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THE ROOT CLASS RESIDUALITY OF THE TREE PRODUCT OF GROUPS WITH AMALGAMATED RETRACTS

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Abstract: Given a root class $\mathscr{K}$ of groups, we prove that the tree product of residually $\mathscr{K}$-groups with amalgamated retracts is a residually $\mathscr{K}$-group. This yields a criterion for the $\mathscr{K}$-residuality of Artin and Coxeter groups with tree structure. We also prove that the HNN-extension X of a residually $\mathscr{K}$-group B is a residually $\mathscr{K}$-group provided that the associated subgroups of X are retracts in B and $\mathscr{K}$ contains at least one nonperiodic group.

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1. Introduction

Consider a tree T with vertex set V and edge set E. Assign to each vertex $v \in V$ some group A_v; and to each edge $e = \{v, w\} \in E$, some group H_e with embeddings $\varphi_{ev} : H_e \to A_v$ and $\varphi_{ew} : H_e \to A_w$. For convenience, denote the subgroups $H_e\varphi_{ev}$ and $H_e\varphi_{ew}$ of A_v and A_w by H_{vw} and H_{wv}.

Assume that the generators and defining relations are available for each group A_v ($v \in V$). Following [1], refer as the *tree product* of A_v ($v \in V$) with subgroups H_{vw} and H_{wv} amalgamated by φ_{ev} and φ_{ew} ($e = \{v, w\} \in E$) to the group

$$G = \langle A_v \ (v \in V);\ H_{vw} = H_{wv} \text{ for } \{v, w\} \in E\rangle,$$

whose generators are those of A_v for all $v \in V$ and whose defining relations are those of A_v for all $v \in V$ together with all possible relations of the form $h\varphi_{ev} = h\varphi_{ew}$ for $e = \{v, w\} \in E$ and $h \in H_e$, where $h\varphi_{ev}$ is the word in the generators of A_v which determines the image of h under φ_{ev}, while $h\varphi_{ew}$ is the word in the generators of A_w which determines the image of h under $h\varphi_{ew}$. The groups A_v ($v \in V$) are called the *vertex groups*, and the subgroups H_{vw} and H_{wv} ($\{v, w\} \in E$) are called *edge subgroups*.

If the tree T consists of two vertices and one edge then the corresponding tree product is the free product of two vertex groups with amalgamated subgroups. Another well-known construction is the free product of an arbitrary family of groups with one amalgamated subgroup; we can regard it as the tree product corresponding to a star graph on assuming that all edge subgroups of the group corresponding to the central vertex coincide.

Given a class $\mathscr{K}$ of groups, a group X is called a *residually $\mathscr{K}$-group* whenever for each nontrivial $x \in X$ there is a homomorphism of X onto a group in $\mathscr{K}$, called a $\mathscr{K}$-group, carrying x into a nontrivial element. This article studies the residuality of tree products with respect to an arbitrary root class of groups.

The concept of root class of groups was introduced by Gruenberg [2]. According to his definition, a nontrivial class $\mathscr{K}$ (containing at least one nontrivial group) is called a *root class* whenever $\mathscr{K}$ is closed under subgroups and direct products of finitely many factors, as well as satisfies the following condition: If $1 \leq Z \leq Y \leq X$ is a subnormal series of X whose factors X/Y and Y/Z lie in $\mathscr{K}$ then X has a normal subgroup T such that $T \subseteq Z$ and $X/T \in \mathscr{K}$. However, in order to understand what root classes really are, the equivalent definition of [3] turns out more convenient: A nontrivial class $\mathscr{K}$ of groups is called

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a *root class* whenever $\mathscr{K}$ is closed under subgroups and extensions, while together with two arbitrary groups X and Y $\mathscr{K}$ contains the Cartesian product of the form $\prod_{y\in Y} X_y$, where X_y is an isomorphic copy of X for each $y \in Y$. The second definition implies immediately that the intersection of root classes yields a root class. As the concrete examples of root classes, we mention the classes of all finite groups, finite p-groups for prime p, soluble groups, and all torsion-free groups.

As [2, 4] established, the free product of every family of residually $\mathscr{K}$-groups is a residually $\mathscr{K}$-group for every root class $\mathscr{K}$. Furthermore, the results presently available on the residuality with respect to particular root classes of groups (see [5–9] for instance) suggest that for more complicated group theory constructions, like free products with amalgamated subgroups or HNN-extensions, any similar universal criterion is most likely unfeasible. The residuality of these constructions is therefore studied under various restrictions on the original groups, their subgroups, and the connecting isomorphisms; see [3, 4, 10–19] for instance. One of these restrictions is the property that the amalgamated subgroup of the generalized free product is a retract in the free factor that includes it.

