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NECESSARY CONDITIONS FOR THE RESIDUAL NILPOTENCY OF CERTAIN GROUP THEORY CONSTRUCTIONS

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Abstract: Consider a graph G of groups such that each vertex group locally satisfies a nontrivial identity and each edge subgroup is properly included into the corresponding vertex groups and its index in at least one of them exceeds 2. We prove that if the fundamental group F of G is locally residually nilpotent then there exists a prime number p such that each edge subgroup is p'-isolated in the corresponding vertex group. We show also that if F is the free product of an arbitrary family of groups with one amalgamated subgroup or a multiple HNN-extension then the same result holds without restrictions on the indices of edge subgroups.

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1. Introduction

Call a group X *locally satisfying a nontrivial identity* whenever each finitely generated subgroup of X satisfies some nontrivial identity that is not necessarily the same for all subgroups. Call a group X *locally residually nilpotent* whenever every finitely generated subgroup Y of X is *residually nilpotent*, meaning that for each nontrivial $y \in Y$ there exists a homomorphism σ of Y onto a nilpotent group with $y\sigma \neq 1$. A group need not be residually nilpotent or satisfy any nontrivial identity to have these properties locally, and we provide an example at the end of this article.

Our goal is to find necessary conditions for the local residual nilpotency for the free constructions composed of the groups locally satisfying some nontrivial identity. As we show, the suitable condition under some restrictions is that all amalgamated or associated subgroups are p'-isolated for a prime p. Recall that a subgroup Y of a group X is called *p'-isolated* in X whenever for any $x \in X$ and any prime $q \neq p$ the condition $x^q \in Y$ implies that $x \in Y$.

Let us turn to defining the group theory constructions in question.

Given a directed graph Γ with underlying undirected graph $\overline{\Gamma}$, call Γ *connected* whenever so is $\overline{\Gamma}$; i.e., each pair of vertices can be connected by a path. Similarly, call Γ *acyclic* whenever so is $\overline{\Gamma}$. Given an edge e of Γ, denote the source and target vertices of e by $e(1)$ and $e(-1)$.

Consider a nonempty connected directed graph $G = (V, E)$ with vertex set V and edge set E. The numbers of vertices and edges might not be finite. Multiple edges and loops are allowed. Associating to each vertex $v \in V$ a group F_v, and to each edge $e \in E$ a group H_e together with two embeddings $\varphi_{+e} : H_e \to F_{e(1)}$ and $\varphi_{-e} : H_e \to F_{e(-1)}$, we obtain the *graph of groups* $\mathscr{G}$ corresponding to G.

Take a maximal (spanning) subtree $T = (V, E_T)$ in G; i.e., a connected acyclic subgraph of G containing all vertices of G. Refer as the *fundamental group* of a graph of groups $\mathscr{G}$ to the group

$$F = \big\langle *F_v,\ t_f;\ H_e\varphi_{+e} = H_e\varphi_{-e},\ t_f^{-1}(H_f\varphi_{+f})t_f = H_f\varphi_{-f} \\ (v \in V,\ e \in E_T,\ f \in E \setminus E_T)\big\rangle, \tag{1}$$

whose generators are those of F_v for $v \in V$ and the additional letters t_f for $f \in E \setminus E_T$, and whose

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defining relations are those of F_v together with all possible relations of the form

$$h\varphi_{+e} = h\varphi_{-e} \quad (e \in E_T,\ h \in H_e),$$
$$t_f^{-1}(h\varphi_{+f})t_f = h\varphi_{-f} \quad (f \in E \setminus E_T,\ h \in H_f).$$

It is known that the fundamental group of $\mathscr{G}$ is independent of the choice of T; see Proposition 20 of [1, Chapter 1] for instance. If $\mathscr{G}$ is a tree then the fundamental group F of $\mathscr{G}$ is called the *tree product* of F_v for $v \in V$ *with amalgamated subgroups* $H_{\varepsilon e}$ for $e \in E$ and $\varepsilon = \pm 1$. Moreover, if, for all $e, f \in E$ and $\varepsilon, \delta \in \{1, -1\}$, the equality $e(\varepsilon) = f(\delta)$ implies that $H_e\varphi_{\varepsilon e} = H_f\varphi_{\delta f}$; then in F all subgroups $H_e\varphi_{\varepsilon e}$ of F for $e \in E$ and $\varepsilon = \pm 1$ coincide. Therefore, every group F satisfying this extra condition is called the *free product* of the family $\{F_v \mid v \in V\}$ of groups *with one amalgamated subgroup*. It is not difficult to show that the resulting group is isomorphic to the free product of the family $\{F_v \mid v \in V\}$ of groups with one amalgamated subgroup defined in accordance with [2].

If $\mathscr{G}$ contains one vertex v and at least one edge, then the fundamental group F of $\mathscr{G}$ has the presentation

$$F = \big\langle F_v,\ t_f;\ t_f^{-1}(H_f\varphi_{+f})t_f = H_f\varphi_{-f}\ (f \in E)\big\rangle \tag{2}$$

and F is called the (multiple) *HNN-extension* of F_v with the family of stable letters $\{t_f \mid f \in E\}$.

The main result is as follows:

Theorem 1. *Consider the fundamental group F of a graph of groups of the form* (1). *Suppose that each group F_v for $v \in V$ locally satisfies a nontrivial identity and the subgroup $H_e\varphi_{\varepsilon e}$ is properly included into $F_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$. If F is a locally residually nilpotent group and for every edge $e \in E$ at least one of the indices $[F_{e(1)} : H_e\varphi_{+e}]$ and $[F_{e(-1)} : H_e\varphi_{-e}]$ exceeds 2 then there exists a prime p such that $H_e\varphi_{\varepsilon e}$ is p'-isolated in $F_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

Theorem 1 implies Theorems 2 and 3 which generalize the main results of [3–5]. In contrast to Theorem 1, no additional restrictions on the indices of amalgamated and associated subgroups appear in Theorems 2 and 3.

