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ON STRONGLY Π-PERMUTABLE SUBGROUPS OF A FINITE GROUP

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Abstract: Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, let $\varnothing \neq \Pi \subseteq \sigma$, and let G be a finite group. A set $\mathscr{H}$ of subgroups of G is said to be a complete Hall Π-set of G if each member $\neq 1$ of $\mathscr{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathscr{H}$ has exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi$ such that $\sigma_i \cap \pi(G) \neq \varnothing$. A subgroup A of G is called (i) Π-permutable in G if G has a complete Hall Π-set $\mathscr{H}$ such that $AH^x = H^xA$ for all $H \in \mathscr{H}$ and $x \in G$; (ii) σ-subnormal in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a σ_k-group for some k for all $i = 1, \dots, t$; and (iii) strongly Π-permutable if A is Π-permutable and σ-subnormal in G. We study the strongly Π-permutable subgroups of G. In particular, we give characterizations of these subgroups and prove that the set of all strongly Π-permutable subgroups of G forms a sublattice of the lattice of all subgroups of G.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group, while $\mathscr{L}(G)$ is the lattice of all subgroups of G. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$, and $\pi' = \mathbb{P} \setminus \pi$. Note that $\pi(n)$ denotes the set of all primes dividing n, and $\pi(G) = \pi(|G|)$. The subgroups A and B of G are *permutable*, if $AB = BA$. In this case we also said that A is *permutable* with B.

In what follows, $\sigma = \{\sigma_i \mid i \in I\}$ is some partition of $\mathbb{P}$; i.e., $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$; and $\Pi \subseteq \sigma$. We let $\sigma(n)$ to stand for $\{\sigma_i \mid \sigma_i \cap \pi(n) \neq \varnothing\}$; and $\sigma(G) = \sigma(|G|)$. A group G is said to be Π-*primary* if G is a σ_i-group for some $\sigma_i \in \Pi$; a Π-*group* if $\sigma(G) \subseteq \Pi$.

A set $\mathscr{H}$ of subgroups of G is said to be a *complete Hall* Π-*set* of G [1] if each member $\neq 1$ of $\mathscr{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$, and $\mathscr{H}$ has exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi$; while G is called Π-*full* if G possesses a complete Hall Π-set.

A subgroup A of G is said to be [1, 2]: (i) Π-*permutable* in G if G has a complete Hall Π-set $\mathscr{H}$ such that $AH^x = H^xA$ for all $H \in \mathscr{H}$ and all $x \in G$; (ii) σ-*subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a σ_k-group for some $k = k(i)$ for all $i = 1, \dots, t$; and (iii) *strongly* Π-*permutable* in G if A is Π-permutable and σ-subnormal in G.

Note in passing that by Theorem B of [2] A is strongly σ-permutable in G if and only if A is σ-permutable in G. The examples and key properties of Π-permutable and, in particular, σ-permutable subgroups were discussed in [1–4]. Basing on the results of [2], we study the properties of strongly Π-permutable subgroups in this paper.

Example 1.1. Let $p > q > r > t$ be primes, where t divides $r - 1$, and let $G = C_p \times (Q \rtimes (C_r \rtimes C_t))$, where $C_r \rtimes C_t$ is a nonabelian group of order rt and Q is a simple $\mathbb{F}_q(C_r \rtimes C_t)$-module faithful for $C_r \rtimes C_t$. Put $\sigma = \{\{p\}, \{q, r\}, \{p, q, r\}'\}$ and $\Pi = \{\{p\}, \{q, r\}\}$.

(i) The subgroup C_r is σ-subnormal and Π-permutable in G, and so C_r is strongly Π-permutable in G. We show now that this subgroup is not σ-permutable in G. Indeed, assume that $C_rC_t^x = C_t^xC_r$ for every $x \in G$. Then $C_r = Q \rtimes C_r \cap C_t^xC_r$ is normal in $C_t^xC_r$. Hence $C_t^G = Q \rtimes (C_r \rtimes C_t) \leq N_G(C_r)$, and

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so $C_r \leq C_G(Q)$, contrary to the fact that Q is faithful for $C_r \rtimes C_t$. This contradiction shows that C_r is not σ-permutable in G.

(ii) The subgroup C_t is Π-permutable in the group $Q \rtimes (C_r \rtimes C_t)$. It is not difficult to check that C_t is not σ-subnormal in this group. Thus, C_t is not strongly Π-permutable in G.

The first result shows that a σ-subnormal subgroup of a Π-full group of G is strongly Π-permutable in G if it permutes with each Hall σ_i-subgroup of G for all $\sigma_i \in \Pi$.

