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MAXIMAL SOLVABLE SUBGROUPS OF SIZE 2 INTEGER MATRICES

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Abstract: Studying the solvable subgroups of 2×2 matrix groups over Z, we find a maximal finite order primitive solvable subgroup of $GL(2, Z)$ unique up to conjugacy in $GL(2, Z)$. We describe the maximal primitive solvable subgroups whose maximal abelian normal divisor coincides with the group of units of a quadratic ring extension of Z. We prove that every real quadratic ring R determines h classes of conjugacy in $GL(2, Z)$ of maximal primitive solvable subgroups of $GL(2, Z)$, where h is the number of ideal classes in R.

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Introduction. Denote by Z the ring of integers; by M, a free two-dimensional Z-module; and by $GL(2, Z)$, the automorphism group of M. As in [1], call a subgroup Γ of $GL(2, Z)$ *reducible* whenever M includes one-dimensional submodules invariant under Γ. Then M has a one-dimensional direct summand invariant under Γ; see [2].

This article studies the structure of maximal solvable subgroups of $GL(2, Z)$.

1. If Γ is a reducible subgroup of $GL(2, Z)$ then Γ is obvious that the matrices g in Γ are all simultaneously of the form

$$g = \begin{bmatrix} \pm 1 & z \\ 0 & \pm 1 \end{bmatrix}, \quad z \in Z. \tag{1}$$

This implies that $GL(2, Z)$ includes a maximal solvable reducible subgroup unique up to conjugacy in $GL(2, Z)$.

Thus, we have to study only the irreducible maximal solvable subgroups of $GL(2, Z)$. Take an irreducible subgroup Γ of $GL(2, Z)$. The group Γ is called *imprimitive* whenever we can express M as the direct sum of submodules permuted by the automorphisms in Γ. If Γ is an imprimitive subgroup of $GL(2, Z)$ then M obviously has some basis

$$u_1, u_2 \tag{2}$$

in which all $g \in \Gamma$ are monomial. Consequently, the matrices g in Γ are of the two forms:

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}. \tag{3}$$

Thus, $GL(2, Z)$ includes a maximal imprimitive solvable subgroup Γ unique up to conjugacy in $GL(2, Z)$ and consisting of eight matrices of the form (3). It remains to consider only the primitive solvable subgroups of $GL(2, Z)$.

2. Take a maximal primitive solvable subgroup Γ of $GL(2, Z)$ and the maximal abelian normal divisor H of G.

Consider the two cases:

(1) H is a reducible subgroup of $GL(2, Z)$;

(2) H is an irreducible subgroup of $GL(2, Z)$.

Assume that H is a reducible maximal abelian normal divisor of Γ. It is obvious that H is conjugate in $GL(2, Z)$ to some subgroup of the form (1) of the full matrix group. Since Γ is also irreducible as

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a subgroup of $GL(2,Q)$, where Q stands for the field of rationals, it follows that H is a totally reducible subgroup of $GL(2,Q)$. The order of H divides 8, and so it must be equal to 2 or 4. Consider these two cases.

(1) The order of H equals 4. Then H is generated by two matrices of the form

$$h_1 = \begin{bmatrix} -1 & z_1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad h_2 = \begin{bmatrix} 1 & -z_1 \\ 0 & -1 \end{bmatrix}, \quad \text{with } z_1 \in Z.$$

(2) The order of H equals 2. Then H consists of two matrices with ± 1 on the main diagonal.

In turn, case 1 splits into the two subcases:

(a) $z_1 \equiv 0 \pmod 2$;

(b) $z_1 \equiv 1 \pmod 2$.

We may assume that in subcase (a)

$$h_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

whereas in subcase (b)

$$h_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

Consider subcase (a). Take $g \in \Gamma \backslash H$, so that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a,b,c,d \in Z \text{ with } ad - bc = \pm 1 \text{ and } gh_1g^{-1} = h_2.$$

Then from $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we find that $g = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ and $bc = \pm 1$. Consequently, Γ is imprimitive, and so subcase (a) is impossible.

Consider subcase (b). There exists $g \in \Gamma \backslash H$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that $gh_1g^{-1} = h_2$. This yields

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} -a & a+b \\ -c & c+d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -c & -d \end{bmatrix}.$$

Hence, the two possibilities are open for g; i.e.,

$$g_1 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

The following two groups correspond to the open possibilities:

$$\Gamma_1 = (g_1)H; \qquad \Gamma_2 = (g_2)H;$$

$$\Gamma_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}; H \right\}; \quad \Gamma_2 = \left\{ \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}; H \right\}.$$

Since ${g_1}^{-1}g_2 = g_1g_2 = h_2$, it follows that $\Gamma_1 = \Gamma_2$. Thus, Γ_1 is conjugate to $G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ by way of the matrix $f = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, and so Γ_1 is primitive. Consequently, subcase (b) is impossible.