Theorem 2. *Consider the free product F of a family $\{F_v \mid v \in V\}$ of groups with one amalgamated subgroup H. Suppose that all F_v locally satisfy a nontrivial identity and at least two of them include H properly. If F is a locally residually nilpotent group then there exists a prime p such that the subgroup H is p'-isolated in F_v for every $v \in V$.*

Theorem 3. *Consider an HNN-extension F of the form* (2). *Suppose that F_v locally satisfies a nontrivial identity, and the subgroup $H_f\varphi_{\varepsilon f}$ is properly included into F_v for all $f \in E$ and $\varepsilon = \pm 1$. If F is a locally residually nilpotent group then there exists a prime p such that $H_f\varphi_{\varepsilon f}$ are p'-isolated in F_v for all edges $f \in E$ and all numbers $\varepsilon = \pm 1$.*

If F is the free product of a family $\{F_v \mid v \in V\}$ of groups with one amalgamated subgroup H and the inequality $H \neq F_v$ holds only for one $v \in V$, then $F = F_v$ and the local residual nilpotency of this group certainly does not imply that the subgroup H is p'-isolated for any prime p. Thus, Theorem 2 is false in this case. The claim of Theorem 3 ceases to be valid either if at least one of the associated subgroups $H_f\varphi_{\varepsilon f}$ for $f \in E$ and $\varepsilon = \pm 1$ coincides with the base group F_v; for example, see [5].

Theorems 1–3 in combination with the description [6, 7] of the isolators of subgroups of nilpotent and residually nilpotent groups of a particular form, as well as the numerous results on the residual p-finiteness of free constructions of groups, can serve as a foundation for finding a criterion for the residual nilpotency of the certain generalized free products and the HNN-extensions of nilpotent and residually nilpotent groups.

2. Some Auxiliary Statements

Proposition 1 [8, Lemma 2]. *If a group X satisfies a nontrivial identity then X satisfies a nontrivial identity of the form*

$$w(y, x_1, x_2) = w_0(x_1, x_2)y^{\varepsilon_1}w_1(x_1, x_2)\dots y^{\varepsilon_n}w_n(x_1, x_2), \tag{3}$$

where $n \geq 1$, $\varepsilon_1, \dots, \varepsilon_n = \pm 1$, and $w_0(x_1, x_2), \dots, w_n(x_1, x_2) \in \{x_1^{\pm 1}, x_2^{\pm 1}, (x_1x_2^{-1})^{\pm 1}\}$.

Proposition 2. *If p and q are coprimes and X is a finite group of order p then for every $x \in X$ there is an integer m such that $x = x^{qm}$.*

Proof. Take $x \in X$. Then $x^p = 1$. Since $(q, p) = 1$, there exist $m, k \in \mathbb{Z}$ with $qm + pk = 1$. So $x = x^{qm+pk} = (x^q)^m(x^p)^k = x^{qm}$, as required.

Proposition 3. *Given a group X with finitely generated subgroups Y and Z, take $x_1 \in Y$ and $x_2 \in Z$ with $[x_1, x_2] \neq 1$, as well as two primes p and q. Suppose that*

(*a*) *for every prime r and every homomorphism ρ of Y onto a finite r-group the relation $r \neq p$ implies that $x_1\rho = 1$;*
(*b*) *for every prime s and every homomorphism σ of Z onto a finite s-group the relation $s \neq q$ implies that $x_2\sigma = 1$.*

If $p \neq q$ then X is not a locally residually nilpotent group.

Proof. Take finite sets M and N of generators of Y and Z respectively and the subgroup U generated by $M \cup N$. Suppose that $p \neq q$ but X is locally residually nilpotent. Then U is a residually (finitely generated nilpotent) group. By Hirsch's Theorem [9], every finitely generated nilpotent group is residually finite. Hence, U is a residually (finite nilpotent) group.

Put $x = [x_1, x_2]$. Take a homomorphism τ of U onto a finite nilpotent group Z sending x to a nontrivial element and a prime divisor t of the order of $x\tau$. The Burnside–Wielandt Theorem, see Theorem 2.7 of [10] for instance, shows that Z decomposes as the direct product of the Sylow subgroups of Z. Denote by T the product of all Sylow subgroups of Z except the subgroup corresponding to t. Then $x\tau \notin T$, and so the composition of τ with the natural homomorphism of Z onto Z/T sends x to a nontrivial element of the finite t-group Z/T.

If $t = q$; then, according to the inequality $p \neq q$ and condition (a), the restriction of τ to Y sends x_1 to the identity element. If $t \neq q$ then $x_2\tau = 1$ by condition (b). Anyway, $x\tau = 1$, which contradicts the choice of τ.

3. Some Properties of the Generalized Free Products of Groups

Throughout this section F stands for a free product of groups A and B with an amalgamated subgroup H.

An expression for $f \in F$ as $f = f_1f_2\dots f_n$ with $n \geq 1$ is called a *reduced form* whenever each f_k for $1 \leq k \leq n$ lies in one of the factors A and B, but adjacent f_k and f_{k+1} do not lie in the same free factors. The number n is called the *length* of this reduced form.

Proposition 4 is a straightforward corollary of the Normal Form Theorem for generalized free products of two groups; see Corollary 4.4.1 of [11] for instance.

Proposition 4. *Each element $f \in F$ with at least one reduced form of length greater than* 1 *lies outside both free factors A and B; in particular, f is nontrivial.*

Proposition 5 [11, Corollary 4.4.3]. *$A \cap B = H$ in F.*

Proposition 6. *Suppose that groups A and B locally satisfy a nontrivial identity, the subgroup H is properly included into each of them, and at least one of the indices $[A : H]$ and $[B : H]$ exceeds* 2.

Suppose also that there exists $a \in A \setminus H$ such that $a^q \in H$ for some prime q. Then there is a finitely generated subgroup $S \leq F$ with $g_1, g_2 \in S$ enjoying the following properties:

- (a) *g_1 and g_2 have reduced forms of length greater than 1; furthermore, the first and the last syllables of this form of g_1 belong to $A \setminus H$, while the first and the last syllables of this form of g_2 belong to $B \setminus H$;*
- (b) *for every prime p and for every homomorphism σ of S onto a finite p-group the relation $p \neq q$ implies that $g_1\sigma = g_2\sigma = 1$.*

Proof. Consider the three mutually exclusive cases:

(1) $[B : H] > 2$;
(2) $[B : H] = 2$ and $q > 2$;
(3) $[B : H] = 2$ and $q = 2$.