Proposition 1.2. *Suppose that G is Π-full and let A be a σ-subnormal subgroup of G. Then the following are equivalent:*

(1) *A is Π-permutable in G.*

(2) *A permutes with every Hall σ_i-subgroup of G for all $\sigma_i \in \Pi$.*

(3) *A/N is Π-permutable in G/N for every normal subgroup N of G lying in A.*

Recall that G is *σ-nilpotent* [1] if $G = H_1 \times \cdots \times H_t$, where $\{H_1, \ldots, H_t\}$ is a complete Hall σ-set of G.

Theorem A. *Suppose that G is Π-full and let A be a σ-nilpotent subgroup of G. Then the following are equivalent:*

(1) *A is strongly Π-permutable in G.*

(2) *Every characteristic subgroup of A is strongly Π-permutable in G.*

(3) *Every Hall σ_i-subgroup of A is strongly Π-permutable in G for all i.*

Recall that a subgroup A of G is said to be *π-permutable* or *π-quasinormal* in G (see [5]) if A permutes with every Sylow p-subgroup P of G; i.e., $AP = PA$ for all $p \in \pi$; $\pi(G)$-permutable subgroups are called also *S-permutable* [6]. Therefore in the classical case of $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from Theorem A the well-known fact:

Corollary 1.3 [6, Theorem 1.2.17]. *Let A be a nilpotent subgroup of G. Then the following are equivalent:*

(i) *A is S-permutable in G.*

(ii) *Every Sylow subgroup of A is S-permutable in G.*

(iii) *Every characteristic subgroup of A is S-permutable in G.*

We say that a subgroup A of G is Π-*modular* (*π-modular* in the case $\sigma = \{\{2\}, \{3\}, \ldots\}$) provided that G is Π-full and $\langle A, H \cap C\rangle = \langle A, H\rangle \cap C$ for every Hall σ_i-subgroup H of G and all $\sigma_i \in \Pi$ and $A \leq C \leq G$; while G is said to be a Π-*full group of Sylow type* [1] if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \Pi$.

Theorem B. *Let A be a σ-subnormal subgroup of G. If G is a Π-full group of Sylow type, then the following are equivalent:*

(1) *A is Π-permutable in G.*

(2) *A is Π-permutable in $\langle A, x\rangle$ for all $x \in G$.*

(3) *A is Π-modular in every subgroup E of G containing A.*

Corollary 1.4. *A subnormal subgroup A of G is π-permutable in G if and only if A is π-permutable in $\langle A, x\rangle$ for all $x \in G$.*

Since every S-permutable subgroup of G is subnormal in G [5], we get from Corollary 1.4 the well-known result:

Corollary 1.5 [7; 6, Theorem 1.2.13]. *A subgroup A of G is S-permutable in G if and only if A is S-permutable in $\langle A, x\rangle$ for all $x \in G$.*

Theorem B allows us to give another characterization of S-permutability.

Corollary 1.6. *A subgroup A of G is S-permutable in G if and only if A is subnormal in G and $\langle A, P \cap C\rangle = \langle A, P\rangle \cap C$ for every Sylow subgroup P of G and every subgroup C of G including A.*

If $\sigma^* = \{\sigma_j^* \mid j \in J\}$ is a partition of $\mathbb{P}$ such that $I \subseteq J$ and $\sigma_j^* \subseteq \sigma_j$ for all $j \in J$, then we write $\sigma^* \leq \sigma$.

We use $\mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G)$ and $\mathscr{L}_{\Pi per}(G)$ to denote the sets of all Π-permutable subgroups A of G with σ^*-nilpotent A^G/A_G and of all strongly Π-permutable subgroups of G, respectively. If $\sigma^* = \{\{2\}, \{3\}, \dots\}$, then let $\mathfrak{N}$ denote $\mathfrak{N}_{\sigma^*}$.

Theorem C. *Suppose that G is Π-full and $\sigma^* \leq \sigma$. Then $\mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G) \subseteq \mathscr{L}_{\Pi per}(G)$ are sublattices of $\mathscr{L}(G)$.*

Corollary 1.7 [5, Theorem 2]. *The set of all π-permutable subnormal subgroups of G forms a sublattice of $\mathscr{L}(G)$.*

Theorem D. *Suppose that G is Π-full and let $\mathscr{L}$ be the set of all Π-subgroups of G belonging to $\mathscr{L}_{\mathfrak{N}\Pi}(G)$. Then $\mathscr{L}$ is a sublattice of $\mathscr{L}(G)$, and $\mathscr{L}$ is modular if and only if every two members of $\mathscr{L}$ are permutable.*

Corollary 1.8 [8, Theorem B]. *The lattice of all S-permutable subgroups of G is modular if and only if every two members of the lattice are permutable.*

2. Proofs of Proposition 1.2 and Theorems A and B

Proof of Proposition 1.2. The implication (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2): It is enough to show that if A is a σ-subnormal subgroup of G and there is a Hall σ_i-subgroup L of G such that $AL^x = L^xA$ for all $x \in G$, then A permutes with all Hall σ_i-subgroups of G. Assume that this is false and let G be a counterexample of minimal order. Then $AH \neq HA$ for some Hall σ_i-subgroup H of G.

