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A GENERAL INTEGRAL OF A QUASILINEAR EQUATION AND APPLICATION TO A NONLINEAR CHARACTERISTIC PROBLEM

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Abstract: We describe a method for constructing general integrals for some nonstrictly hyperbolic quasilinear equations and prove a nonlinear analog of Asgeirsson's mean value theorem. Using a general integral, we study the nonlinear version of the Goursat characteristic problem.

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1. Introduction

The theory of construction of general integrals has a long history stemming from the fundamental works by Monge [1] and Darboux [2, 3]. Even though the construction of general solutions to partial differential equations presents fundamental difficulties, the integrals play an essential role at least because their combination with other methods facilitates the study of the problems posed for these equations (see [4]).

Using the theory of characteristics in this article, we construct general integrals for some nonstrictly hyperbolic equations. The construction of a general integral of an equation is not connected to some problems (initial-value, boundary-value, or any other). The structure of general solutions depends only on the equation under study, and the solution itself is in a sense equivalent to the equation. In our earlier articles [5, 6], we used representations of general integrals for some classes of quasilinear equations to solve some initial-value and characteristic problems. Note the articles by Gvazava [7, 8] and Bitsadze [9], in which some general representations of solutions are constructed for a few classes of quasilinear equations on using the method of characteristics. We consider the new class of equations whose coefficients of the second derivatives contain squares of the first derivatives of the sought solution; both families of characteristics depend on an unknown function. This class of equations is noticeable also for the fact that it admits parabolic degeneration that depends on the unknown solution. One of the equations of this class admits order degeneration either. The above properties of the class of equations hamper the construction of the general integrals as well as the statement and study of the characteristic problem. In the present article we were able to construct general integrals in the form of the sum of two arbitrary functions for some equations from this class. Also, we formulate an analog of the Asgeirsson principle and study the characteristic problem for a whole class of equations. Since no general theory of nonlinear equations is available, studying some particular classes and particular equations is valuable, as this widens the class of quasilinear equations for which the general integrals are constructed and the various problems are posed correctly.

Here we present a systematic description of construction of the integrals on using the method of characteristics as well as application of the integrals to solving the nonlinear characteristic problem.

2. Construction of General Integrals

1. On the plane of variables x and y, we consider the class of nonstrictly hyperbolic quasilinear equations

$$L(u) = \Phi(x, y, u, u_x, u_y), \tag{1}$$

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where

$$L(u) \equiv \big(u_y^2 - u_y\big)u_{xx} - (2u_xu_y + u_y - u_x - 1)u_{xy} + \big(u_x^2 + u_x\big)u_{yy} \tag{2}$$

and Φ is in general a given function of five variables, defined and continuously differentiable on the plane of the independent variables (x, y) and defined for all finite values of u, u_x, and u_y. The characteristic roots

$$\lambda_1 = -\frac{p+1}{q}, \quad \lambda_2 = -\frac{p}{q-1} \tag{3}$$

of the operator L depend on the derivatives u_x and u_y of the unknown solution and, at each point, define two characteristic directions (here and below we use the Monge notation: $p \equiv u_x$ and $q \equiv u_y$). These directions can coincide, which is expressed by the condition

$$p - q + 1 = 0. \tag{4}$$

For the solutions satisfying (4), the equation under study is parabolic. Consequently, condition (4) determines the class of parabolic solutions to (1). If condition (4) is not met everywhere for some concrete solution, then (1) is hyperbolic along the latter and the condition

$$p - q + 1 \neq 0 \tag{5}$$

determines the class of hyperbolic solutions to (1). In the case when condition (4) holds only at isolated points or on lines, (1) is an equation of hyperbolic type with parabolic degeneration. Therefore, (1) belongs to the class of nonstrictly hyperbolic equations.

Relation (1) describes a rather wide class of equations. It is well known (see [3]) that, to construct a general integral, it suffices to get some intermediate integrals that, in turn, are obtained on using the first integrals for the characteristic differential relations and characteristic invariants.

The differential relations (see, for example, [3, 7]) corresponding to the characteristic root λ_1 have the form

$$\begin{cases} (p+1)\,dx + q\,dy = 0, \\ dp - \frac{p}{q-1}\,dq - \frac{\Phi}{q^2-q}\,dx = 0, \\ du = p\,dx + q\,dy. \end{cases} \tag{6}$$

The differential relations corresponding to the characteristic root λ_2 can be written as follows:

$$\begin{cases} p\,dx + (q-1)\,dy = 0, \\ dp - \frac{p+1}{q}\,dq - \frac{\Phi}{q^2-q}\,dx = 0, \\ du = p\,dx + q\,dy. \end{cases} \tag{7}$$

Note in particular that under condition (4) these two systems of characteristics coincide becoming a sole system, which is typical for parabolic degeneration.