Consider now case (2); i.e., $H = \{\pm E_2\}$. Take a maximal abelian normal divisor A/H of Γ/H. Since the orders of elements of the quotient group divide its order, A/H is a 2-group. Hence, we can express A as $A = (a_1, b_1)$, where $(a_1, b_1) = -1$, $a_1^2 = \pm 1$, and $b_1^2 = \pm 1$. We may assume that $a_1^2 = b_1^2 = 1$. Given $g \in \Gamma$, we have

$$ga_1g^{-1} = \lambda a_1^{\alpha} b_1^{\beta} = c_1, \quad gb_1g^{-1} = \mu a_1^{\gamma} b_1^{\delta} = d_1, \tag{4}$$

where $\alpha, \mu = \pm 1$ and $\alpha, \beta, \gamma, \delta \in Z_2 = Z/(2)$. Clearly, $c_1^2 = a_1^2$ and $d_1^2 = b_1^2$. This and (4) yield $\alpha + \beta + \alpha\beta = \gamma + \delta + \gamma\delta$. Since $\alpha, \beta, \gamma, \delta \in Z_2$, it follows that $\alpha = \beta = \gamma = \delta$. The latter is impossible.

Thus, $GL(2, Z)$ lacks maximal primitive solvable subgroups with reducible maximal abelian normal divisors.

3. Assume that H is irreducible. The maximal abelian normal divisor of a primitive group is the group of all invertible elements of an integral domain whose dimension is a divisor of 2. Since Γ is maximal, this implies that H is the multiplicative group of the ring $R = Z(\Theta)$, where Θ is a root of the polynomial $x^2 + \alpha x + \beta$ irreducible over Z. The regular expression of the roots Θ_1 and Θ_2 of the polynomial in the basis $[1, \Theta]$ is as follows:

$$\Theta_1 = \begin{bmatrix} 0 & -\beta \\ 1 & -\alpha \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} -\alpha & \beta \\ -1 & 0 \end{bmatrix}. \tag{5}$$

By the corollary to Theorem 3 of [3] and the maximality of Γ, the group Γ/H is isomorphic to the relative automorphism group of $Z(\Theta)$. Thus, $\Gamma : H = 2$. Consequently, we can always choose some element g in the nonabelian group Γ so that $g\Theta_1 g^{-1} = \Theta_2$. Given $t \in \Gamma$, we have $t\Theta_1 t^{-1} = \Theta_i$ for $i = 1, 2$; thus, $\Gamma = (g)H$.

Let us find the matrix of g. Denote by $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in Z$ with $ad - bc = \pm 1$, the matrix corresponding to g in the basis $[1, \Theta]$. It is clear that $g^2 \in H$. From $g\Theta_1 = \Theta_2 g$ we find that

$$g = \begin{bmatrix} a & \alpha a + \beta c \\ c & -a \end{bmatrix}. \tag{6}$$

Write $g^2 = \omega + v\Theta_1$, where $\omega, v \in Z$. Then $g^2 = g(\omega + v\Theta_1)g^{-1} = \omega + v\Theta_2 = \omega + v\Theta_1$. Consequently, $(\Theta_2 - \Theta_1)v = 0$. Since the roots Θ_1 and Θ_2 are distinct, we infer that $v = 0$. But then $g^2 = \pm E_2$. Consider both cases.

Assume that $g^2 = E_2$. Then $a^2 - \alpha ac + \beta c^2 = 1$. If $Z(\Theta)$ is an imaginary quadratic extension of Z, while the quadratic form $a^2 - \alpha ac + \beta c^2$ (i.e., $\{1, -\alpha, \beta\}$) has discriminant Δ, then for $\Delta = -3$ this form is brought into reduced form by the unimodular substitution $S = \begin{bmatrix} 1 & \frac{1+\alpha}{2} \\ 0 & 1 \end{bmatrix}$. Consequently, $(1, 0)$ will be a solution to the equation $a^2 - \alpha ac + \beta c^2 = 1$. We can find the remaining solutions by multiplying S on the left by the automorphisms of the quadratic form $\{1, 1, 1\}$. These solutions are the pairs $(-1, 0)$, $\left(\frac{1+\alpha}{2}, 1\right)$, $\left(\frac{-1+\alpha}{2}, 1\right)$, $\left(\frac{-1-\alpha}{2}, -1\right)$, and $\left(\frac{1-\alpha}{2}, -1\right)$. Hence, for $\Delta = -3$ the condition $g\Theta_1 g^{-1} = \Theta_2$ is met by the following three pairs of matrices:

$$g_{1,2} = \pm \begin{bmatrix} 1 & -\alpha \\ 0 & -1 \end{bmatrix}, \quad g_{3,4} = \pm \begin{bmatrix} \frac{1+\alpha}{2} & \frac{3-\alpha}{2} - \beta \\ 1 & -\frac{1-\alpha}{2} \end{bmatrix}, \quad g_{5,6} = \pm \begin{bmatrix} \frac{1-\alpha}{2} & \frac{-3-\alpha}{2} + \beta \\ -1 & \frac{-1+\alpha}{2} \end{bmatrix}. \tag{7}$$

For $\Delta = -4$ the solutions to the equation $a^2 - \alpha ac + \beta c^2 = 1$ are the pairs $(\pm 1, 0)$, $\left(-\frac{\alpha}{2}, -1\right)$, and $\left(\frac{\alpha}{2}, 1\right)$. Consequently, in this case

$$g_{1,2} = \pm \begin{bmatrix} 1 & -\alpha \\ 0 & -1 \end{bmatrix}, \quad g_{3,4} = \pm \begin{bmatrix} \frac{\alpha}{2} & 2 - \beta \\ 1 & -\frac{\alpha}{2} \end{bmatrix}.$$

If $\Delta < 0$ but $\Delta \neq -3$ and $\Delta \neq -4$ then the equation $a^2 - \alpha ac + \beta c^2 = 1$ has two solutions. However, in this case H consists only of the matrices $\pm E_2$, and by the above $GL(2, Z)$ lacks the primitive subgroups having this H as their maximal abelian normal divisor. Consequently, in the case of the positive definite quadratic form $\{1, -\alpha, \beta\}$ and $g^2 = E_2$ we have to consider only the discriminants $\Delta = -3$ and $\Delta = -4$.

Since no positive definite quadratic form can represent the number -1, it follows that $g^2 \neq -E_2$ for $\Delta < 0$.

By Dirichlet's Unit Theorem (see [4]) the multiplicative group of each imaginary quadratic extension of Z is a finite cyclic group. Consider a finite maximal primitive solvable subgroup of $GL(2,Z)$. As we mentioned above, it suffices to consider the cases $\Delta = -3$ and $\Delta = -4$. Since $\Gamma = (g)H$, it follows that $\Gamma = \left\{ g, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ for $\Delta = -4$. It is easy to see that the order of Γ in this case equals 8. There are only five isomorphism types of order 8 groups (see [5, Introduction]) and Γ belongs to one of them (the dihedral group) only for $\alpha = 0$. However, in this case Γ is not primitive.

For $\Delta = -3$ the three pairs of matrices g were constructed, see (7), but only for $\alpha = -1$ they constitute, together with $H = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$, groups (of type 3 in [5, § 72]). Furthermore, all these groups coincide. Hence, for the resulting order 12 group we have the following

Theorem 1. *Up to conjugacy in $GL(2,Z)$, the order 12 group*

$$\Gamma = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\} \tag{8}$$

is the unique maximal primitive solvable subgroup of finite order of $GL(2,Z)$.

4. Consider the case that $Z(\Theta)$ is a real quadratic extension of Z; i.e., the discriminant Δ is positive.

If $\Delta = 4k$, where k is a positive integer and $g^2 = E_2$; then the quadratic form $\{1, -\alpha, \beta\}$ representing the unit is equivalent to the form $\{1, 0, -k\}$ and is taken into the form by the permutation $S = \begin{bmatrix} 1 & \frac{-\alpha}{2} \\ 0 & 1 \end{bmatrix}$. The undetermined Fermat equation $x^2 - ky^2 = 1$ has infinitely many integer solutions; therefore, there exist infinitely many matrices g of the form (6). Finding these matrices reduces to finding the numerators and denominators of the continued fractions of $\sqrt{k}$.

The case $\Delta = 4k$ and $g^2 = -E_2$ reduces to solving the undetermined Fermat equation too.