Recall that a subgroup Y of a group X is called a *retract* whenever X has a normal subgroup Z with $X = YZ$ and $Y \cap Z = 1$. In other words, a subgroup Y is a retract in X whenever there exists a homomorphism of X onto Y which acts identically on Y. Note also that Y is a retract in X if and only if X is the split extension of some group Z by Y.

Boler and Evans established [20] that the free product of two residually finite groups with amalgamated retracts is a residually finite group. A similar result is obtained in [21] for residual p-finiteness. Indeed, Theorem 1 of [22] generalizes the two assertions.

Theorem 1 [22, Theorem 1]. *Consider an arbitrary root class $\mathscr{K}$ of groups and the free product P of groups A_1 and A_2 with subgroups $H_1 \le A_1$ and $H_2 \le A_2$ amalgamated by some isomorphism $\varphi : H_1 \to H_2$. If A_1 and A_2 are residually $\mathscr{K}$-groups and the subgroups H_i are retracts in A_i then P is a residually $\mathscr{K}$-group.*

In [11] Theorem 1 was extended to the case of free products of arbitrary families of groups with one amalgamated subgroup. Moreover, [17] obtained a sufficient condition, generalizing Theorem 1, for the residuality with respect to an arbitrary root class of the free product of two groups with amalgamated subgroups only one of which is a retract in the corresponding free factor.

The following statement is the main result of this article:

Theorem 2. *Consider an arbitrary root class $\mathscr{K}$ of groups, a tree $T = (V, E)$, and the tree product G of groups A_v for $v \in V$ with subgroups H_{vw} and H_{wv} amalgamated by isomorphisms φ_{ev} and φ_{ew} for $e = \{v, w\} \in E$. If all A_v are residually $\mathscr{K}$-groups and, for each edge $\{v, w\} \in E$, the subgroup H_{vw} is a retract in A_v and the subgroup H_{wv} is a retract in A_w then G is a residually $\mathscr{K}$-group.*

By way of application of Theorem 2, we obtain criteria for the residuality with respect to an arbitrary root class for Artin and Coxeter groups with tree structures.

Take an arbitrary set V and a family E of two-element subsets of V. Recall (see [23] for instance) that the *Artin group* is the group with generators a_v for $v \in V$ and defining relations

$$a_v a_w a_v \ldots = a_w a_v a_w \ldots \quad \text{for } \{v, w\} \in E,$$

where the words on both sides are of length $m_e \ge 2$, with $e = \{v, w\} \in E$, and consist of alternating symbols a_v and a_w. Henceforth we write these relations as $\langle a_v a_w\rangle^{m_e} = \langle a_w a_v\rangle^{m_e}$ for $e = \{v, w\} \in E$.

Thus, the Artin group is determined by the presentation

$$G = \langle a_v\ (v \in V);\ \langle a_v a_w\rangle^{m_e} = \langle a_w a_v\rangle^{m_e} \text{ for } e = \{v, w\} \in E\rangle.$$

The *Coxeter group* $\overline{G}$ is usually considered along with the Artin group. We obtain $\overline{G}$ by adding the relation $a_v^2 = 1$ for $v \in V$ to the defining relations of G. The equalities $a_v^2 = 1$ imply that $a_v = a_v^{-1}$ for all $v \in V$, and so each relation $\langle a_v a_w\rangle^{m_e} = \langle a_w a_v\rangle^{m_e}$ for $e = \{v, w\} \in E$ can be rewritten as $(a_v a_w)^{m_e} = 1$. Hence, $\overline{G}$ is presented as

$$\overline{G} = \big\langle a_v\ (v \in V);\ a_v^2 = 1\ (v \in V),\ (a_v a_w)^{m_e} = 1 \text{ for } e = \{v, w\} \in E\big\rangle.$$

Clearly, we can regard the pair of sets (V, E) as an undirected graph with vertex set V and edge set E. If this graph is a tree then the Artin (Coxeter) group is called the Artin (Coxeter) group *with tree structure* [24]. As we show below, the Artin and Coxeter groups with tree structure turn out to be tree products with amalgamated retracts, and so we can use Theorem 2 to study their root class residuality.

Henceforth, given an arbitrary root class $\mathcal{K}$ of groups, we denote by $\pi(\mathcal{K})$ the set of all prime divisors of the orders of finite groups in $\mathcal{K}$, and by $\mathcal{Z}(\mathcal{K})$ the set of positive integers with all prime divisors in $\pi(\mathcal{K})$.

Corollary 1. *Consider an arbitrary root class $\mathcal{K}$ and the Artin group*

$$G = \langle a_v \ (v \in V);\ \langle a_v a_w \rangle^{m_e} = \langle a_w a_v \rangle^{m_e} \text{ for } e = \{v, w\} \in E \rangle$$

with tree structure.