Since each of the systems (6) and (7) consists of three equations and contains the five variables x, y, u, p, and q, we should not treat them as ordinary systems of differential equations; but we can pose the problem of constructing the first integrals for them.

Denote the first integral for (6) by $\xi(x, y, u, p, q) = \text{const}$. From $d\xi = 0$ and (6) we obtain

$$\begin{cases} M_1(\xi) := \xi_x - \frac{p+1}{q}\xi_y - \xi_u + \frac{\Phi}{q^2-q}\xi_p = 0, \\ M_2(\xi) := p\xi_p + (q-1)\xi_q = 0. \end{cases} \tag{8}$$

These two equations are linearly independent and compatible, but they do not define a complete system (see [10]). To make this system complete, extend it with the use of the Poisson bracket:

$$M_3(\xi) := M_1(M_2(\xi)) - M_2(M_1(\xi)) = \xi_y = 0. \tag{9}$$

Further application of the Poisson bracket to the extended system yields the new equations which can be represented as linear combinations of the previous equations:

$$\begin{cases} M_4(\xi) := M_1(M_3(\xi)) - M_3(M_1(\xi)) = 0, \\ M_5(\xi) := M_2(M_3(\xi)) - M_3(M_2(\xi)) = 0. \end{cases} \tag{10}$$

In case the homogeneous system constituted by five equations (8)–(10) is linearly independent, it does not admit nontrivial solutions. Consequently, the first integral for (6) cannot depend on x, y, p, q, and u; so it must be constant. In these cases, it is impossible to construct an integral for (1) by using characteristics. Similar arguments can be carried out for (7).

From the whole class (1), we will consider the particular equations for which it is possible to construct general integrals by using characteristics.

2. Start with equation (1) for $\Phi \equiv 0$; i.e.,

$$\big(u_y^2 - u_y\big)u_{xx} - (2u_x u_y + u_y - u_x - 1)u_{xy} + \big(u_x^2 + u_x\big)u_{yy} = 0. \tag{11}$$

Theorem 1. *The general integral for* (11) *is representable as*

$$f(u+x) + g(u-y) = x, \tag{12}$$

where f and g are arbitrary twice continuously differentiable functions on R^1.

PROOF. For equation (11), the system

$$\begin{cases} (p+1)\,dx + q\,dy = 0, \\ (1-q)\,dp + p\,dq = 0 \end{cases} \tag{13}$$

corresponds to the characteristic root λ_1 and the system defined by the root λ_2 has the form

$$\begin{cases} p\,dx + (q-1)\,dy = 0, \\ q\,dp - (p+1)\,dq = 0. \end{cases} \tag{14}$$

As in (6) and (7), we add the compatibility equation

$$du = p\,dx + q\,dy \tag{15}$$

to both systems (13) and (14). Let us start with (13). Introducing the notation $\xi(x,y,u,p,q)$ for the first integral of (13), we obtain the system of two linear differential equations of the first order:

$$\begin{cases} L_1(\xi) := \xi_x - \xi_u - \frac{p+1}{q}\xi_y = 0, \\ L_2(\xi) := \xi_p + \frac{q-1}{p}\xi_q = 0. \end{cases} \tag{16}$$

To make (16) complete in the Jacobi sense, extend (16) with the Poisson bracket

$$L_3(\xi) := L_1(L_2(\xi)) - L_2(L_1(\xi)) = \frac{p-q+1}{q^2(1-q)}\xi_y = 0. \tag{17}$$

As we can see, (17) is linearly independent of the equations of (16). Further application of the Poisson bracket to the extended system yields the new equation representable as linear combinations of the previous equations:

$$L_4(\xi) := L_1(L_3(\xi)) - L_3(L_1(\xi)) \equiv 0, \quad L_5(\xi) := L_2(L_3(\xi)) - L_3(L_2(\xi)) \equiv 0.$$