For $\Delta = 4k+1$ the quadratic form $\{1, -\alpha, \beta\}$ is equivalent to the form $\{1, 1, -k\}$ for $g^2 = E_2$ and the form $\{-1, 1, k\}$ for $g^2 = -E_2$. The Diophantine equations $x^2 + xy - ky^2 = 1$ and $x^2 - xy - ky^2 = 1$ have infinitely many solutions. Denote the roots of $x^2 + x - k = 0$ by ξ_1 and ξ_2; so that, for instance, the first of these two equations rearranges as $(x - \xi_1 y) \cdot (x - \xi_2 y) = 1$. However, if the norm $N(X - \xi_1 Y)$ equals 1 then $X - \xi_1 Y$ is a unit of the real quadratic ring $Z(\Theta)$, and so it can be expressed as some power v of the principal unit ε_0 of $Z(\Theta)$: $X - \xi_1 Y = \varepsilon_0^v$.

This completely settles the question of constructing the matrices g of the form (6) in all cases.

If the order of H is not finite then H is the multiplicative group of a real quadratic ring. By Dirichlet's Unit Theorem, this multiplicative group is an extension of the cyclic group of order 2 consisting of $\pm E_2$ by the infinite cyclic group. This implies that we can express all elements of H as $\varepsilon = a + \Theta c$, where $a, c \in Z$ with $a^2 - \alpha ac + \beta c^2 = \pm 1$ and Θ is a root of the irreducible equation $x^2 + \alpha x + \beta = 0$. The regular expression for ε in the fundamental basis is

$$\varepsilon = \begin{bmatrix} a & -\beta c \\ c & a - \alpha c \end{bmatrix}. \tag{9}$$

Multiply now the matrices g of (6) and ε of (9):

$$g\varepsilon = \begin{bmatrix} a^2 - \alpha ac + \beta c^2 & -\alpha\beta c^2 - \alpha a^2 + \alpha^2 ac \\ 0 & -a^2 + \alpha ac - \beta c^2 \end{bmatrix}.$$

Hence, $g = \pm \begin{bmatrix} 1 & -\alpha \\ 0 & -1 \end{bmatrix} \cdot \varepsilon^{-1}$. Consequently, we arrive at

Theorem 2. *The maximal primitive solvable subgroups of $GL(2,Z)$ whose maximal abelian normal divisor H coincides with the group of units of a real quadratic extension of Z by a root of some polynomial $x^2 + \alpha x + \beta$ irreducible over the field of rationals are of the form*

$$\Gamma = \left\{ \begin{bmatrix} 1 & -\alpha \\ 0 & -1 \end{bmatrix}; H \right\}.$$

Suppose that the real quadratic ring R whose group of units is H has h ideal classes. Since the characteristic polynomial of the matrix of the regular expression for the principal unit of R is rationally irreducible; by [6, pp. 393–395], we have $r_i^{-1}Hr_i = H_i \subset GL(2,Z)$, where r_i is the transition matrix from the basis for R to the basis for the ideal I_i of R, for $i = 1, 2, \dots, h$.

The groups H and H_i are not conjugate in $GL(2,Z)$. Indeed, denote by N the group of units of R; by $[\omega_1, \omega_2] = [\omega]$, a basis for R; and by $\left[v_1^i, v_2^i\right] = [v_i]$, a basis for I_i. Then the expressions for N in the bases $[\omega]$ and $[v_i]$ are of the form

$$[\omega \mathrm{N}] = H[\omega], \quad [v_i \mathrm{N}] = H_i[v_i].$$

If $H = s^{-1}H_i s$, where $s \in GL(2,Z)$; then $s[\omega\mathrm{N}] = H_i s[\omega]$. However, H_i expresses N in the basis $[v_i]$; therefore, $[v_i] = s[\omega]$. The resulting contradiction justifies the claim.

The transition from a special basis of the ring to a special basis of the ideal (see [4]) is realized by transformation with some matrix $r = \begin{bmatrix} m_1 & m_2 \\ 0 & m_3 \end{bmatrix}$; hence, the matrix $g = \begin{bmatrix} 1 & -\alpha \\ 0 & -1 \end{bmatrix}$ of this transformation goes to the matrix

$$r^{-1}gr = \begin{bmatrix} 1 & 2\frac{m_2}{m_1} - \alpha\frac{m_3}{m_1} \\ 0 & -1 \end{bmatrix}. \tag{10}$$

It is clear from (10) that $r^{-1}gr$ belongs to $GL(2,Z)$ only if the following divisibility condition holds:

$$m_1/(2m_2 - \alpha m_3). \tag{11}$$

This implies that if (11) holds then $r^{-1}\Gamma r$ is a maximal primitive solvable subgroup of $GL(2,Z)$ and $r^{-1}\Gamma r$ is not conjugate with Γ in $GL(2,Z)$. If (11) is violated then H_i is a maximal irreducible primitive solvable subgroup of $GL(2,Z)$.