(1) *If $\mathcal{K}$ contains at least one nonperiodic group then G is a residually $\mathcal{K}$-group.*

(2) *Suppose that $\mathcal{K}$ consists of periodic groups.*

(i) *If m_e is even for all $e \in E$ then G is a residually $\mathcal{K}$-group if and only if $m_e/2$ lies in $\mathcal{Z}(\mathcal{K})$ for all $e \in E$.*

(ii) *If there is at least one odd number among m_e for $e \in E$ then G is a residually $\mathcal{K}$-group if and only if m_e lies in $\mathcal{Z}(\mathcal{K})$ for all $e \in E$ and $2 \in \mathcal{Z}(\mathcal{K})$.*

Corollary 2. *Consider an arbitrary root class $\mathcal{K}$. The Coxeter group*

$$\overline{G} = \left\langle a_v \ (v \in V);\ a_v^2 = 1 \ (v \in V),\ (a_v a_w)^{m_e} = 1 \text{ for } e = \{v, w\} \in E \right\rangle$$

with tree structure is a residually $\mathcal{K}$-group if and only if m_e lies in $\mathcal{Z}(\mathcal{K})$ for all $e \in E$ and $2 \in \mathcal{Z}(\mathcal{K})$.

As we mentioned above, the classes of finite groups and soluble groups are root classes; and, therefore, so is their intersection, the class of finite soluble groups. Thus, Corollaries 1 and 2 imply that the Artin and Coxeter groups with tree structures are residually finite soluble groups. It is also easy to deduce a criterion for the residual p-finiteness of these groups: In this case $\pi(\mathcal{K}) = \{p\}$, while $\mathcal{Z}(\mathcal{K})$ amounts to the set all nonnegative powers of p. As a comment to Corollaries 1 and 2, observe that if m_e lies in $\mathcal{Z}(\mathcal{K})$ for all $e \in E$ and at least one m_e is even then $2 \in \mathcal{Z}(\mathcal{K})$ automatically.

Apart from residuality criteria for Artin and Coxeter groups, we use Theorem 2 to prove the following assertion:

Theorem 3. *Consider a root class $\mathcal{K}$ of groups containing at least one nonperiodic group and the HNN-extension $X = \langle B, t;\ t^{-1}Ht = K,\ \varphi \rangle$ of a residually $\mathcal{K}$-group B with subgroups $H \leq B$ and $K \leq B$ related by some isomorphism $\varphi : H \to K$. If H and K are retracts in B then X is a residually $\mathcal{K}$-group.*

It is well known that each free group is residually torsion-free soluble [25]. The class of torsion-free soluble groups is a root class because it is the intersection of root classes: soluble groups and torsion-free groups. Thus, Theorem 3 implies the following two assertions:

Corollary 3. *Consider the HNN-extension X of a soluble group B with subgroups $H \leq B$ and $K \leq B$ associated by some isomorphism $\varphi : H \to K$. If H and K are retracts in B then X is a residually soluble group. If B is torsion-free then X is residually torsion-free soluble.*

Corollary 4. *Consider the HNN-extension X of a free group B with subgroups $H \leq B$ and $K \leq B$ associated by some isomorphism $\varphi : H \to K$. If H and K are retracts in B then X is a residually torsion-free soluble group.*

The rest of the article contains the proofs of Theorems 2 and 3 as well as Corollaries 1 and 2.

2. Some Properties of the Tree Products of Groups

Given some tree $T = (V, E)$, consider the tree product G of groups A_v $(v \in V)$ with subgroups H_{vw} and H_{wv} amalgamated by some isomorphisms φ_{ev} and φ_{ew} $(e = \{v, w\} \in E)$. Given a subtree $T' =$

(V', E') of T, we can define a tree product by assigning to the vertices and edges of T' the same groups and embeddings as for T. Call this tree product *corresponding* to T'. As [1] shows, if $T' = (V', E')$ is a subtree of T and G' is the corresponding tree product then the identity mapping of the generators of G' into G extends to an embedding. In particular, A_v embeds into G by the identity mapping of the generators for all $v \in V$, which enables us to regard each vertex group as a subgroup of the tree product.

Take a subset E' of E and the set $\{T_i \mid i \in I\}$ of the connected components of the graph obtained from T by removing all edges in E'. Call $\{T_i \mid i \in I\}$ the *partition* of T into subtrees corresponding to E'.