Consequently, the homogeneous system $L_k(\xi) = 0$, $k = 1,2,3$, where L_k are defined by (16) and (17), is Jacobi complete. By the Jacobi Theorem (see [3]), since the system of three equations is complete; it admits $n-3$ different first integrals, where n is the number of variables. In our case $n = 5$; therefore, system (13), (15) has two, and only two, independent twice continuously differentiable first integrals which can be easily obtained by direct integration of equations of the system. By integrating the second equation of (13), we obtain the first integral

$$\xi_1 = \frac{p}{1-q}, \tag{18}$$

and the second equation of (13) together with (15) defines the integral

$$\xi = u + x. \tag{19}$$

Studying (14) and (15) similarly, we conclude that the system also has exactly two independent twice continuously differentiable first integrals determined explicitly:

$$\eta = u - y, \quad \eta_1 = \frac{p+1}{q}. \tag{20}$$

Both ξ and ξ_1 are the first integrals for (13), (15). Each function depending on ξ and ξ_1 is also an integral for the same system. By the similar arguments applied to (14), (15), we can conclude that each function of η and η_1 is an integral for the corresponding system. Hence, there exist pairwise functional correlations between the characteristic invariants ξ, ξ_1 and the invariants η, η_1:

$$\xi_1 = F(\xi), \quad \eta_1 = G(\eta), \tag{21}$$

where F and G are arbitrary functions of the class $C^2(R^1)$. Consequently, (11) admits exactly two intermediate integrals presented in terms of characteristic invariants in form (21). By (18)–(20), in initial variables they have the form

$$\frac{p}{1-q} = F(u+x), \quad \frac{p+1}{q} = G(u-y). \tag{22}$$

Obtaining p and q from (22) and inserting the corresponding expressions into (15), we derive

$$du = F\frac{G-1}{F+G}\,dx + \frac{F+1}{F+G}\,dy.$$

Hence,

$$\frac{du - dy}{G-1} + \frac{du + dx}{F+1} = dx. \tag{23}$$

Introduce the notations

$$\frac{1}{G(u-y)-1} \equiv g'(u-y), \quad \frac{1}{F(u+x)+1} \equiv f'(u+x).$$

Using these in (23), we easily arrive at (12) by integrating the obtained relation.

Prove that (12) is a general integral for (11) if the arbitrary functions f and g belong to $C^2(R^1)$. Indeed, obtain f' and g' from the system

$$\begin{cases} (u_x+1)f'(u+x) + u_x g'(u-y) = 0, \\ u_y f'(u+x) + (u_y - 1)g'(u-y) = 0, \end{cases} \tag{24}$$

derived by differentiation of (12) with respect to x and y. Differentiating (24) leads to the system of three equations for f'' and g'':

$$\begin{gathered} (u_x+1)^2 f''(u+x) + u_{xx}f'(u+x) + u_x^2 g''(u-y) + u_{xx}g'(u-y) = 0, \\ u_y^2 f''(u+x) + u_{yy}f'(u+x) + (u_y-1)^2 g''(u-y) + u_{yy}g'(u-y) = 0, \\ (u_x+1)u_y f''(u+x) + u_{xy}f'(u+x) + u_x(u_y-1)g''(u-y) + u_{xy}g'(u-y) = 0, \end{gathered}$$

where the first derivatives $f'(u+x)$ and $g'(u-y)$ are already defined from (24). If we determine $f''(u+x)$ and $g''(u-y)$ from arbitrary two equations of the last system and put them into the third equation, then we obtain (11). Thus, the general integral for (11) has the form (12). The theorem is proven.

REMARK. It is easy to see that the general integrals for (11) have the representation

$$f(u+x)+g(u-y)=y, \tag{25}$$

$f(u+x)+g(u-y)=u$. Owing to arbitrariness of f and g, all these representations are equivalent.

3. Consider the equation containing the derivatives of the first order of the unknown function u on the right-hand side:

$$L(u)=-\frac{1}{y}p(p+1)(p-q+1). \tag{26}$$

Here L is defined by (2). Note first that the right-hand side of this equation with lower-order derivatives is not bounded in the neighborhood of the line $y=0$. In such case, the equation under consideration can be classified as a nonlinear version of the Euler–Darboux equation (see [2, 11, 12]). If we rewrite (26) as follows: $yL(u)=-p(p+1)(p-q+1)$; then we can see that on the line $y=0$ the order of this equation degenerates. Owing to these peculiarities, in the initial-value and characteristic problems there appear the effects of coincidence of nonlinearity, order degeneration, and type degeneration (see [11]).

Theorem 2. *The general integral for* (26) *is representable by the formula*

$$f(u+x)+g(u-y)-y^2=0, \tag{27}$$

where f and g are arbitrary twice continuously differentiable functions on R^1.

