., vol. 18, no. 2, 227–247 (1982).

M. Sh. Shabozov
University of Central Asia, Tajik National University, Dushanbe, Tajikistan
E-mail address: shabozov@mail.ru

M. R. Langarshoev
Tajik National University, Dushanbe, Tajikistan
E-mail address: mukhtor77@mail.ru