Clearly, for distinct $i, j \in I$ the tree T has at most one edge connecting some vertex of T_i to some vertex of T_j and that if this edge exists then it lies in E'. It is also obvious that each edge of E' joins two vertices of distinct subtrees. Therefore, we can define the graph $\mathscr{T}$ with the vertex set $\mathscr{V} = \{T_i \mid i \in I\}$ and the edge set $\mathscr{E}$ corresponding bijectively to E', with $\{T_i, T_j\} \in \mathscr{E}$ if and only if there exists an edge $e \in E'$ joining some vertex of T_i to some vertex of T_j. It is easy to see that $\mathscr{T}$ is a tree. Call it the *tree product determined* by E'.

Denote by G_i the tree product corresponding to the subtree T_i. As we indicated above, for each $i \in I$ we may regard all vertex groups of T_i as subgroups of G_i. Hence, if $e \in E'$ joins a vertex v of T_i with a vertex w of T_j then φ_{ev} and φ_{ew} determine embeddings of H_e into G_i and G_j respectively. Consequently, we can define the tree product $\mathscr{G}$ by assigning to $T_i \in \mathscr{V}$ the group G_i; while to the edge $\{T_i, T_j\} \in \mathscr{E}$ corresponding to some edge $e = \{v, w\} \in E'$, the group H_e and the embeddings φ_{ev} and φ_{ew}. Refer to the so-constructed group $\mathscr{G}$ as the *tree product determined* by E'.

The following proposition is not difficult to verify:

Proposition 1 [1]. *Consider the tree product G corresponding to some tree T and a subset E' of edges of T. Then the tree product $\mathscr{G}$ determined by E' admits the same presentation as G. In particular, G and $\mathscr{G}$ are isomorphic.*

Now, for $v \in V$, fix a normal subgroup R_v of each vertex group A_v. Refer to the system of subgroups $\mathscr{R} = \{R_v \mid v \in V\}$ as a system of *compatible normal subgroups* provided that for every $e = \{v, w\} \in E$ the subgroups R_v and R_w are H_e-compatible, meaning that H_e has a normal subgroup R_e satisfying $R_v \cap H_{vw} = R_e\varphi_{ev}$ and $R_w \cap H_{wv} = R_e\varphi_{ew}$.

It is easy to see that if $\mathscr{R} = \{R_v \mid v \in V\}$ is a system of compatible normal subgroups of A_v for $v \in V$ then for every $e = \{v, w\} \in E$ the mapping $\overline{\varphi_{ev}} : H_e/R_e \to H_{vw}R_v/R_v$, carrying the coset hR_e with $h \in H_e$ to the coset $(h\varphi_{ev})R_v$, and the mapping $\overline{\varphi_{ew}} : H_e/R_e \to H_{wv}R_w/R_w$, carrying the coset hR_e with $h \in H_e$ to the coset $(h\varphi_{ew})R_w$, are well defined and are isomorphisms of subgroups. Therefore, along with the tree product G, for the same tree T we can consider the tree product

$$G_{\mathscr{R}} = \langle A_v/R_v \ (v \in V);\ H_{vw}R_v/R_v = H_{wv}R_w/R_w \text{ for } \{v, w\} \in E\rangle$$

of the groups A_v/R_v $(v \in V)$ with the subgroups $H_{vw}R_v/R_v$ and $H_{wv}R_w/R_w$ amalgamated by $\overline{\varphi_{ev}}$ and $\overline{\varphi_{ew}}$ $(e = \{v, w\} \in E)$.

We can verify directly that the mapping of words extending the identity mapping of the generators of G into $G_{\mathscr{R}}$ carries all defining relations of G into equalities valid in $G_{\mathscr{R}}$, and so this mapping is a surjective homomorphism.

Thus, we have

Proposition 2. *Given a tree $T = (V, E)$, consider the corresponding tree product G of groups A_v for $v \in V$ and a system $\mathscr{R} = \{R_v \mid v \in V\}$ of compatible normal subgroups of A_v. Then there exists a surjective homomorphism $\rho_{\mathscr{R}} : G \to G_{\mathscr{R}}$ whose action on each subgroup A_v for $v \in V$ coincides with the action of the natural homomorphism $A_v \to A_v/R_v$.*

Proposition 3. *Given a tree $T = (V, E)$, consider the corresponding tree product G of A_v $(v \in V)$ with the subgroups H_{vw} and H_{wv} amalgamated by the isomorphisms φ_{ev} and φ_{ew} $(e = \{v, w\} \in E)$. Suppose furthermore that H_{vw} is a retract in A_v and H_{wv} is a retract in A_w for all $\{v, w\} \in E$. Then for every $u \in V$ and every normal subgroup R_u of A_u there exists a homomorphism of G onto the* *quotient A_u/R_u extending the natural homomorphism of A_u onto A_u/R_u. In particular, each subgroup B_u of A_u that is a retract in A_u turns out to be a retract in G.*

Proof. Fix $u \in V$ and a normal subgroup R_u of A_u. Since T is a tree, for each vertex $v \in V$ there exists a unique shortest chain from u to v. Refer to its length as the *distance* from u to v.