PROOF. For (26), the system

$$\begin{cases} (p+1)\,dx+q\,dy=0,\\ yq(q-1)\,dp-ypq\,dq+p(p+1)(p-q+1)\,dx=0,\\ du=p\,dx+q\,dy \end{cases} \tag{28}$$

corresponds to the characteristic root $\lambda_1=-\frac{p+1}{q}$ and the system defined by the root $\lambda_2=\frac{p}{1-q}$ has the form

$$\begin{cases} p\,dx+(q-1)\,dy=0,\\ yq(q-1)\,dp-y(p+1)(q-1)\,dq+p(p+1)(p-q+1)\,dx=0,\\ du=p\,dx+q\,dy. \end{cases} \tag{29}$$

For the first integral $\xi(x,y,u,p,q)$ for (28), we obtain the two linear differential equations of the first order:

$$\begin{cases} L_1(\xi):=\xi_x-\xi_u-\frac{p+1}{q}\xi_y+\frac{(p+1)(p-q+1)}{yq}\xi_q=0,\\ L_2(\xi):=\xi_p+\frac{q-1}{p}\xi_q=0. \end{cases} \tag{30}$$

As in the case of (11), the construction of the first integrals for (30) is reduced to integration of some complete system (in the Jacobi sense). To make the system complete, extend it by the Poisson bracket:

$$L_3(\xi):=L_1(L_2(\xi))-L_2(L_1(\xi))=\frac{p-q+1}{p(p+1)^2}(\xi_x-\xi_u)=0.$$

Since we consider the hyperbolic case (condition (5) holds), $L_3(\xi)$ can be written as

$$\xi_x-\xi_u=0. \tag{31}$$

As we can see, (31) is linearly independent of the equations of (30). Further application of the Poisson bracket to the extended system yields a new equation representable by linear combinations of the previous equations. Consequently, the homogeneous system $L_k(\xi) = 0$, $k = 1, 2, 3$, where L_k are defined by formulas (30) and (31), is Jacobi complete.

Rewrite the system in equivalent simplified form:

$$\begin{cases} X_1(\xi) := \xi_x - \xi_u = 0, \\ X_2(\xi) := \xi_p + \frac{(q-1)y}{p(p-q+1)}\xi_y = 0, \\ X_3(\xi) := \xi_q + \frac{y}{p-q+1}\xi_y = 0 \end{cases} \tag{32}$$

and introduce the new group of variables z_k, $k = 1, 2, \ldots, 5$, as follows:

$$z_1 = x, \quad z_2 = y, \quad z_3 = u + x, \quad z_4 = p, \quad z_5 = q.$$

In terms of z_k (32) takes the form

$$\begin{cases} Y_1(\xi) := \xi_{z_1} = 0, \\ Y_2(\xi) := \xi_{z_4} + \frac{(z_5-1)z_2}{z_4(z_4-z_5+1)}\xi_{z_2} = 0, \\ Y_3(\xi) := \xi_{z_5} + \frac{z_2}{z_4-z_5+1}\xi_{z_2} = 0. \end{cases} \tag{33}$$

The first equation of (33) shows that the integral ξ is independent of z_1, so four of the arguments remain. The second equation is equivalent to the following system of ordinary differential equations of the first order:

$$\frac{dz_4}{1} = \frac{z_4(z_4 - z_5 + 1)\,dz_2}{(z_5 - 1)z_2} = \frac{dz_3}{0} = \frac{dz_5}{0},$$

for which the two integrals $z_3 = c$ and $z_5 = c$ are obtained directly. The third integral $\frac{z_4 z_2}{z_4 - z_5 + 1} = c$ can be derived from the equation

$$\frac{dz_4}{z_4(z_4 - z_5 + 1)} = \frac{dz_2}{(z_5 - 1)z_2},$$

where z_5 plays the role of a parameter.

After the new regular transformation of the variables

$$t_1 = z_1, \quad t_2 = \frac{z_4 z_2}{z_4 - z_5 + 1}, \quad t_3 = z_3, \quad t_4 = z_4, \quad t_5 = z_5,$$

the equations $Y_2 = 0$ and $Y_3 = 0$ take the form

$$R_2(\xi) = \frac{\partial \xi}{\partial t_4} = 0, \quad R_3(\xi) = \frac{\partial \xi}{\partial t_5} = 0.$$

Thus, we established that $t_2 = \frac{z_4 z_2}{z_4 - z_5 + 1}$ is the first integral for the system under consideration and, in terms of the initial variables, the latter can be represented as $\xi_1 = \frac{py}{p-q+1}$.

Consequently, the system has two, and only two, independent twice continuously differentiable first integrals, both of which are presented in explicit form by the formulas

$$\xi = u + x, \quad \xi_1 = \frac{py}{p - q + 1}. \tag{34}$$

These first integrals can be obtained directly from the system of differential characteristic relations. However, we carried out complete analysis of the characteristic system in order to make sure that it does not have any other first integrals.