Case 1. $[B : H] > 2$. Take three representatives 1, b_1, and b_2 of the distinct right cosets of H in B. The group $U = sgp\{a^q, b_1, b_2\}$ is a finitely generated subgroup of B, and so U satisfies a nontrivial identity that we may assume to be of the form (3) by Proposition 1. Put

$$S = sgp\{a, b_1, b_2\},$$
$$g_1 = a^{-1}w(a, b_1, b_2)a = a^{-1}w_0(b_1, b_2)a^{\varepsilon_1}w_1(b_1, b_2)\dots a^{\varepsilon_n}w_n(b_1, b_2)a,$$
$$g_2 = w(a, b_1, b_2) = w_0(b_1, b_2)a^{\varepsilon_1}w_1(b_1, b_2)\dots a^{\varepsilon_n}w_n(b_1, b_2).$$

By Proposition 1, the elements $w_r(b_1, b_2)$ for $r \in \{0, \dots, n\}$ are of the form $b_1^{\pm 1}$, $b_2^{\pm 1}$, or $\left(b_1 b_2^{-1}\right)^{\pm 1}$. Since b_1 and b_2 belong to the distinct right cosets of H in B and are nontrivial, $w_r(b_1, b_2) \in B \setminus H$ for every $r \in \{0, \dots, n\}$. By assumption, $a \in A \setminus H$. Thus, the above expressions for g_1 and g_2 are reduced and property (a) holds for them.

Take some homomorphism σ of S onto a finite p-group with $p \neq q$. Since $a^q \in U$, Proposition 2 implies that $a\sigma \in U\sigma$. Since U satisfies (3), this yields $g_1\sigma = g_2\sigma = 1$. Thus, property (b) holds for the subgroup S and the elements g_1 and g_2 of S.

Case 2. $[B : H] = 2$ and $q > 2$. Take $b \in B \setminus H$. Then $U = sgp\{a^q, b\}$ is a finitely generated subgroup of B and so U satisfies a nontrivial identity that, as above, we may assume to be of the form (3). Put

$$S = sgp\{a, b\},$$
$$g_1 = w(b, a, a^2) = w_0(a, a^2)b^{\varepsilon_1}w_1(a, a^2)\dots b^{\varepsilon_n}w_n(a, a^2),$$
$$g_2 = b^{-1}w(b, a, a^2)b = b^{-1}w_0(a, a^2)b^{\varepsilon_1}w_1(a, a^2)\dots b^{\varepsilon_n}w_n(a, a^2)b.$$

By Proposition 1, $w_r(a, a^2) \in \{a^{\pm 1}, a^{\pm 2}\}$ for all $r \in \{0, \dots, n\}$. By assumption, $a \in A \setminus H$, while the inequality $q > 2$ with q prime also implies that $a^2 \in A \setminus H$. Consequently, $w_r(a, a^2) \in A \setminus H$ for all $r \in \{0, \dots, n\}$. Thus, the expressions for g_1 and g_2 are reduced and property (a) holds for them. Property (b) is verified in the same fashion as in Case 1.

Case 3. $[B : H] = 2$ and $q = 2$. Take $b \in B \setminus H$. Since $[B : H] = 2$, the hypotheses imply that $[A : H] > 2$. Take three representatives 1, a_1, and a_2 of the distinct right cosets of H in A.

Since $[B : H] = 2$, it follows that $b^2 \in H$, and so $U = sgp\{b^2, a_1, a_2\}$ is a finitely generated subgroup of A. Hence, U satisfies a nontrivial identity that we may again assume to be of the form (3). Put

$$S = sgp\{a_1, a_2, b\},$$
$$g_1 = w(b, a_1, a_2) = w_0(a_1, a_2)b^{\varepsilon_1}w_1(a_1, a_2)\dots b^{\varepsilon_n}w_n(a_1, a_2),$$
$$g_2 = b^{-1}w(b, a_1, a_2)b = b^{-1}w_0(a_1, a_2)b^{\varepsilon_1}w_1(a_1, a_2)\dots b^{\varepsilon_n}w_n(a_1, a_2)b.$$

As in Case 1, we verify that $w_r(a_1, a_2) \in A \setminus H$ for every $r \in \{0, \dots, n\}$. Moreover, $b \in B \setminus H$. Therefore, the above expressions for g_1 and g_2 are reduced and property (a) holds.

Take some homomorphism σ of S onto a finite p-group with $p \neq q$. The relation $q = 2$, the containment $b^2 \in U$, and Proposition 2 imply that $b\sigma \in U\sigma$. Since U satisfies (3), this implies that $g_1\sigma = g_2\sigma = 1$ and so property (b) also holds.

4. Some Properties of HNN-Extensions

Consider an HNN-extension F of the form (2). An expression for $g \in F$ as

$$g = g_0 t_{f_1}^{\varepsilon_1} g_1 \dots g_{n-1} t_{f_n}^{\varepsilon_n} g_n$$

with $g_0, \dots, g_n \in F_v$, $f_1, \dots, f_n \in E$, and $\varepsilon_1, \dots, \varepsilon_n = \pm 1$, where $n \geq 0$, is called a *reduced form* whenever it avoids the sequences $t_f^{-\varepsilon}$, h, and t_f^{ε}, where $f \in E$, $\varepsilon = \pm 1$, and $h \in H_{\varepsilon f}$. The number n is called the *length* of this reduced form. The following proposition can be deduced from Britton's Lemma for HNN-extensions with one stable letter; see [12, Chapter IV, § 2] for instance.

Proposition 7. *If F is an HNN-extension of the form* (2) *then every $g \in F$ with at least one reduced form of nonzero length lies outside the base group F_v; in particular, g is nontrivial.*

Proposition 8. *Consider an HNN-extension F of the form* (2)*. Suppose that the graph G contains only one edge e. Suppose also that the subgroup $H_e\varphi_{+e}$ is properly included into some subgroup A of F_v locally satisfying a nontrivial identity, the subgroup $H_e\varphi_{-e}$ is properly included into some subgroup B of F_v locally satisfying a nontrivial identity, and at least one of the indices $[A : H_e\varphi_{+e}]$ and $[B : H_e\varphi_{-e}]$ exceeds* 2*. If there exist a prime q and $a \in A \setminus H_e\varphi_{+e}$ with $a^q \in H_e\varphi_{+e}$* or *$b \in B \setminus H_e\varphi_{-e}$ with $b^q \in H_e\varphi_{-e}$ then there is a finitely generated subgroup $S \leq F$ with $g_1, g_2 \in S$ enjoying the following properties:*

(a) *g_1 and g_2 have reduced forms of nonzero length and, furthermore, the form of g_1 starts with t_e^{-1} and ends with t_e, while the form of g_2 starts with t_e and ends with t_e^{-1};*

(b) *for every prime p and every homomorphism σ of S onto a finite p-group the relation $p \neq q$ implies that $g_1\sigma = g_2\sigma = 1$.*

Proof. Switching, if necessary, the direction of the edge e to the opposite one and swapping the subgroups A and B, we may assume that there exists $a \in A \setminus H_e\varphi_{+e}$ with $a^q \in H_e\varphi_{+e}$. Consider the three mutually exclusive cases:

(1) $[B : H_e\varphi_{-e}] > 2$;

(2) $[B : H_e\varphi_{-e}] = 2$ and $q > 2$;

(3) $[B : H_e\varphi_{-e}] = 2$ and $q = 2$.