We show first that $A_G = 1$. Indeed, assume that $R = A_G \neq 1$. Then LR/R is a Hall σ_i-subgroup of G/R such that $(A/R)(LR/R)^{xR} = (A/R)(L^xR/R) = AL^x/R = L^xA/R = (LR/R)^{xR}(A/R)$ for all $xR \in G/R$. Moreover, A/R is a σ-subnormal subgroup of G/R by [2, Lemma 2.6(4)]. Hence the hypothesis holds for $(G/R, A/R, LR/R)$, and so for a Hall σ_i-subgroup HR/R of G/R we see that

$$AH/R = (A/R)(HR/R) = (HR/R)(A/R) = HA/R$$

by the choice of G. But then $AH = HA$; a contradiction. Therefore, $A_G = 1$.

Note that A is not a σ_i-group since otherwise $A = A \cap H$ by [2, Lemma 2.6(7)], and so $AH = H = HA$. Hence $O^{\sigma_i}(A) \neq 1$.

Let $x \in G$ and $E = AL^x = L^xA$. Since $|E : A|$ is evidently a σ_i-number and A is σ-subnormal in E by [2, Lemma 2.6(1)], $O^{\sigma_i}(A) = O^{\sigma_i}(E)$ by [2, Lemma 2.6(8)]. Therefore $L^x \leq N_G(O^{\sigma_i}(A))$. Thus $L^G \leq N_G(O^{\sigma_i}(A))$. Assume that $L^GA < G$. The hypothesis holds for (L^GA, A, L). On the other hand, $H \cap L^G$ is a Hall σ_i-subgroup of L^G and so $H \leq L^GA$. The choice of G implies that $AH = HA$. This contradiction shows that $L^GA = G$ and so $O^{\sigma_i}(A)$ is normal in G. But then $O^{\sigma_i}(A) \leq A_G = 1$ and so A is a σ_i-group; a contradiction. This completes the proof of the fact that (1) implies (2).

(3) $\Rightarrow$ (1): Let H be a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$. Then HN/N is a Hall σ_i-subgroup of G/N and A/N is σ-subnormal in G/N, and so $AH/N = (A/N)(HN/N) = (HN/N)(A/N) = HA/N$ by applying (1) $\Rightarrow$ (2) to G/N. Hence (3) implies (1).

(1) $\Rightarrow$ (3): By hypothesis, G has a complete Hall Π-set $\mathscr{H} = \{H_1, \dots, H_t\}$ such that $AH^x = H^xA$ for all $H \in \mathscr{H}$ and $x \in G$. Then $\{H_1N/N, \dots, H_tN/N\}$ is a complete Hall Π-set of G/N. Moreover, $(A/N)(NH_i/N) = (NH_i/N)(A/N)$ for all i and A/N is Π-permutable in G/N. The proposition is proved.

Proof of Theorem A. (1) $\Rightarrow$ (2): It is enough to show that if B is a characteristic subgroup of A and H is a Hall σ_i-subgroup of G such that $AH = HA$, then $BH = HB$. Assume the contrary and let G be a counterexample with $|G| + |B| + |A|$ minimal. By hypothesis, $A = A_1 \times \dots \times A_t$, where $\{A_1, \dots, A_t\}$ is a complete Hall σ-set of A. Hence $B = (A_1 \cap B) \times \dots \times (A_t \cap B)$, where $\{A_1 \cap B, \dots, A_t \cap B\}$ is a complete Hall σ-set of B. We can assume without loss of generality that A_k is a σ_k-subgroup of A for all $k = 1, \dots, t$.

It is clear that $A_i \cap B$ is characteristic in A for all $i = 1, \dots, t$. Therefore, if $A_i \cap B < B$, then $(A_i \cap B)H = H(A_i \cap B)$ by the choice of G. So for some j, $j = 1$, say, $A_1 \cap B = B$ since otherwise

$$BH = ((A_1 \cap B) \times \cdots \times (A_t \cap B))H = H((A_1 \cap B) \times \cdots \times (A_t \cap B)) = HB.$$

Thus $B \leq A_1$. It is clear that A_1 is a σ-subnormal subgroup of G, and so in the case that $i = 1$ we have $B \leq A_1 \leq H$ by [2, Lemma 2.6(7)]. But then $BH = H = HB$; a contradiction. Thus $i > 1$.