Consider (29) for the second root λ_2. For the first integral η for this system, we have the system of two equations

$$\begin{cases} L_1(\eta) := \frac{1-q}{p}\eta_x + \eta_y + \eta_u - \frac{p-q+1}{y}\eta_q = 0, \\ L_2(\eta) := \eta_p + \frac{q}{p+1}\eta_q = 0. \end{cases} \tag{35}$$

Extending (35) by the Poisson bracket to a Jacobi complete system and simplifying the obtained equations, we obtain

$$\begin{cases} X_1(\eta) := \eta_x = 0, \\ X_3(\eta) := \frac{qy}{(p+1)(p-q+1)}(\eta_y + \eta_u) + \eta_p = 0, \\ X_2(\eta) := \frac{y}{p-q+1}(\eta_y + \eta_u) - \eta_q = 0. \end{cases} \tag{36}$$

System (36) is integrated in much the same way as (32). It can be proved that it also has two, and only two, twice continuously differentiable first integrals

$$\eta = u - y, \quad \eta_1 = \frac{p+1}{p-q+1}y. \tag{37}$$

Every function of two variables ξ and ξ_1 satisfies system (28). Similarly, the general integral for (29) is representable by an arbitrary function of η and η_1. Hence, there exist functional correlations between the characteristic invariants ξ and ξ_1 and the invariants η and η_1, which we write as follows:

$$\xi_1 = \frac{1}{2}f'(\xi), \quad \eta_1 = \frac{1}{2}g'(\eta). \tag{38}$$

Here f and g are arbitrary functions in $C^2(R^1)$. Consequently, (26) admits exactly two intermediate integrals presented in terms of the characteristic invariants (38). These intermediate integrals in initial variables have the form

$$y\frac{\partial u}{\partial x} = \frac{1}{2}f'(u+x)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + 1\right), \tag{39}$$

$$y\left(\frac{\partial u}{\partial x} + 1\right) = \frac{1}{2}g'(u-y)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + 1\right). \tag{40}$$

We follow the classical scheme of construction of general integrals for (26) using the intermediate integrals (see, for example, [11]). From (39) and (40), we obtain the values

$$p = -\frac{f'(\xi)}{f'(\xi) + g'(\eta)}, \quad q = -\frac{g'(\eta)}{f'(\xi) + g'(\eta)}$$

and then insert them into (15):

$$du = -\frac{f'(\xi)}{f'(\xi) + g'(\eta)}\,dx + \frac{g'(\eta) + 2y}{f'(\xi) + g'(\eta)}\,dy.$$

By integrating the above equality, we easily arrive at

$$f(\xi) + g(\eta) = y^2, \tag{41}$$

which, by the first relations from (34) and (37), takes form (27); i.e., it is equivalent to the representation of the general integral for (26).

By analogy to the case of (11), we can show that if we differentiate (27) twice and eliminate the arbitrary functions f and g and their derivatives; then we obtain equation (26); i.e., (27) is indeed the general integral for (26), and Theorem 2 is proven completely.

3. Some Properties of Equation (1)

Asgeirsson's Mean Value Theorem is valid for hyperbolic equations (see [13]). For example, for the wave equation $u_{xx} - u_{yy} = 0$, it can be stated as follows: The sums of the values of the solution on the opposite vertices of an arbitrary characteristic quadrangle are equal. Equation (1) has its own nonlinear analog of this property. The latter can be easily deduced from the invariants of (1). Denote the vertices of an arbitrary characteristic curvilinear quadrangle by (x_i, y_i), $i = 1, \ldots, 4$. (The opposite sides of this "quadrangle" are intervals from one family of characteristics.) We denote by u_i, $i = 1, \dots, 4$, the values of the solution at (x_i, y_i), $i = 1, \dots, 4$. The characteristic quadrangle is constituted by intersections of the two pairs of the characteristic lines. One pair consists of the two nonintersecting lines that belong to one family of characteristics for the root λ_1; denote them by λ_{1a} and λ_{1b}. The second pair consists of two nonintersecting lines of the family for the root λ_2; denote them by λ_{2a} and λ_{2b}.