Construct a system $\mathscr{R} = \{R_v \mid v \in V\}$ of compatible normal subgroups of A_v for $v \in V$ by induction on the distance l from u to v for which it is necessary to define R_v.

If $l = 0$ then $R_v = R_u$. Assume that $l > 0$ and for all vertices at distance less than l from u the subgroups have already been constructed. Take a vertex v at distance l from u, the vertex w directly preceding v in the shortest (u, v)-chain, and the edge e joining w and v. Then w lies at distance $l - 1$ from u, and so a compatible normal subgroup R_w in the vertex group A_w is available. Find in A_v a normal subgroup R_v such that R_w and R_v are H_e-compatible.

Put $R_{wv} = R_w \cap H_{wv}$ and $R_{vw} = R_{wv}\varphi_{ew}^{-1}\varphi_{ev}$. Since H_{vw} is a retract in A_v, there exists a normal subgroup N_{vw} of A_v satisfying $A_v = H_{vw}N_{vw}$ and $H_{vw} \cap N_{vw} = 1$. Put $R_v = R_{vw}N_{vw}$ and verify that R_v is the required subgroup.

Verify firstly that R_v is a normal subgroup of A_v. Since R_w is normal in A_w, it follows that R_{wv} is normal in H_{wv}, and consequently R_{vw} is normal in H_{vw}. Moreover, N_{vw} is normal in A_v. Therefore,

$$R_v^{A_v} = (R_{vw}N_{vw})^{A_v} = R_{vw}^{A_v}N_{vw} = R_{vw}^{H_{vw}N_{vw}}N_{vw} = R_{vw}^{N_{vw}}N_{vw}.$$

The normality of N_{vw} in A_v also implies that $[R_{vw}, N_{vw}] \subseteq N_{vw}$. Hence, $R_{vw}^{N_{vw}} \subseteq R_{vw}N_{vw} = R_v$. Therefore, $R_v^{A_v} \subseteq R_vN_{vw} = R_v$, and so R_v is a normal subgroup of A_v.

Verify that R_w and R_v are H_e-compatible subgroups; i.e., H_e includes a normal subgroup R_e satisfying $R_w \cap H_{wv} = R_e\varphi_{ew}$ and $R_v \cap H_{vw} = R_e\varphi_{ev}$. Put $R_e = R_{wv}\varphi_{ew}^{-1}$. Then $R_w \cap H_{wv} = R_{wv} = R_e\varphi_{ew}$ and $R_{vw} = R_{wv}\varphi_{ew}^{-1}\varphi_{ev} = R_e\varphi_{ev}$. As we indicated above, R_{wv} is normal in H_{wv} and so R_e is normal in H_e. It remains to show that $R_v \cap H_{vw} = R_{vw}$.

Since $R_v = R_{vw}N_{vw}$, we have $R_{vw} \subseteq R_v \cap H_{vw}$. The reverse inclusion follows because $H_{vw} \cap N_{vw} = 1$. Indeed, if $h \in R_v \cap H_{vw}$ then $h = h_1h_2$ for some $h_1 \in R_{vw}$ and $h_2 \in N_{vw}$. Since $R_{vw} \subseteq H_{vw}$, this implies that $h_2 = h_1^{-1}h \in H_{vw} \cap N_{vw} = 1$ and $h = h_1 \in R_{vw}$.

Therefore, R_v is defined for each vertex v at distance l from u and, consequently, for every $v \in V$.

For the same tree T and the constructed system $\mathscr{R}$ of compatible normal subgroups of vertex groups, along with the tree product G we can consider the tree product $G_{\mathscr{R}}$ of the quotient groups A_v/R_v for $v \in V$ with $H_{vw}R_v/R_v$ and $H_{wv}R_w/R_w$ amalgamated by the isomorphisms $\overline{\varphi_{ev}}$ and $\overline{\varphi_{ew}}$ ($e = \{v, w\} \in E$). By Proposition 2, there exists a surjective homomorphism $\rho_{\mathscr{R}} : G \to G_{\mathscr{R}}$ whose action on A_v for $v \in V$ coincides with the action of the natural homomorphism $A_v \to A_v/R_v$.