We show now that $A_1 H = HA_1$. Note first that A_i is σ-subnormal in G, and so $A_i \leq H$ by [2, Lemma 2.6(7)]. Moreover, $A = A_1 \times V \times A_i$, where $V = A_2 \cdots A_{i-1} A_{i+1} \cdots A_t$, and so

$$HA = AH = (A_1 \times V \times A_i)H = (A_1 \times V)H = H(A_1 \times V),$$

where $A_1 \times V$ is a σ-subnormal σ_i'-subgroup of G. Then $A_1 \times V$ is σ-subnormal in $(A_1 \times V)H$ by [2, Lemma 2.6(1)]. Hence $H \leq N_G(A_1 \times V)$ by [2, Lemma 2.6(8)]. Since A_1 is a characteristic subgroup of $A_1 \times V$; therefore, $H \leq N_G(A_1)$. But B is a characteristic subgroup of A_1 since B is characteristic in A by hypothesis and $A = A_1 \times \cdots \times A_t$. Hence, $H \leq N_G(B)$ and so $BH = HB$; a contradiction. The implication is proved.

(2) $\Rightarrow$ (3): This is evident.

(3) $\Rightarrow$ (1): Since $A_1, \cdots, A_t$ are strongly Π-permutable in G by hypothesis, A is Π-permutable in G. Moreover, A is σ-subnormal in G by [2, Lemma 2.6(3)], and so A is strongly Π-permutable in G. The theorem is proved.

Proof of Theorem B. (1) $\Rightarrow$ (2): Let $x \in G$ and $L = \langle A, x \rangle$. Then L is Π-full and every Hall σ_i-subgroup V of L lies in some Hall σ_i-subgroup H of G for all $\sigma_i \in \Pi$ since G is a Π-full group of Sylow type by hypothesis. Then $AH = HA$ by Proposition 1.2, and so $AH \cap L = A(H \cap L) = AV = VA$; hence A is Π-permutable in L.

(2) $\Rightarrow$ (1): Assume the contrary and let G be a counterexample with $|G| + |A|$ minimal. Then $AH \neq HA$ for some $\sigma_i \in \Pi$ and some Hall σ_i-subgroup H of G.

Suppose that $\langle A, H \rangle < G$. Since the hypothesis holds for $(\langle A, H \rangle, A)$ by [2, Lemma 2.6(1)], A is Π-permutable in $\langle A, H \rangle$ by the choice of G. But then A permutes with every Hall σ_i-subgroup of $\langle A, H \rangle$ by Proposition 1.2 and so $AH = HA$. This contradiction shows that $\langle A, H \rangle = G$.

Now, let $x \in H$ and $L = \langle A, x \rangle$. Then A is Π-permutable in L by hypothesis, and so for every Hall σ_i-subgroup E of L we have $AE = EA$. It follows that $|AE : A|$ is a σ_i-number and so $O^{\sigma_i}(AE) = O^{\sigma_i}(A)$ by [2, Lemma 2.6(8)]. Hence $E^L \leq N_L(O^{\sigma_i}(A))$ and so $x \in N_G(O^{\sigma_i}(A))$. Therefore, $O^{\sigma_i}(A)$ is normal in G. But since $AH \neq HA$, $A \nleq H$ and so $O^{\sigma_i}(A) \neq 1$ by [2, Lemma 2.6(11)]. Thus, $A_G \neq 1$.

Note that $\langle A/A_G, xA_G \rangle = \langle A, x \rangle / A_G$, where A/A_G is Π-permutable in $\langle A, x \rangle / A_G$ by Proposition 1.2. It is clear also that A/A_G is σ-subnormal in G/A_G. Hence the hypothesis holds for G/A_G, and so A/A_G is Π-permutable in G/A_G by the choice of G. Thus, A is Π-permutable in G by Proposition 1.2. This contradiction completes the proof of the implication.

(3) $\Rightarrow$ (1): Assume that this implication is false and let G be a counterexample of minimal order. Then $AH \neq HA$ and so $A \neq G$ for some $\sigma_i \in \Pi$ and some Hall σ_i-subgroup H of G.

By hypothesis, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a σ_k-group for some $k = k(i)$ for all $i = 1, \dots, n$. We can assume without loss of generality that $M = A_{n-1} < G$.

It is clear that the hypothesis holds for (M, A), and so A is Π-permutable in M by the choice of G. Moreover, the Π-modularity of A in G implies that $M = M \cap \langle A, H \rangle = \langle A, (M \cap H) \rangle$. On the other hand, $M \cap H$ is a Hall σ_i-subgroup of M by [2, Lemma 2.6(7)]. Hence $M = A(M \cap H) = (M \cap H)A$. If $H \leq M_G$, then $A(M \cap H) = AH = HA$ and so $H \nleq M_G$.