Case 1. $[B : H_e\varphi_{-e}] > 2$. Take three representatives 1, b_1, and b_2 of the distinct right cosets of $H_e\varphi_{-e}$ in B. The group $U = sgp\{t_e^{-1}a^q t_e, b_1, b_2\}$ is a finitely generated subgroup of B and so U satisfies a nontrivial identity that we may assume to be of the form (3) by Proposition 1. Put

$$\begin{aligned}
S &= sgp\{a, b_1, b_2, t_e\},\\
g_1 &= t_e^{-1} w\big(t_e^{-1} a t_e, b_1, b_2\big) t_e\\
&= t_e^{-1} w_0(b_1, b_2) t_e^{-1} a^{\varepsilon_1} t_e w_1(b_1, b_2) \dots t_e^{-1} a^{\varepsilon_n} t_e w_n(b_1, b_2) t_e,\\
g_2 &= t_e w\big(t_e^{-1} a t_e, b_1, b_2\big) t_e^{-1}\\
&= t_e w_0(b_1, b_2) t_e^{-1} a^{\varepsilon_1} t_e w_1(b_1, b_2) \dots t_e^{-1} a^{\varepsilon_n} t_e w_n(b_1, b_2) t_e^{-1}.
\end{aligned}$$

As in the proof of Proposition 6, we establish that $w_r(b_1, b_2) \in B \setminus H_e\varphi_{-e}$ for every $r \in \{0, \dots, n\}$. Together with $a \in A \setminus H_e\varphi_{+e}$ this implies that the expressions for g_1 and g_2 are reduced and property (a) holds.

Take some homomorphism σ of S onto a finite p-group with $p \neq q$. Since $t_e^{-1} a^q t_e \in U$, Proposition 2 yields $\big(t_e^{-1} a t_e\big)\sigma \in U\sigma$. Since U satisfies (3), this shows that $\big(t_e g_1 t_e^{-1}\big)\sigma = \big(t_e^{-1} g_2 t_e\big)\sigma = 1$. Thus, property (b) also holds for the subgroup S with g_1 and g_2.

Case 2. $[B : H_e\varphi_{-e}] = 2$ and $q > 2$. Take $b \in B \setminus H_e\varphi_{-e}$. Then $U = sgp\{t_e^{-1} a^q t_e, b\}$ is a finitely generated subgroup of B and so U satisfies a nontrivial identity that, as above, we may assume to be of the form (3). Put

$$\begin{aligned}
S &= sgp\{a, b, t_e\},\\
g_1 &= w\big(b, t_e^{-1}at_e, t_e^{-1}a^2t_e\big)\\
&= w_0\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big)b^{\varepsilon_1}w_1\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big)\dots b^{\varepsilon_n}w_n\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big),\\
g_2 &= t_e^2w\big(b, t_e^{-1}at_e, t_e^{-1}a^2t_e\big)t_e^{-2}\\
&= t_e^2w_0\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big)b^{\varepsilon_1}w_1\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big)\dots b^{\varepsilon_n}w_n\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big)t_e^{-2}.
\end{aligned}$$

According to Proposition 1, for each $r \in \{0, \dots, n\}$ we have

$$w_r\big(t_e^{-1}at_e, t_e^{-1}a^2t_e\big) \in \big\{t_e^{-1}a^{\pm 1}t_e, t_e^{-1}a^{\pm 2}t_e\big\}.$$

Since $a \in A \setminus H_e\varphi_{+e}$ and $q > 2$ is prime, $a^2 \in A \setminus H_e\varphi_{+e}$. Thus, the expressions for g_1 and g_2 are reduced and property (a) holds. Property (b) is established in the same fashion as in case 1.

Case 3. $[B : H_e\varphi_{-e}] = 2$ and $q = 2$. The hypotheses and $[B : H_e\varphi_{-e}] = 2$ imply that $[A : H_e\varphi_{+e}] > 2$. Take three representatives 1, a_1, and a_2 of the distinct right cosets of $H_e\varphi_{+e}$ in A, as well as $b \in B\setminus H_e\varphi_{-e}$. Since $[B : H_e\varphi_{-e}] = 2$, the subgroup $H_e\varphi_{-e}$ is normal in B. Therefore, $b^{-1}\big(t_e^{-1}a^qt_e\big)b \in H_e\varphi_{-e}$ and $t_eb^{-1}\big(t_e^{-1}a^qt_e\big)bt_e^{-1} \in H_e\varphi_{+e}$. Consequently, $U = sgp\big\{t_eb^{-1}t_e^{-1}a^qt_ebt_e^{-1}, a_1, a_2\big\}$ is a finitely generated subgroup of A and so U satisfies a nontrivial identity that we may again assume to be of the form (3). Put

$$\begin{aligned}
S &= sgp\{a, b, a_1, a_2, t_e\},\\
g_1 &= t_e^{-1}w\big(t_eb^{-1}t_e^{-1}at_ebt_e^{-1}, a_1, a_2\big)t_e\\
&= t_e^{-1}w_0(a_1, a_2)t_eb^{-1}t_e^{-1}a^{\varepsilon_1}t_ebt_e^{-1}w_1(a_1, a_2)\dots t_eb^{-1}t_e^{-1}a^{\varepsilon_n}t_ebt_e^{-1}w_n(a_1, a_2)t_e,\\
g_2 &= t_ew\big(t_eb^{-1}t_e^{-1}at_ebt_e^{-1}, a_1, a_2\big)t_e^{-1}\\
&= t_ew_0(a_1, a_2)t_eb^{-1}t_e^{-1}a^{\varepsilon_1}t_ebt_e^{-1}w_1(a_1, a_2)\dots t_eb^{-1}t_e^{-1}a^{\varepsilon_n}t_ebt_e^{-1}w_n(a_1, a_2)t_e^{-1}.
\end{aligned}$$