To verify that $A_v\rho_{\mathscr{R}} \subseteq A_u\rho_{\mathscr{R}}$ for each vertex $v \in V$, induct on the distance l from u to v. If $l = 0$ then the claim is obvious. For $l > 0$ take the vertex w directly preceding v in the shortest (u, v)-chain and the edge e joining w and v, assuming that $A_w\rho_{\mathscr{R}} \subseteq A_u\rho_{\mathscr{R}}$. Since $A_v = H_{vw}N_{vw}$, we can express each $a \in A_v$ as $a = hb$ for suitable $h \in H_{vw}$ and $b \in N_{vw}$. Since the action of $\rho_{\mathscr{R}}$ on the vertex group A_v coincides with that of the natural homomorphism $A_v \to A_v/R_v$ and $N_{vw} \subseteq R_v$ by the construction of R_v, we infer that $a\rho_{\mathscr{R}} = h\rho_{\mathscr{R}}$. Consequently, $A_v\rho_{\mathscr{R}} = H_{vw}\rho_{\mathscr{R}}$. Since $\rho_{\mathscr{R}}$ carries all defining relations of G into equalities, $x\varphi_{ew}\rho_{\mathscr{R}} = x\varphi_{ev}\rho_{\mathscr{R}}$ for each $x \in H_e$.

Therefore, $H_{wv}\rho_{\mathscr{R}} = H_{vw}\rho_{\mathscr{R}}$ and so $A_v\rho_{\mathscr{R}} = H_{wv}\rho_{\mathscr{R}} \subseteq A_w\rho_{\mathscr{R}} \subseteq A_u\rho_{\mathscr{R}}$.

The above shows that $G_{\mathscr{R}} \subseteq A_u\rho_{\mathscr{R}}$, and so $G_{\mathscr{R}} = A_u/R_u$. Thus, $\rho_{\mathscr{R}}$ is the required homomorphism.

In particular, if B_u is a retract in A_u then the homomorphism of A_u onto B_u acting identically on B_u extends to a homomorphism of G onto B_u. Thus, B_u is a retract in G as well. $\square$

3. Proof of Theorem 2

Assume firstly that the tree T is finite. In this case the proof proceeds by induction on the number of edges in T. If T lacks edges and consists of a sole vertex then the tree product G coincides with the sole vertex group which is a residually $\mathscr{K}$-group by assumption. Verify that if the tree products G are residually $\mathscr{K}$-groups for trees T with at most n edges then the same holds for $n + 1$ edges.

Fix some leaf v of T and the unique edge $e = \{v, w\}$ incident to v. Denote by T_e the subtree of T obtained by removing e together with v and by G_e, the tree product corresponding to T_e. By the inductive assumption, G_e is a residually $\mathscr{K}$-group. By Proposition 3, the edge subgroup H_{wv} is a retract in G_e. Consequently, the tree product $\mathscr{G}$ determined by e is the free product of residually $\mathscr{K}$-groups A_v and G_e with amalgamated retracts, and so it is a residually $\mathscr{K}$-group by Theorem 1. It remains to observe that the original tree product G is isomorphic to $\mathscr{G}$ according to Proposition 1.

Assume that T is an infinite tree and take some nontrivial element g of G. To complete the proof, we find a homomorphism of G onto a residually $\mathscr{K}$-group carrying g into a nontrivial element. Clearly, each homomorphism of this kind extends to a homomorphism of G onto a $\mathscr{K}$-group acting on g similarly.

Take some word ω defining g. Denote by T_0 the smallest subtree of T the union of whose vertex groups contains all generators of G corresponding to the letters in ω. The tree T_0 is obviously finite. Take the tree product G_0 corresponding to T_0. The above argument for the case of finite trees shows that G_0 is a residually $\mathscr{K}$-group. As indicated above, we may assume that G_0 is a subgroup of G; moreover, $g \in G_0$ by construction.

Denote by E' the set of all edges of T joining some vertex of T_0 to some vertex outside T_0. Partition T into the set $\{T_i \mid i \in I\}$ of subtrees corresponding to E'. Take the tree $\mathscr{T} = (\mathscr{V}, \mathscr{E})$ and the tree product $\mathscr{G}$ determined by E'. It is easy to see that $\mathscr{T}$ is a star whose central vertex is the subtree T_0.

Take an arbitrary edge $\{T_0, T_i\} \in \mathscr{E}$ of $\mathscr{T}$ and a corresponding edge e of E' joining some vertex v of T_0 with some vertex w of T_i. Since the subgroup H_{vw} is a retract in A_v and the subgroup H_{wv} is a retract in A_w, Proposition 3 shows that H_{vw} is a retract in G_0 and H_{wv} is a retract in G_i.