Note now that $HM = MH$. Indeed, this is clear if M is normal in G. Otherwise, G/M_G is a σ_k-group for some k and so $G = MH = HM$ since $H \nleq M_G$ and H is a Hall σ_i-subgroup of G. Therefore, $HA = H(M \cap H)A = HM = MH = A(M \cap H)H = AH$. This contradiction completes the proof of the fact that (3) implies (1).

(1) $\Rightarrow$ (3): Let $A \leq C \leq E$ and let V be a Hall σ_i-subgroup of E, where $\sigma_i \in \Pi$. Then $AV = VA$ (see the proof of (1) $\Rightarrow$ (2)). Hence, $\langle A, V \cap C \rangle = A(V \cap C) = AV \cap C = \langle A, V \rangle \cap C$, and so A is Π-modular in E. The theorem is proved.

3. Proofs of Theorems C and D

Lemma 3.1 [9, A, Lemma 1.6]. *Let A, B, and E be subgroups of G such that $AE = EA$ and $BE = EB$. Then $\langle A, B\rangle E = E\langle A, B\rangle$.*

The following can be proved directly:

Lemma 3.2. *Let A and B be normal subgroups of G. Then*
(1) *if G is σ-nilpotent, then all quotients and all subgroups of G are σ-nilpotent;*
(2) *if G/A and G/B are σ-nilpotent, then $G/(A \cap B)$ is σ-nilpotent;*
(3) *if A and B are σ-nilpotent, then AB is σ-nilpotent.*

Proof of Theorem C. We show first that $\mathscr{L}_{\Pi per}(G)$ is a sublattice of $\mathscr{L}(G)$. In fact, by Lemma 3.1 and [2, Lemma 2.6(3)], it is enough to show that if A and B are σ-subnormal subgroups of G such that for a Hall σ_i-subgroup H of G we have $AH = HA$ and $BH = HB$, then $(A \cap B)H = H(A \cap B)$. Assume the contrary and let G be a counterexample of minimal order. Then G is not a σ_i-group, since otherwise $H = G$ and so $G = (A \cap B)H = H(A \cap B)$.

Let $E = AH \cap BH$. Then $A \cap E$ and $B \cap E$ are σ-subnormal subgroups of E by [2, Lemma 2.6(1)]. Moreover, $AH \cap E = H(A \cap E) = (A \cap E)H$. Similarly, $(B \cap E)H = H(B \cap E)$. Hence the hypothesis holds for $(E, A \cap E, B \cap E, H)$. Assume that $E < G$. Then the choice of G implies that $A \cap B = (A \cap E) \cap (B \cap E)$ is permutable with H. Hence $E = G$, and so $G = AH = BH$. Thus $|G : A|$ and $|G : B|$ are σ_i-numbers. Hence, $O^{\sigma_i}(A) = O^{\sigma_i}(G) = O^{\sigma_i}(B)$ by [2, Lemma 2.6(8)]. Therefore, since G is not a σ_i-group, it follows that $V = A_G \cap B_G \neq 1$. Moreover, A/V and B/V are σ-subnormal subgroups of G/V by [2, Lemma 2.6(4)]. Also, $(A/V)(HV/V) = AH/V = HA/V = (HV/V)(A/V)$ and $(B/V)(HV/V) = (HV/V)(B/V)$, where HV/V is a Hall σ_i-subgroup of G/V. Hence the choice of G implies that

$$(A \cap B/V)(HV/V) = ((A/V) \cap (B/V))(HV/V)$$
$$= (HV/V)((A/V) \cap (B/V)) = (HV/V)(A \cap B/V).$$

But then $(A \cap B)H = (A \cap B)HV = HV(A \cap B) = H(A \cap B)$. This contradiction completes the proof of the fact that $\mathscr{L}_{\Pi per}(G)$ is a sublattice of $\mathscr{L}(G)$.

Note now that the condition $\sigma^* \leq \sigma$ implies that every σ^*-subnormal subgroup of G is σ-subnormal in G. On the other hand, every subgroup A of G with $A^G/A_G \in \mathfrak{N}_{\sigma^*}$ is clearly σ^*-subnormal in G, and so $\mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G) \subseteq \mathscr{L}_{\Pi per}(G)$.

Finally, we show that for every two subgroups $A, B \in \mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G)$ we have $A \cap B, \langle A, B\rangle \in \mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G)$. Since $\mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G) \subseteq \mathscr{L}_{\Pi per}(G)$, it is enough to show that for every two subgroups A and B of G with $A^G/A_G \in \mathfrak{N}_{\sigma^*}$ and $B^G/B_G \in \mathfrak{N}_{\sigma^*}$ we have $(A \cap B)^G/(A \cap B)_G \in \mathfrak{N}_{\sigma^*}$ and $\langle A, B\rangle^G/\langle A, B\rangle_G \in \mathfrak{N}_{\sigma^*}$.