For clarity, observe that the lines λ_{1a} and λ_{2a} intersect at (x_1, y_1), the lines λ_{1a} and λ_{2b} intersect at (x_2, y_2), the lines λ_{1b} and λ_{2b} intersect at (x_3, y_3), and the lines λ_{1b} and λ_{2a} intersect at (x_4, y_4). Consequently, (x_1, y_1) and (x_2, y_2) lie on the characteristic λ_{1a}. Hence, $u_1 + x_1 = u_2 + x_2$ at these points. Similarly, since (x_3, y_3) and (x_4, y_4) lie on the characteristic λ_{1b}; therefore, $u_3 + x_3 = u_4 + x_4$ at these points. At the same time, (x_1, y_1) and (x_4, y_4) lie on the characteristic λ_{2a}; hence, $u_1 - y_1 = u_4 - y_4$. Similarly, $u_2 - y_2 = u_3 - y_3$, since (x_2, y_2) and (x_3, y_3) lie on the characteristic λ_{2b}.

The characteristic quadrangle been taken arbitrarily, by eliminating u_1, u_2, u_3, and u_4 in the obtained relations, we conclude that the sums of abscissas and ordinates of the opposite vertices of an arbitrary characteristic quadrangle are equal: $x_1 + x_3 + y_1 + y_3 = x_2 + x_4 + y_2 + y_4$. This is the nonlinear analog of Asgeirsson's mean value principle. This simple property, together with other methods, essentially facilitates the study of the problems stated for (1).

Remark. Observe that in the particular case of (11) a stronger property is valid too. Namely, the sums of the abscissas and ordinates of the opposite vertices are equal:

$$x_1 + x_3 = x_2 + x_4, \quad y_1 + y_3 = y_2 + y_4. \tag{42}$$

4. The Goursat Problem for Equation (11)

Observe that the linear statements of the characteristic problems are not valid in the case of nonlinear equations. The main reason for this is dependence of the characteristic families on the unknown solution. In this case we should prescribe how the data will be provided. There are various options for that. For example, we can specify some combinations of the solution or its derivatives on the unknown characteristics which will be determined together with the solution.

Consider the following nonlinear characteristic Goursat problem:

The Goursat problem (the general case). Let the Jordan arcs γ and δ, starting at the common point (x_0, y_0), strictly monotone, smooth, and nonclosed, be given in explicit form by the functions $y = \varphi(x)$ and $y = \psi(x)$, where $\varphi \in C^2[x_0, x_1]$ and $\psi \in C^2[x_0, x_2]$. Without loss of generality, we assume that $x_2 > x_1$. Let φ and ψ satisfy the conditions

$$\begin{cases} \varphi'(x) \neq 0; -1, & x \in [x_0, x_1], \\ \psi'(x) \neq 0; -1, & x \in [x_0, x_2], \\ \varphi(x) \neq \psi(x), & x \in (x_0, x_1], \\ \varphi'(x) \neq \psi'(x), & x \in (x_0, x_1]. \end{cases} \tag{43}$$

We need to find a solution $u(x, y)$ to (11) and its domain for a given value u_0 of $u(x, y)$ at the point (x_0, y_0) if γ is a characteristic from the family for the root λ_1 and δ is a characteristic from the family for the root λ_2.

Suppose that the above defined functions φ and ψ, together with (43), satisfy the following conditions: the functional equations

$$u_0 + x_0 - x - \varphi(x) = z, \quad u_0 - \psi(x_0) + x + \psi(x) = t$$

are uniquely solvable for x on $[x_0, x_1]$ and $[x_0, x_2]$ respectively and their solutions

$$\begin{aligned} x = \tau(z), &\quad z \in [u_0 - \varphi(x_0), u_0 + x_0 - x_1 - \varphi(x_1)], \\ x = \nu(t), &\quad t \in [u_0 + x_0, u_0 + x_2 - \psi(x_0) + \psi(x_2)], \end{aligned}$$

are twice continuously differentiable. We will construct a solution to the Goursat problem and find the domain of the solution.

By condition of the problem, γ belongs to the family of characteristics for the root λ_1. Thereby, everywhere on this arc the value of the invariant $u+x$ is constant; it is the same as at (x_0, y_0); i.e.,

$$(u(x,\varphi(x))+x)|_\gamma = u_0 + x_0.$$

It is easy to define the values of the characteristic invariant $u-y$ from another family on the same arc; i.e.,

$$(u(x,\varphi(x))-y)|_\gamma = u_0 + x_0 - x - \varphi(x).$$

Using these values, from the representation of the general integral (25), we obtain

$$f(u_0+x_0)+g(u_0+x_0-x-\varphi(x)) = \varphi(x). \tag{44}$$

Similarly, since δ belongs to the family of characteristics for the root λ_2 by the condition of the problem; everywhere on this arc the invariant $u-y$ is constant. Consequently,