Hence, $\mathscr{G}$ amounts to a tree product with amalgamated retracts. It is obvious that G_0 is a retract in G_0; hence, Proposition 3 shows once again that G_0 is a retract in $\mathscr{G}$. By Proposition 1, G and $\mathscr{G}$ are isomorphic. Thus, there exists a homomorphism of G onto G_0 acting identically on G_0, as required. $\square$

4. Some Statements on the Root Classes of Groups and Residuality

Throughout this section, for a root class $\mathscr{K}$ of groups and a group X denote by $\mathscr{K}^*(X)$ the set of all normal subgroups of X the quotients by which lie in $\mathscr{K}$. It is clear that X is a residually $\mathscr{K}$-group if and only if the intersection of all subgroups in $\mathscr{K}^*(X)$ is trivial.

Proposition 4. *The following hold for each root class $\mathscr{K}$ of groups and each group X:*
(1) *The intersection of finitely many subgroups in $\mathscr{K}^*(X)$ belongs to this family.*
(2) *If X is a finite residually $\mathscr{K}$-group then X is in $\mathscr{K}$.*

PROOF. Indeed, if $N_1, N_2, \dots, N_k$ are subgroups in $\mathscr{K}^*(X)$ then by Remak's Theorem the quotient group $X/(N_1 \cap N_2 \cap \dots \cap N_k)$ embeds into the direct product of the quotient groups $X/N_1, X/N_2, \dots, X/N_k$, and so belongs to $\mathscr{K}$ because this class is closed under subgroups and direct products of finitely many factors.

Assume that X is a finite residually $\mathscr{K}$-group. Obviously $X \in \mathscr{K}$ if $X = 1$. If $X \neq 1$ then for each nontrivial element $x \in X$ choose a subgroup $N_x \in \mathscr{K}^*(X)$ that avoids x and denote by N the intersection of all these subgroups. Then $N = 1$ and Proposition 1 yields $N \in \mathscr{K}^*(X)$. Thus, $X \in \mathscr{K}$. $\square$

Proposition 5. *Consider a root class $\mathscr{K}$ consisting of periodic groups. Denote by P the free product of a cyclic group A and an abelian group B with two subgroups $H \leq A$ and $K \leq B$ amalgamated by some isomorphism $\varphi : H \to K$; furthermore, $K \neq B$ and $[A : H] < \infty$. If P is a residually $\mathscr{K}$-group then $[A : H] \in \mathscr{Z}(\mathscr{K})$.*

PROOF. Assume that P is a residually $\mathscr{K}$-group and A is generated by an element a. Put $t = [A : H]$. If $t = 1$ then the claim is obvious. Assume that $t > 1$.

Suppose that there exists an integer $s \in [1; t-1]$ such that $a^s \in HN$ for every subgroup $N \in \mathscr{K}^*(P)$. Since $K \neq B$, there is $b \in B \setminus K$. Consider $c = [a^s, b]$, which has in P a reduced form of length 4 and so is nontrivial by Theorem 4.4 of [26]. At the same time, for each subgroup $N \in \mathscr{K}^*(P)$ there is $h \in H$ with $a^s \equiv h \pmod{N}$, and so $c \equiv [h, b] = 1 \pmod{N}$ because B is abelian. This contradicts the property that P is a residually $\mathscr{K}$-group.

Hence, for every $s \in \{1, \ldots, t-1\}$ there is a subgroup $N_s \in \mathscr{K}^*(P)$ satisfying $a^s \notin HN_s$. Put $N = \bigcap_{s=1}^{t-1} N_s$. Then Proposition 4 yields $N \in \mathscr{K}^*(P)$ and it is easy to see that $N \cap A \subseteq H$.

Since $A/(N \cap A) \cong AN/N \leq P/N \in \mathscr{K}$ and $\mathscr{K}$ is closed under subgroups, $A/(N \cap A) \in \mathscr{K}$. Since $\mathscr{K}$ consists of periodic groups, the cyclic group $A/(N \cap A)$ must be finite, and so all prime divisors of the order of the group lie in $\pi(\mathscr{K})$. Since $N \cap A \leq H \leq A$, we see that $[A : H]$ divides $[A : N \cap A]$. Thus, $[A : H] \in \mathscr{Z}(\mathscr{K})$. $\square$

Proposition 6. *For each root class $\mathscr{K}$ of groups the following hold:*

(1) *Every free group is a residually $\mathscr{K}$-group* [4, Theorem 1].

(2) *The free product of each family of residually $\mathscr{K}$-groups is a residually $\mathscr{K}$-group* [2, 4].

(3) *Every extension of a residually $\mathscr{K}$-group by a $\mathscr{K}$-group is a residually $\mathscr{K}$-group* [2, Lemma 1.5].