As above, we establish that $w_r(a_1, a_2) \in A\setminus H_e\varphi_{+e}$ for every $r \in \{0, \dots, n\}$. Moreover, $a \in A\setminus H_e\varphi_{+e}$ and $b \in B\setminus H_e\varphi_{-e}$. Hence, the expressions for g_1 and g_2 are reduced and property (a) holds. Property (b) is established in the same fashion as in case 1, with the only difference that instead of $t_e^{-1}at_e$ and $t_e^{-1}a^qt_e$ we should use $t_eb^{-1}\big(t_e^{-1}at_e\big)bt_e^{-1}$ and $t_eb^{-1}\big(t_e^{-1}a^qt_e\big)bt_e^{-1}$ respectively.

5. Some Properties of the Fundamental Groups of Arbitrary Graphs of Groups

Consider a fundamental group F of some graph of groups of the form (1) and the tree product P corresponding to a maximal subtree T. Then F amounts to the multiple HNN-extension of P with the family of stable letters $\{t_f \mid f \in E \setminus E_T\}$ and Theorem 2 of [13] shows that P may be assumed to be a subgroup of F. For an edge $e \in E \setminus E_T$, the group F is a multiple HNN-extension of the group

$$P(e) = \big\langle P,\ t_e;\ t_e^{-1}(H_e\varphi_{+e})t_e = H_e\varphi_{-e}\big\rangle$$

with the family of stable letters $\{t_f \mid f \in E \setminus (E_T \cup \{e\})\}$. Consequently, the HNN-extension $P(e)$ also turns out to be a subgroup of F. Finally, if $\widetilde{T} = (\widetilde{V}, \widetilde{E})$ is a subtree of T with the same groups and mappings assigned to vertices and edges, while $\widetilde{P}$ is the tree product corresponding to $\widetilde{T}$; then, according to Theorem 1 of [14], the identity mapping of the generators of $\widetilde{P}$ to P determines an isomorphic embedding. Hence, we may also assume that $\widetilde{P}$ is a subgroup of F. These arguments enable us to apply Propositions 6 and 8 to the above subgroups of F. In the proof of Proposition 9 we will use this possibility tacitly.

Proposition 9. *Consider the fundamental group F of a graph of groups of the form* (1)*. Suppose that the group F_v locally satisfies a nontrivial identity for each $v \in V$ and the subgroup $H_e\varphi_{\varepsilon e}$ is properly included into $F_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$. Suppose also that F is a locally residually nilpotent group and for every edge $e \in E$ at least one of the indices $[F_{e(1)} : H_e\varphi_{+e}]$ and $[F_{e(-1)} : H_e\varphi_{-e}]$ exceeds 2. If there exist* (*not necessarily distinct*) *edges $e, f \in E$ and numbers ε, $\delta = \pm 1$, some elements $x_e \in F_{e(\varepsilon)} \setminus H_e\varphi_{\varepsilon e}$ and $x_f \in F_{f(\delta)} \setminus H_f\varphi_{\delta f}$, as well as primes p and q such that $x_e^p \in H_e\varphi_{\varepsilon e}$ and $x_f^q \in H_f\varphi_{\delta f}$ then $p = q$.*

Proof. Consider the three mutually exclusive cases:

(1) $e, f \in E_T$;
(2) either $e \in E_T$ and $f \notin E_T$ or $e \notin E_T$ and $f \in E_T$;
(3) $e, f \notin E_T$.

Case 1. $e, f \in E_T$. At least two among the vertices $e(1)$, $e(-1)$, $f(1)$, and $f(-1)$ are distinct. It is obvious that, as we change the direction of any number of edges in E_T, the presentation of F remains the same. Therefore, without loss of generality we may assume that $e(-1) \neq f(1)$ and that in the tree T there is a path from $f(1)$ to $e(-1)$ passing through $e(1)$ and $f(-1)$.

According to Proposition 6, applied first to the generalized free product

$$P_e = \langle F_{e(1)} * F_{e(-1)};\ H_e\varphi_{+e} = H_e\varphi_{-e}\rangle$$

and x_e and then to the generalized free product

$$P_f = \langle F_{f(1)} * F_{f(-1)};\ H_f\varphi_{+f} = H_f\varphi_{-f}\rangle$$

and x_f, there exist finitely generated subgroups $S_1 \leq P_e$ and $S_2 \leq P_f$ with $g_1 \in S_1$ and $g_2 \in S_2$ enjoying the following properties:

(a) g_1 and g_2 in P_e and P_f respectively have reduced forms of length greater than 1, and furthermore the first and the last syllables of this form of g_1 belong to $F_{e(-1)} \setminus H_e\varphi_{-e}$, while the first and the last syllables of this form of g_2 belong to $F_{f(1)} \setminus H_f\varphi_{+f}$;
(b) for every prime r and every homomorphism ρ of S_1 onto a finite r-group the relation $r \neq p$ implies that $g_1\rho = 1$; for every prime s and every homomorphism σ of S_2 onto a finite s-group the relation $s \neq q$ implies that $g_2\sigma = 1$.

Verify that $g = [g_1, g_2]$ is distinct from 1. Applying Proposition 3 to F, the subgroups S_1 and S_2, the elements g_1 and g_2, and the numbers p and q, we infer that $p = q$, as required.

Take the subtree $\widetilde{T} = (\widetilde{V}, \widetilde{E})$ of T which amounts to the path connecting $e(-1)$ and $f(1)$, as well as the tree product $\widetilde{P}$ corresponding to $\widetilde{T}$. If $e = f$ then $\widetilde{P} = P_e = P_f$ and property (a) shows that in this group g has a reduced form of length at least 12. By Proposition 4, this implies that $g \neq 1$.

Suppose that $e \neq f$. Take the tree $\widetilde{T}_1$ obtained from $\widetilde{T}$ by removing the edge e and the vertex $e(-1)$ as well as the tree $\widetilde{T}_2$ obtained from $\widetilde{T}_1$ by removing the edge f and the vertex $f(1)$. Denote by $\widetilde{P}_1$ and $\widetilde{P}_2$ the tree products corresponding to $\widetilde{T}_1$ and $\widetilde{T}_2$. Then $\widetilde{P}$ is the free product of P_e and $\widetilde{P}_1$ with amalgamated subgroup $F_{e(1)}$, while $\widetilde{P}_1$ is the free product of $\widetilde{P}_2$ and P_f with amalgamated subgroup $F_{f(-1)}$.