Since we avoid restrictions on the group of units of the real quadratic extension of Z, this implies the next

Theorem 3. *The group $GL(2,Z)$ includes infinitely many maximal primitive solvable subgroups not conjugate in $GL(2,Z)$. Every real quadratic ring R determines h classes of conjugacy in $GL(2,Z)$ of maximal primitive solvable subgroups of $GL(2,Z)$, where h is the number of ideal classes in R.*

5. Consider two examples of infinite maximal solvable subgroups of $GL(2,Z)$.

Example 1. Determine the maximal primitive irreducible solvable groups in $GL(2,Z)$ related to the extension of Z by a root of the polynomial $x^2 - x - 1$.

The principal unit $\varepsilon_0 = \frac{1}{2}(1+\sqrt{5})$ of the ring $Z(\sqrt{5})$ has the following regular expression in the fundamental basis of this ring:

$$\varepsilon_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The ring $Z(\sqrt{5})$ has only one ideal class; therefore, we can uniquely express its group of units H as a subgroup of $GL(2,Z)$:

$$H = \left\{-E_2; \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\} = (E_2)^k \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n.$$

Consequently,

$$\Gamma = \left\{-E_2; \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\} = (E_2)^k \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^s \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$$

is a unique maximal irreducible solvable subgroup of $GL(2,Z)$ whose maximal abelian normal divisor coincides with the group of units of $Z(\sqrt{5})$.

Example 2. If H_1 is the multiplicative group of the extension R of Z by a root of the polynomial $x^2 - 6x - 1$ then the presence in $Z(\sqrt{10})$ of two ideal classes implies the existence of a subgroup H_2 of $GL(2,Z)$ rationally conjugate to H_1 but not conjugate to H_1 in $GL(2,Z)$.

The principal unit ε_0 of $Z(\sqrt{10})$ is determined by the least integer solution to Pell's equation: $\frac{u^2-10v^2}{4} = \pm 1$. Since the expansion of $\sqrt{10}$ as a continued fraction is

$$\sqrt{10} = 3 + (\sqrt{10} - 3) = 3 + \frac{1}{\sqrt{10}+3},$$

$$\sqrt{10} + 3 = 6 + (\sqrt{10} - 3) = 6 + \frac{1}{\sqrt{10}+3}, \quad \sqrt{10} = \{3, \bar{6}\},$$

and the convergent numerators and denominators constitute the scheme

6		
6	1	0
1	0	1

,

the principal unit of this extension is $\varepsilon_0 = 3 + \sqrt{10}$. The fundamental basis of the ring $Z(\sqrt{10})$ is of the form $[w_1, w_2] = [1, \sqrt{10}]$, while $[v_1, v_2] = [3, 2 + \sqrt{10}]$ is a basis for an ideal of this ring. Indeed, $3\sqrt{10} = -2 \cdot 3 + 3 \cdot (2 + \sqrt{10})$ and $(2 + \sqrt{10}) \cdot \sqrt{10} = 2 \cdot 3 + 2 \cdot (2 + \sqrt{10})$. The transition matrix from the first basis to the second one is $r = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$. To the unit ε_0 in the basis $[w_1, w_2]$ of the ring R there corresponds the matrix $\Theta_1 = \begin{bmatrix} 3 & 10 \\ 1 & 3 \end{bmatrix}$; whereas in the basis $[v_1, v_2]$ of the ideal, the matrix $\Theta_2 = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Condition (10) is violated since 3 does not divide 10. Thus, the extension $Z(\sqrt{10})$ determines, up to conjugacy in $GL(2, Z)$, the two maximal irreducible solvable subgroups of $GL(2, Z)$:

$$\Gamma_1 = \left\{ -E_2; \begin{bmatrix} 1 & 6 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 3 & 10 \\ 1 & 3 \end{bmatrix} \right\} = (-E_2)^k \cdot \begin{bmatrix} 1 & 6 \\ 0 & -1 \end{bmatrix}^s \cdot \begin{bmatrix} 3 & 10 \\ 1 & 3 \end{bmatrix}^n;$$

$$\Gamma_2 = \left\{ -E_2; \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}; \right\} = (-E_2)^k \cdot \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^n.$$

It is clear from these examples that the problem of constructing a maximal primitive solvable subgroup Γ of $GL(2, Z)$ whose maximal abelian normal divisor coincides with the multiplicative group of $Z(\sqrt{d})$ with $d > 0$ reduces to finding the principal unit of this extension.

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