$$(u(x,\psi(x))-y)|_\delta = u_0 - y_0.$$

Using this, we can define the value of the characteristic invariant $u+x$ from the first family on the same arc:

$$(u(x,\psi(x))+x)|_\delta = u_0 - y_0 + x + \psi(x).$$

From (25) we derive

$$f(u_0-y_0+x+\psi(x))+g(u_0-\psi(x_0)) = \psi(x). \tag{45}$$

To determine the arbitrary functions f and g, we obtain the system consisting of (44) and (45). Put

$$u_0+x_0-x-\varphi(x) = z, \tag{46}$$

$$u_0-y_0+x+\psi(x) = t \tag{47}$$

and define x as a function of z of (46). By assumptions, this can be done and (46) has the inverse $x=\tau(z)$. Similarly, define x as a function of t from the functional equation (47); i.e., $x=\nu(t)$. Consequently,

$$f(u_0+x_0)+g(z) = \varphi(\tau(z)), \quad z\in[u_0-\varphi(x_0), u_0+x_0-x_1-\varphi(x_1)], \tag{48}$$

$$f(t)+g(u_0-\psi(x_0)) = \psi(\nu(t)), \quad t\in[u_0+x_0, u_0+x_2-\psi(x_0)+\psi(x_2)]. \tag{49}$$

Inserting in (48) and (49) $u-y$ and $u+x$ instead of z and t and taking the general integral for the equation into account, we obtain the integral for the problem in the form

$$\psi(\nu(u+x))+\varphi(\tau(u-y))-f(u_0+x_0)-g(u_0-y_0) = y.$$

Since

$$f(u_0+x_0)+g(u_0-y_0) = y_0,$$

which follows from the representation of the general integral taken at (u_0, y_0); we finally arrive at

$$\psi(\nu(u+x))+\varphi(\tau(u-y)) = y+y_0. \tag{50}$$

Thus, solving the Goursat problem reduces to solving (50). Integral (50) for the characteristic problem enables us to define all characteristics starting at the points of the characteristics $\varphi(x)$ and $\psi(x)$.

First, consider the characteristics from the family of the root λ_2 which start from the points of γ. Take a point $(\alpha,\varphi(\alpha))$ on γ arbitrarily. At this point, the value of the solution u is known: $u(\alpha,\varphi(\alpha)) = u_0+x_0-\alpha$. Consequently, we know the value of the invariant $u-y$ of the family of characteristics for the root λ_2. Along the characteristic of the family of λ_2 starting at the point $(\alpha,\varphi(\alpha))$, the invariant of the family of characteristics for the root λ_2 is constant and equal to $u_0 + x_0 - \varphi(\alpha) - \alpha$. Denote this characteristic by $\delta(\alpha)$. Then

$$(u+x)|_{\delta(\alpha)} = u - y + y + x = u_0 + x_0 - \alpha - \varphi(\alpha) + y + x.$$

Inserting the value of the combination of $u+x$ and $u-y$ into (50), which is the integral for the problem, we arrive at the equation for the characteristic curve $\delta(\alpha)$ in implicit form:

$$\psi(\nu(u_0 + x_0 - \alpha - \varphi(\alpha) + x + y)) + \varphi(\alpha) = y + y_0.$$

By similar arguments, we come to the equation for the characteristic curve $\gamma(\beta)$ from the family for the root λ_1, starting at $(\beta, \psi(\beta))$:

$$\psi(\beta) + \varphi(\tau(u_0 - y_0 + \beta + \psi(\beta) - x - y)) = y + y_0.$$

Remark. Integral (50) can be represented in equivalent form. For this, rewrite (46) and (47) as follows:

$$u_0 + x_0 - \tau(z) - \varphi(\tau(z)) = z, \quad u_0 - y_0 + \nu(t) + \psi(\nu(t)) = t.$$

Hence,

$$\varphi(\tau(z)) = u_0 + x_0 - \tau(z) - z, \quad \psi(\nu(t)) = t - u_0 + y_0 - \nu(t).$$

Put these expressions into (50) and after some transformations obtain

$$\tau(u-y) + \nu(u+x) = x + x_0. \tag{51}$$

It is easy to verify that the solution of the Goursat problem, implicitly defined by (51), satisfies the conditions of the problem.