In view of the G-isomorphisms

$$A^G(A_GB_G)/A_GB_G \simeq A^G/(A^G \cap A_GB_G) = A^G/A_G(A^G \cap B_G) \simeq (A^G/A_G)/(A_G(A^G \cap B_G)/A_G),$$

we get that $A^G(A_GB_G)/A_GB_G \in \mathfrak{N}_{\sigma^*}$ since $\mathfrak{N}_{\sigma^*}$ is a homomorph by Lemma 3.2(1). Similarly, we can show that $B^G(A_GB_G)/A_GB_G \in \mathfrak{N}_{\sigma^*}$. Moreover, $A^GB^G/A_GB_G = (A^G(A_GB_G)/A_GB_G)(B^G(A_GB_G)/A_GB_G)$ and so $A^GB^G/A_GB_G \in \mathfrak{N}_{\sigma^*}$ by Lemma 3.2(3).

Note that $\langle A, B\rangle^G = A^GB^G$ and $A_GB_G \leq \langle A, B\rangle_G$. Therefore we get that $\langle A, B\rangle^G/\langle A, B\rangle_G \in \mathfrak{N}_{\sigma^*}$ by Lemma 3.2(1).

Note now that $(A \cap B)_G = A_G \cap B_G$. On the other hand, from the isomorphism

$$(A^G \cap B^G)/(A_G \cap B^G) = (A^G \cap B^G)/(A_G \cap B^G \cap A^G) \simeq A_G(B^G \cap A^G)/A_G \leq A^G/A_G$$

it follows that $(A^G \cap B^G)/(A_G \cap B^G) \in \mathfrak{N}_{\sigma^*}$ by Lemma 3.2(1). Similarly, $(B^G \cap A^G)/(B_G \cap A^G) \in \mathfrak{N}_{\sigma^*}$. But then $(A^G \cap B^G)/(A_G \cap B_G) \in \mathfrak{N}_{\sigma^*}$ by Lemma 3.2(2). It clear also that $(A \cap B)^G \leq A^G \cap B^G$. Hence $(A \cap B)^G/(A \cap B)_G \in \mathfrak{N}_{\sigma^*}$. Therefore $\mathscr{L}_{\mathfrak{N}_{\sigma^*}\Pi}(G)$ is a sublattice of $\mathscr{L}(G)$. The theorem is proved.

Lemma 3.3 [10, Lemma 5.2]. *Let $\mathscr{L}$ be a modular sublattice of $\mathscr{L}(G)$, and $U, V, N \in \mathscr{L}$ with $N \trianglelefteq \langle U, V\rangle$. If U permutes both with $V \cap UN$ and VN, then U permutes with V.*

We use $\mathscr{L}_{i\mathfrak{N}\Pi}(G)$ to denote the set of all Π-permutable σ_i-subgroups A of G with A^G/A_G nilpotent.

Proposition 3.4. *Suppose that G is Π-full and let $\mathscr{L} = \mathscr{L}_{i\mathfrak{N}\Pi}(G)$, where $\sigma_i \in \Pi$. Then*
(i) *$\mathscr{L}$ is a sublattice of $\mathscr{L}_{\mathfrak{N}\Pi}(G)$, and*
(ii) *if $\mathscr{L}$ is modular, then $AB = BA$ for all $A, B \in \mathscr{L}$.*

Proof. (i) Let $A, B \in \mathscr{L}$. Then $\langle A, B\rangle$ and $A \cap B$ are Π-permutable subgroups of G with $\langle A, B\rangle^G/\langle A, B\rangle_G \in \mathfrak{N}$ and $(A \cap B)^G/(A \cap B)_G$ both nilpotent by Theorem C. Moreover, the hypothesis implies that for some Hall σ_i-subgroup H of G and each $x \in G$ we have $H^x = AH^x = H^xA$, and so $A \leq H_G \leq O_{\sigma_i}(G)$. Similarly, $B \leq O_{\sigma_i}(G)$. Thus $\langle A, B\rangle$ and $A \cap B$ are σ_i-subgroups of G. Hence $\mathscr{L}$ is a sublattice of $\mathscr{L}_{\mathfrak{N}\Pi}(G)$.

(ii) Suppose the contrary and let G be a counterexample with $|G|+|A|+|B|$ minimal. Thus $AB \neq BA$ but $A_1B_1 = B_1A_1$ for all $A_1, B_1 \in \mathscr{L}$ such that $A_1 \leq A$, $B_1 \leq B$ and either $A_1 \neq A$ or $B_1 \neq B$. Let $\mathscr{H} = \{H_1, \ldots, H_t\}$ be a complete Hall Π-set of G. We can assume without loss of generality that H_1 is a σ_i-group.