Since in P_e the element g_1 has a reduced form of length greater than 1, by Proposition 4 it lies outside the free factor $F_{e(1)}$. For the same reason $g_2 \notin F_{f(-1)}$. By Proposition 5, $\widetilde{P}_2 \cap P_f = F_{f(-1)}$ in $\widetilde{P}_1$. Since $F_{e(1)} \leq \widetilde{P}_2$, it follows that $g_2 \notin F_{e(1)}$. Hence, in $\widetilde{P}$ regarded as a generalized free product of P_e and $\widetilde{P}_1$ the element g has a reduced form of length 4 and by Proposition 4 it is distinct from 1.

Case 2. Either $e \in E_T$ and $f \notin E_T$, or $e \notin E_T$ and $f \in E_T$. Denote by P the tree product corresponding to the maximal subtree T. Without loss of generality we may assume that $e \in E_T$ and $f \notin E_T$. Therefore, by Proposition 6 applied to the generalized free product

$$P_e = \langle F_{e(1)} * F_{e(-1)};\ H_e\varphi_{+e} = H_e\varphi_{-e}\rangle$$

and x_e, as well as Proposition 8 applied to the HNN-extension

$$P(f) = \langle P,\ t_f;\ t_f^{-1}(H_f\varphi_{+f})t_f = H_f\varphi_{-f}\rangle$$

and x_f, there exist finitely generated subgroups $S_1 \leq P_e$ and $S_2 \leq P(f)$ with $g_1 \in S_1$ and $g_2 \in S_2$ enjoying the following properties:

(a) g_1 has a reduced form of length greater than 1 in P_e;

(b) g_2 has a reduced form of nonzero length beginning with t_f^{-1} and ending with t_f in $P(f)$;

(c) for every prime r and every homomorphism ρ of S_1 onto a finite r-group the relation $r \neq p$ implies that $g_1\rho = 1$; for every prime s and every homomorphism σ of S_2 onto a finite s-group the relation $s \neq q$ implies that $g_2\sigma = 1$.

Consider the connected components T_1 and T_2 of the graph obtained from T by removing the edge e and the corresponding tree products P_1 and P_2. Then the group P amounts to the tree product of P_1, P_e, and P_2 with amalgamated subgroups $F_{e(1)}$ and $F_{e(-1)}$. Proposition 5 yields $P_1 \cap P_e \subseteq F_{e(1)} \cup F_{e(-1)}$ and $P_2 \cap P_e \subseteq F_{e(1)} \cup F_{e(-1)}$. Since in P_e the element g_1 has a reduced form of length greater than 1, Proposition 4 yields $g_1 \notin F_{e(1)} \cup F_{e(-1)}$. This implies that $g_1 \notin F_v$ for each vertex $v \in V$ and in particular $g_1 \notin H_f\varphi_{-f}$. Thus, the expression

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in P(f)$$

is reduced, has nonzero length, and $[g_1, g_2] \neq 1$ by Proposition 7. As in case 1, this implies that $p = q$ by Proposition 3.

Case 3. $e, f \notin E_T$. Consider again the tree product P corresponding to the maximal subtree T. According to Proposition 8 applied first to the HNN-extension

$$P(e) = \langle P,\ t_e;\ t_e^{-1}(H_e\varphi_{+e})t_e = H_e\varphi_{-e}\rangle$$

and x_e and then to the HNN-extension

$$P(f) = \langle P,\ t_f;\ t_f^{-1}(H_f\varphi_{+f})t_f = H_f\varphi_{-f}\rangle$$

and x_f, there are finitely generated subgroups $S_1 \leq P(e)$ and $S_2 \leq P(f)$ with $g_1 \in S_1$ and $g_2 \in S_2$ enjoying the following properties:

(a) the elements g_1 and g_2 in $P(e)$ and $P(f)$ respectively have reduced forms of nonzero length; furthermore, the form of g_1 starts with t_e^{-1} and ends with t_e, while the form of g_2 starts with t_f^{-1} and ends with t_f;

(b) for every prime r and every homomorphism ρ of S_1 onto a finite r-group the relation $r \neq p$ implies that $g_1\rho = 1$; for every prime s and every homomorphism σ of S_2 onto a finite s-group the relation $s \neq q$ implies that $g_2\sigma = 1$.

Take $c \in F_{e(-1)} \setminus H_e\varphi_{-e}$ and

$$g = [c^{-1}g_1c, g_2] = c^{-1}g_1^{-1}cg_2^{-1}c^{-1}g_1cg_2.$$

Then the expression for g is reduced and has nonzero length in the group F regarded as an HNN-extension of P; this is obvious if $e \neq f$, and otherwise follows from the fact that $c \in P \setminus H_e\varphi_{-e}$. Hence, Proposition 7 yields $g \neq 1$. Applying Proposition 3 to F, the subgroups $c^{-1}S_1c$ and S_2, the elements $c^{-1}g_1c$ and g_2, and the numbers p and q, we infer that $p = q$.

6. Proof of Theorem 1

If for all $e \in E$ and $\varepsilon = \pm 1$ the subgroup $H_e\varphi_{\varepsilon e}$ is isolated in $F_{e(\varepsilon)}$ then we can take any prime as p. Therefore, assume henceforth that $H_e\varphi_{\varepsilon e}$ is not isolated in $F_{e(\varepsilon)}$ for some $e \in E$ and $\varepsilon = \pm 1$. Then there exist $x_e \in F_{e(\varepsilon)} \setminus H_e\varphi_{\varepsilon e}$ and a prime p such that $x_e^p \in H_e\varphi_{\varepsilon e}$.

Assume that $f \in E$, $\delta = \pm 1$, $x_f \in F_{f(\delta)} \setminus H_f\varphi_{\delta f}$, and a prime q is such that $x_f^q \in H_f\varphi_{\delta f}$. Then $p = q$ by Proposition 9. This implies that if $H_f\varphi_{\delta f}$ is not isolated in $F_{f(\delta)}$ for some $f \in E$ and $\delta = \pm 1$ then $H_f\varphi_{\delta f}$ is p'-isolated. In particular, this holds for $H_e\varphi_{\varepsilon e}$.