The Goursat problem for rectilinear characteristics. Consider the case when φ and ψ are linear functions:

$$\varphi = ax + b, \quad \psi = cx + d. \tag{52}$$

By (43), require that the following conditions are met:

$$a \neq 0; -1, \quad c \neq 0; -1, \quad a \neq \pm c. \tag{53}$$

Let us state the problem: *Find a solution $u(x, y)$ to* (11) *by the given value u_0 at the point $(x_0, y_0) = \left(\frac{d-b}{a-c}, \frac{ad-bc}{a-c}\right)$ and define the domain of $u(x,y)$ if $\varphi(x)$, $x > x_0$, is a characteristic from the family for the root λ_1 and $\psi(x)$, $x > x_0$, is a characteristic from the family for the root λ_2.*

Theorem 3. *Let* (52) *satisfy* (53). *Then there exists a solution to the above characteristic problem for* (11) *in explicit form:*

$$u = \frac{(1+a)c}{a-c}x - \frac{1+c}{a-c}y + \frac{(a+c)u_0 + bc + d}{a-c} \tag{54}$$

defined in the domain bounded by the following four characteristics: φ, ψ, and

$$y = cx + b + a\alpha - c\alpha, \tag{55}$$

$$y = ax + d - a\beta + c\beta. \tag{56}$$

PROOF. Use the integral for the problem written as in (51). From (46) τ is obtained by inversion of $u_0 + x_0 - x - \varphi(x)$. In our case, $u_0 + x_0 - x - ax - b = z$; whence

$$x = \frac{u_0 + x_0 - b - z}{1 + a} \equiv \tau(z).$$

Similarly, from (47) we have

$$\nu(t) \equiv \frac{t - d - u_0 + y_0}{1 + c}.$$

Using the obtained expressions in (51), we obtain the integral for the problem

$$\frac{u_0 + x_0 - b - (u - y)}{1 + a} + \frac{u + x - d - u_0 + y_0}{1 + c} = x + x_0;$$

whence we find the sought function in explicit form (54). By repeating the above arguments for constructing characteristic curves of both families in the general case, the equations of these curves can be easily derived; in our case they are straight lines (55) and (56). We can see that the characteristics from one family (φ and (55); ψ and (56)) are parallel straight lines. Hence, the domain of the solution to the problem does not contain singular points, which is not excluded in the general case. The theorem is proven.

Corollary. *In the case of φ and ψ being linear together with* (42)*, the same property as for the wave equation holds:*

$$u(x_1, y_1) + u(x_3, y_3) = u(x_2, y_2) + u(x_4, y_4). \tag{57}$$

EXAMPLE 1. Consider the case when the characteristics

$$\varphi(x) = x + 1,\ x \in [0, \alpha], \quad \psi(x) = 2x + 1,\ x \in [0, \beta],$$

in the Goursat problem start at the common point $(0, 1)$ where $u(0, 1) = u_0$.

In this case, the conditions of Theorem 3 are met and the solution of the problem has the form $u = 3y - 4x + u_0 - 3$. The domain of the solution is the characteristic quadrangle bounded by the segments of straight lines

$$y = x + 1, \quad y = 2x + 1, \quad y = 2x - \alpha + 1, \quad y = x + \beta + 1.$$

It is easy to notice that at the vertices of the characteristic quadrangle $A(0, 1)$, $B(\alpha, \alpha + 1)$, $C(\alpha + \beta, \alpha + 2\beta + 1)$, and $D(\beta, 2\beta + 1)$ an analog of Asgeirsson's Theorem for (11) holds: the sums of abscissas of the opposite vertices A, C and B, D are equal; the sums of ordinates of the opposite vertices are equal; and (57) holds together with (42).

Rectilinearity of characteristics is not necessary for existence of a regular solution to the Goursat problem for (11), which the following example justifies.

EXAMPLE 2. Let the functions $\varphi(x) = x + 2$, $x \in [2, x_1]$, and $\psi(x) = x^2$, $x \in [2, x_2]$, start at the common point $(2, 4)$ and $u(2, 4) = 4$.

As we can see, (43) holds and the solution to the problem has the form $u = y - 2x + 1 + \sqrt{4y - 4x + 1}$. As for the families of characteristic curves, on the one hand, it is a family of parallel straight lines starting at points of the parabola $y = x^2$, $x \in [2, x_2]$, and on the other hand, it is a family of branches of the parabolas starting at the points of $y = x + 2$, $x \in [2, x_1]$. Hence, a regular solution to the problem is determined in the curvilinear quadrangle bounded by the curves

$$y = x + 2,\ y = x^2,\ y = x + x_2^2 + x_2 + 1 - \sqrt{x_2^2 + x_2},\ \sqrt{17 - 8x_1 + 4x + y} = 2x - 2x_1 + 5.$$

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