(1) $(AN/N)(BN/N) = (BN/N)(AN/N)$ for any nonidentity normal σ_i-subgroup N of G. Hence $A_G = 1 = B_G$.

Note first that $\{H_1N/N, \ldots, H_tN/N\}$ is a complete Hall Π-set of G/N, and so G/N is Π-full. Moreover, AN/N and BN/N are evidently Π-permutable σ_i-subgroups of G/N. From the isomorphisms

$$(A^GN/N)/(A_GN/N) \simeq A^GN/A_GN \simeq A^G/(A^G \cap A_GN) = A^G/A_G(A^G \cap N)$$

$$\simeq (A^G/A_G)/(A_G(A^G \cap N)/A_G)$$

we get that $(A^GN/N)/(A_GN/N)$ is nilpotent since A^G/A_G is nilpotent. On the other hand, $(AN/N)^{G/N}$ $= (AN)^G/N = A^GN/N$ and $A_GN/N \leq (AN/N)_{G/N}$. Hence $(AN/N)^{G/N}/(AN/N)_{G/N}$ is nilpotent, and so $AN/N \in \mathscr{L}_{i\mathfrak{N}\Pi}(G/N)$. Similarly, $BN/N \in \mathscr{L}_{i\mathfrak{N}\Pi}(G/N)$.

Now, let H/N be a σ_i-subgroup of G/N. Then H is a σ_i-group. Moreover, Proposition 1.2 implies that H/N is Π-permutable in G/N if and only if H is Π-permutable in G. Finally, H^G/H_G is nilpotent if and only if $(H/N)^{G/N}/(H/N)_{G/N}$ is nilpotent in view of the isomorphism $H^G/H_G \simeq (H^G/N)/(H_G/N) = (H/N)^{G/N}/(H/N)_{G/N}$. Therefore, $\mathscr{L}_{i\mathfrak{N}\Pi}(G/N)$ is isomorphic to the interval $[G/N]$ in the modular lattice $\mathscr{L}$. Thus, $ANB/N = (AN/N)(BN/N) = (BN/N)(AN/N) = BNA/N$ by the minimality of $|G| + |A| + |B|$, and so $ANB = BNA$; hence $A_G = 1 = B_G$ since $AB \neq BA$.

(2) A^GB^G is nilpotent, and so $t > 1$.

Claim (1) implies that A^G and B^G are nilpotent, and so A^GB^G is also nilpotent. Assume now that $t = 1$ and let $W = O_{\sigma_i}(A^GB^G)$. Then $A, B \leq O_{\sigma_i}(A^GB^G)$ and every subgroup L of W is Π-permutable in G since $W \leq O_{\sigma_i}(G) \leq H$ for all Hall σ_i-subgroups H of G. It is clear also L^G/L_G is nilpotent. Hence $\mathscr{L}(W)$ is a sublattice of the modular lattice $\mathscr{L}$. But then $AB = BA$ by [11, Lemma 2.3.2]; a contradiction. Hence we have (2).

Now, let $O = H_2^G \cdots H_t^G$, $V = \langle A, B\rangle O$, and $R = \langle A, B\rangle \cap O$.

(3) $R = 1$.

Assume the contrary. Then $BRA = \langle A, B\rangle R$ by Claim (1).

We show now that $BRA = BR$. Suppose that this is false. Then $A \cap BR < A$. Moreover, $A \cap BR \in \mathscr{L}$ by Part (i) since $A, B, R \in \mathscr{L}$, and so the minimality of $|G| + |A| + |B|$ implies that B permutes with $A \cap BR$. Moreover, B permutes with RA since $B(RA) = \langle A, B\rangle R$, and so $AB = BA$ by Lemma 3.3. This contradiction shows that $A \leq BR$, and so $BRA = BR$.

By hypothesis, $BH_j^x = H_j^xB$ and so $H_j^x \leq N_G(B)$ for all $x \in G$ and $j > 1$ by [2, Lemma 2.6(8)] since H_j^x is a σ_i'-group. Hence $O \leq N_G(B)$. But then $R \leq O \leq N_G(B)$. Thus B is normal in BR. It follows that $A \leq N_G(B)$, and so $AB = BA$. This contradiction shows that $R = 1$.

Final contradiction for (ii). From Claim (3) it follows that $V = \langle A, B\rangle \times O$, and so every subgroup of $\langle A, B\rangle$ is O-invariant. It follows that every subgroup of $\langle A, B\rangle$ is Π-permutable in G. Note that $\langle A, B\rangle \leq O_{\sigma_i}(G)$ and for every subgroup L of $\langle A, B\rangle$ we have also that L^G/L_G is nilpotent by Claim (2). Hence $\mathscr{L}(\langle A, B\rangle)$ is a sublattice of $\mathscr{L}$. But then $AB = BA$ by [11, Lemma 2.3.2] since $\langle A, B\rangle$ is nilpotent; a contradiction. Hence (ii) holds.