7. Proof of Theorem 2

If $[F_v : H] \leq 2$ for every vertex $v \in V$ then H is $2'$-isolated in F_v for each $v \in V$. Therefore, assume henceforth that there exists $v \in V$ such that $[F_v : H] > 2$.

With the necessary notational changes not affecting the presentation of F, we may assume that all edges of G are directed away from v. This implies in particular that if two edges are incident to one vertex then the latter is the target of one and the source of the other.

For some edge $e \in E$ denote by φ_e the isomorphism $\varphi_{+e}^{-1}\varphi_{-e} : H_e\varphi_{+e} \to H_e\varphi_{-e}$. If two edges $e, f \in E$ satisfy $e(-1) = f(1)$ then $H_e\varphi_{-e} = H_f\varphi_{+f}$ by the definition of free product with one amalgamated subgroup and hence the composition of isomorphisms $\varphi_e\varphi_f$ is defined.

Consider a star tree G' with the same set of vertices as G, center at v, and edges going out to all remaining vertices. Assume also that the same groups are assigned to the vertices of G' as to those of G. If $e = (v, w)$ is an edge of G' and $e_1, e_2, \dots, e_n$ is a path in G from v to a vertex w then associate the group $H_e = H_{e_1}$ and the embeddings $\varphi_{+e} = \varphi_{+e_1}$ and $\varphi_{-e} = \varphi_{+e_1}\varphi_{e_1}\varphi_{e_2}\dots\varphi_{e_n}$ to the edge e. This yields the graph of groups $\mathscr{G}'$; furthermore, for every edge $e = (v, w)$ in $\mathscr{G}'$ the isomorphic embeddings of the groups F_v and F_w assigned to the endpoints of e into F send $H_e\varphi_{+e}$ and $H_e\varphi_{-e}$ onto H.

Verify that the fundamental groups F and F' of $\mathscr{G}$ and $\mathscr{G}'$ are isomorphic. Indeed, if, as above, $e = (v, w)$ is an edge of G' and $e_1, e_2, \dots, e_n$ is a path in G from v to w then the relations $h = h\varphi_{e_i}$ for $i \in \{1, \dots, n\}$ and $h \in H_{e_i}\varphi_{+e_i}$ valid in F imply that $h = h\varphi_{e_1}\varphi_{e_2}\dots\varphi_{e_n}$ for $h \in H_{e_1}\varphi_{+e_1}$, and so $h\varphi_{+e} = h\varphi_{-e}$ for all $h \in H_e$. Conversely, if $g = (u, w)$ is an edge of G and $e_1, e_2, \dots, e_n = g$ is a path in G from v to w, while $e = (v, w)$ and $f = (v, u)$ are edges of G' then the relations $h\varphi_{+e} = h\varphi_{-e}$ for $h \in H_e$ and $h\varphi_{+f} = h\varphi_{-f}$ for $h \in H_f$ of the group F' imply that $h = h\varphi_{e_1}\varphi_{e_2}\dots\varphi_{e_n}$ and $h = h\varphi_{e_1}\varphi_{e_2}\dots\varphi_{e_{n-1}}$ for $h \in H_{e_1}\varphi_{+e_1}$, whence $h = h\varphi_{e_n}$ for all $h \in H_{e_n}\varphi_{+e_n}$, and so $h\varphi_{+g} = h\varphi_{-g}$ for all $h \in H_g$. Since the remaining defining relations and all generators in F and F' are the same, the required isomorphism follows.

If $H_e\varphi_{-e} = F_{e(-1)}$ for some edge e of G' then all defining relations of $F_{e(-1)}$ follow from those of $F_{e(1)}$. Thus, all generators and defining relations of $F_{e(-1)}$, as well as all possible relations of the form $h\varphi_{+e} = h\varphi_{-e}$ for $h \in H_e$ can be excluded from the presentation of F'. This operation is equivalent to removing from the graph of groups $\mathscr{G}'$ the edge e together with the vertex $e(-1)$ and associated edge and vertex groups.

Removing from $\mathscr{G}'$ all edges of the form described above, we obtain some graph of groups $\mathscr{G}''$ whose fundamental group F'' is isomorphic to F and satisfies the hypotheses of Theorem 1. Consequently, there exists a prime p such that $H_e\varphi_{+e}$ is p'-isolated in $F_{e(1)} = F_v$ and $H_e\varphi_{-e}$ is p'-isolated in $F_{e(-1)}$ for every edge e of $\mathscr{G}''$.

Take some vertex w of $\mathscr{G}$ distinct from v and such that F satisfies the relation $H \neq F_w$; by the hypotheses of the theorem, at least one such vertex exists. Then the edge $e = (v, w)$ of $\mathscr{G}'$ remains as we pass to $\mathscr{G}''$, and since the embeddings of F_v and F_w into F send $H_e\varphi_{+e}$ and $H_e\varphi_{-e}$ onto H, we infer that H is p'-isolated in F_v and F_w. The proof of Theorem 2 is complete.

8. Proof of Theorem 3

If for each $e \in E$ at least one of the indices $[F_v : H_e\varphi_{+e}]$ and $[F_v : H_e\varphi_{-e}]$ exceeds 2 then the claim follows from Theorem 1. Assume that $[F_v : H_e\varphi_{+e}] = 2$ and $[F_v : H_e\varphi_{-e}] = 2$ for some $e \in E$. Take $x_e \in F_v \setminus H_e\varphi_{+e}$ and $y_e \in F_v \setminus H_e\varphi_{-e}$. Since $[F_v : H_e\varphi_{+e}] = [F_v : H_e\varphi_{-e}] = 2$, we see that $x_e^2 \in H_e\varphi_{+e}$ and

$$[H_e\varphi_{-e} : H_e\varphi_{+e} \cap H_e\varphi_{-e}] \leq 2.$$

Hence, $t_e^{-1}x_e^2t_e \in H_e\varphi_{-e}$, $t_e^{-1}x_e^4t_e \in H_e\varphi_{+e} \cap H_e\varphi_{-e}$, and $t_e^{-2}x_e^4t_e^2 \in H_e\varphi_{-e}$. Therefore, $U = sgp\{t_e^{-1}x_e^2t_e, t_e^{-2}x_e^4t_e^2, y\}$ is a finitely generated subgroup of F_v and so satisfies a nontrivial identity that we may assume to be of the form (3) by Proposition 1.

Put

$$\begin{aligned} S_1 &= sgp\{x_e, y_e, t_e\}, \\ g_1 &= w\big(y_e, t_e^{-1}x_e t_e, t_e^{-2}x_e t_e^2\big) \\ &= w_0\big(t_e^{-1}x_e t_e, t_e^{-2}x_e t_e^2\big) y_e^{\varepsilon_1} w_1\big(t_e^{-1}x_e t_e, t_e^{-2}x_e t_e^2\big) \dots y_e^{\varepsilon_n} w_n\big(t_e^{-1}x_e t_e, t_e^{-2}x_e t_e^2\big). \end{aligned}$$

By Proposition 1, for each $r \in \{0, \dots, n\}$ we have

$$w_r\big(t_e^{-1}x_e t_e, t_e^{-2}x_e t_e^2\big) \in \big\{t_e^{-1}x_e^{\pm 1}t_e,\ t_e^{-2}x_e^{\pm 1}t_e^2,\ \big(t_e^{-1}x_e t_e^{-1}x_e^{-1}t_e^2\big)^{\pm 1}\big\}.$$

Since $x_e \in F_v \setminus H_e\varphi_{+e}$ and $y_e \in F_v \setminus H_e\varphi_{-e}$, the above expression for g_1 is reduced.

Take an odd prime r and some homomorphism ρ of S_1 onto a finite r-group. Since $(r, 4) = (r, 2) = 1$, the containments $t_e^{-1}x_e^2 t_e \in U$ and $t_e^{-2}x_e^4 t_e^2 \in U$ combined with Proposition 2 imply that $\big(t_e^{-1}x_e t_e\big)\rho \in U\rho$ and $\big(t_e^{-2}x_e t_e^2\big)\rho \in U\rho$. Since U satisfies (3), this implies that $g_1\rho = 1$. Thus, the subgroup S_1 and its element g_1 enjoy the following properties:

- (a_1) in F the element g_1 has a reduced form of nonzero length beginning with t_e^{-1} and ending with t_e;
- (b_1) for every prime r and every homomorphism ρ of S_1 onto a finite r-group the relation $r \neq 2$ implies that $g_1\rho = 1$.

Take $f \in E \setminus \{e\}$, $\delta = \pm 1$, $x_f \in F_v \setminus H_f\varphi_{\delta f}$, and a prime q such that $x_f^q \in H_f\varphi_{\delta f}$ and at least one of the indices $[F_v : H_f\varphi_{+f}]$ and $[F_v : H_f\varphi_{-f}]$ exceeds 2. Then, according to Proposition 8 applied to the HNN-extension

$$F_v(f) = \big\langle F_v,\ t_f;\ t_f^{-1}(H_f\varphi_{+f})t_f = H_f\varphi_{-f}\big\rangle$$

and x_f, there is a finitely generated subgroup $S_2 \leq F_v(f)$ with $g_2 \in S_2$ enjoying the properties:

- (a_2) in $F_v(f)$ the element g_2 has a reduced form of nonzero length beginning with t_f^{-1} and ending with t_f;
- (b_2) for every prime s and every homomorphism σ of S_2 onto a finite s-group the relation $s \neq q$ implies that $g_2\sigma = 1$.

Put $g = [g_1, g_2]$. Since $e \neq f$, the expression for g in F is reduced and of nonzero length. Consequently, $g \neq 1$ by Proposition 7 and $q = 2$ by Proposition 3. Thus, all associated subgroups of the HNN-extension F are $2'$-isolated in F_v.

9. An Example

Consider the rank 2 free group $\Phi = \langle a, b\rangle$. Given $i \geq 1$, denote by $\gamma_i\Phi$ the ith term of the lower central series of Φ. Put $N_i = \Phi/\gamma_{i+1}\Phi$. Choose any nontrivial element c_i in $\gamma_i\Phi/\gamma_{i+1}\Phi$ and take the group

$$D = \langle N_i\ (i \geq 1);\ [N_i, N_j] = 1,\ c_i = c_j\ (i \neq j)\rangle$$

which amounts to the quotient of the direct product of the groups N_i for $i \geq 1$ with respect to the normal closure of the set of elements of the form $c_i c_j^{-1}$ for $i \neq j$. It is not difficult to show that, since c_i for each $i \geq 1$ lies in the center of N_i and generates in N_i an infinite cyclic subgroup, the identity mapping of the generators of N_i to D for all i extends to an isomorphic embedding, and so we may assume that each N_i is a subgroup of D [15]. These embeddings carry all c_i to the same $c \in D$.

Since the direct product of the nilpotent groups N_i for $i \geq 1$ is obviously a locally nilpotent group, the quotient D inherits the same property. In particular, D locally satisfies a nontrivial identity and is a locally residually nilpotent group.

Verify that D does not satisfy any nontrivial identity. Indeed, take an arbitrary nontrivial identity $w(x_1, \dots, x_n)$. Since Φ violates w, there are $f_1, \dots, f_n \in \Phi$ with $w(f_1, \dots, f_n) \neq 1$. Since $\bigcap_{i\geq 1} \gamma_i\Phi = 1$, see [16], it follows that $w(f_1, \dots, f_n) \notin \gamma_{i+1}\Phi$ for some $i \geq 1$, and so $w(f_1\varepsilon_i, \dots, f_n\varepsilon_i) \neq 1$, where $\varepsilon_i : \Phi \to N_i$ is a natural homomorphism. Consequently, N_i violates w and, since N_i embeds into D, so does the latter.

Verify that for each homomorphism of D onto a nilpotent group the image of c comes out trivial; consequently, D is not a residually nilpotent group. Suppose that σ is a homomorphism of D onto a nilpotent group of class k and denote by σ_{k+1} the restriction of σ to N_{k+1} and by $\varepsilon_{k+1} : \Phi \to N_{k+1}$ the natural homomorphism. Since $\Phi\varepsilon_{k+1}\sigma_{k+1}$ is of nilpotency class at most k, we see that $\gamma_{k+1}\Phi \leq \ker \varepsilon_{k+1}\sigma_{k+1}$. Since $c = c_{k+1} \in \gamma_{k+1}\Phi/\gamma_{k+2}\Phi$, this implies that $c\sigma = c\sigma_{k+1} = 1$, as required.

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