Lemma 3.5. *Suppose that G is Π-full and $\sigma^* \leq \sigma$. If A is a Π-subgroup of G such that $A^G/A_G \in \mathfrak{N}_{\sigma^*}$, then A^G is a Π-group.*

Proof. Assume that the lemma is false and let G be a counterexample of minimal order. The hypothesis holds for $(G/A_G, A/A_G)$, and so $A_G = 1$ since otherwise $(A/A_G)^{G/A_G}/(A/A_G)_{G/A_G} = (A^G/A_G)/(A_G/A_G) \simeq A^G/A_G$ is a Π-group by the choice of G and so A^G is a Π-group. Therefore $A \in \mathfrak{N}_{\sigma^*}$ implying that $A \in \mathfrak{N}_\sigma$ since $\sigma^* \leq \sigma$. Hence $A = A_1 \times \dots \times A_t$, where $\{A_1 \times \dots \times A_t\}$ is a complete Hall σ-set of A, and so the choice of (G, A) implies that $A = A_1$ is a σ_i-group for some $\sigma_i \in \Pi$. By hypothesis, G has a Hall σ_i-subgroup H and then $A \leq H_G$ by [2, Lemma 2.6(7)]. But then A^G is a Π-group; a contradiction. The lemma is proved.

Proof of Theorem D. By Theorem C and Lemma 3.5, $\mathscr{L}$ is a sublattice of $\mathscr{L}(G)$. Moreover, if its every two subgroups $A, B \in \mathscr{L}(G)$ are permutable, then $\mathscr{L}$ is modular (see the proof of Proposition 3.4).

We show now that if $\mathscr{L}$ is modular, then every two subgroups $A, B \in \mathscr{L}(G)$ are permutable. Assume the contrary and let G be a counterexample with $|G| + |A| + |B|$ minimal.

Let R be a minimal normal subgroup of G. Then $\mathscr{L}_{\mathfrak{N}\Pi}(G/R)$ is isomorphic to the interval $[G/R]$ of $\mathscr{L}$ by Proposition 1.2 (see the proof of Proposition 3.4). Therefore the minimality of G implies that $(AR/R)(BR/R) = (BR/R)(AR/R)$. It follows that $RAB = \langle A, B\rangle R$ is a subgroup of G, and so $A_G = 1 = B_G$ since $AB \neq BA$. Hence A^G and B^G are nilpotent. The minimality of $|G| + |A| + |B|$ implies that for some i we have $A, B \leq O_{\sigma_i}(G)$ and so $A, B \in \mathscr{L}_{i\mathfrak{N}\Pi}(G)$. But $\mathscr{L}_{i\mathfrak{N}\Pi}(G)$ is a sublattice of $\mathscr{L}_{\mathfrak{N}\Pi}(G)$ by Proposition 3.4(i). Therefore $AB = BA$ by Proposition 3.4(ii); a contradiction. The theorem is proved.

References

1. Skiba A. N., "On some results in the theory of finite partially soluble groups," Commun. Math. Stat., vol. 4, no. 2, 281–309 (2016).
2. Skiba A. N., "On σ-subnormal and σ-permutable subgroups of finite groups," J. Algebra, vol. 436, 1–16 (2015).
3. Skiba A. N., "Some characterizations of finite σ-soluble $P\sigma T$-groups," J. Algebra, vol. 495, 114–129 (2018).
4. Guo W. and Skiba A. N., "On Π-quasinormal subgroups of finite groups," Monatsh. Math., vol. 185, 443–453 (2018).
5. Kegel O. H., "Sylow-Gruppen und Subnormalteiler endlicher Gruppen," Math. Z., Bd 78, 205–221 (1962).
6. Ballester-Bolinches A., Esteban-Romero R., and Asaad M., *Products of Finite Groups*, Walter de Gruyter, Berlin and New York (2010).
7. Ballester-Bolinches A. and Esteban-Romero R., "On finite soluble groups in which Sylow permutability is a transitive relation," Acta Math. Hungar., vol. 101, 193–202 (2003).
8. Skiba A. N., "On finite groups for which the lattice of S-permutable subgroups is distributive," Arch. Math., vol. 109, 9–17 (2017).
9. Doerk K. and Hawkes T., *Finite Soluble Groups*, Walter de Gruyter, Berlin and New York (1992).
10. Kimber T., "Modularity in the lattice of Σ-permutable subgroups," Arch. Math., vol. 83, 193–203 (2004).
11. Schmidt R., *Subgroup Lattices of Groups*, Walter de Gruyter, Berlin (1994).

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