If a root class $\mathscr{K}$ contains at least one nonperiodic group then it contains an infinite cyclic group because it is closed under subgroups. Hence, Proposition 6 directly implies the following:

Proposition 7. *Consider a root class $\mathscr{K}$ of groups containing at least one nonperiodic group. Then each extension of a residually $\mathscr{K}$-group by an infinite cyclic group is a residually $\mathscr{K}$-group. In particular, each extension of a free group by an infinite cyclic group is a residually $\mathscr{K}$-group.*

Proposition 8. *Given a root class $\mathscr{K}$ of groups, a finite soluble group is in $\mathscr{K}$ if and only if the order of the group belongs to $\mathscr{Z}(\mathscr{K})$.*

Proof. The necessity follows from the definition of $\mathscr{Z}(\mathscr{K})$. Let us verify sufficiency.

Take a finite soluble group X whose order q belongs to $\mathscr{Z}(\mathscr{K})$. It is known that X possesses a subnormal series with cyclic factors of prime orders. Since each of these orders divides q and $q \in \mathscr{Z}(\mathscr{K})$, they all belong to $\pi(\mathscr{K})$. Since $\mathscr{K}$ is a hereditary class, by the definition of $\pi(\mathscr{K})$ and Sylow's Theorem all factors of the series are in $\mathscr{K}$. Since $\mathscr{K}$ is closed under extensions, we see that $X \in \mathscr{K}$. $\square$

5. The Root Class Residuality of Two-Generator Artin and Coxeter Groups

In order to prove the criterion for the root class residuality of a two-generator Artin group with respect to an arbitrary root class, we need the two auxiliary propositions:

Proposition 9. *Consider a root class $\mathscr{K}$ of groups and the group $G = \langle x, y;\ x^m = y^2 \rangle$ for some integer $m > 1$.*

(1) *If $\mathscr{K}$ contains at least one nonperiodic group then G is a residually $\mathscr{K}$-group.*

(2) *If $\mathscr{K}$ consists of periodic groups then G is a residually $\mathscr{K}$-group if and only if $m \in \mathscr{Z}(\mathscr{K})$ and $2 \in \mathscr{Z}(\mathscr{K})$.*

Proof. Observe first of all that G amounts to the free product of the infinite cyclic groups $X = \langle x \rangle$ and $Y = \langle y \rangle$ with amalgamated subgroups generated by x^m and y^2. Therefore, necessity in (2) follows from Proposition 5.

Consider the mapping of the generators of G into the infinite cyclic group with generator z acting as $x \mapsto z^2$ and $y \mapsto z^m$. Extended to a mapping on words, it carries the defining relation of G into a valid equality, and so determines a homomorphism σ that is obviously injective on the free factors X and Y. Hence, Neumann's Theorem [27] implies that the kernel of σ is a free group. Thus, G amounts to an extension of the free group by an infinite cyclic group, and so (1) holds by Proposition 7.

Let us establish sufficiency in (2). To this end, take some $g \in G \setminus 1$ and find a homomorphism of G onto a residually $\mathscr{K}$-group carrying g into a nontrivial element.

Denote by H the infinite cyclic subgroup of G generated by x^m, or equivalently by y^2, and observe that it is central in G. If $g \in H$ then the homomorphism σ constructed above is the required one because σ acts injectively on H, while $G\sigma$ is infinite and cyclic, and consequently it is a residually $\mathscr{K}$-group by Proposition 6. If g lies outside H then it remains nontrivial under the natural homomorphism ρ of G onto the quotient group G/H. The latter admits the presentation $\langle x, y;\ x^m = 1,\ y^2 = 1 \rangle$ and so amounts to the free product of the finite cyclic groups $X_m = \langle x;\ x^m = 1 \rangle$ and $Y_2 = \langle y;\ y^2 = 1 \rangle$. Since $m \in \mathscr{Z}(\mathscr{K})$

and $2 \in \mathscr{Z}(\mathscr{K})$, by Proposition 8 X_m and Y_2 are in $\mathscr{K}$. Hence, by Proposition 6, the free product of X_m and Y_2 is a residually $\mathscr{K}$-group, and ρ is the required homomorphism. $\square$

Proposition 10. *Consider a root class $\mathscr{K}$ of groups and the group $G = \langle x, y;\ x^{-1}y^k x = y^k\rangle$ for some positive integer k.*

(1) *If $\mathscr{K}$ contains at least one nonperiodic group then G is a residually $\mathscr{K}$-group.*

(2) *If $\mathscr{K}$ consists of periodic groups then G is a residually $\mathscr{K}$-group if and only if $k \in \mathscr{Z}(\mathscr{K})$.*

Proof. Use the same strategy as in the proof of Proposition 9.

The group G amounts to the HNN-extension of the infinite cyclic group $Y = \langle y\rangle$ with a stable letter x and the coinciding associated subgroups that are generated by y^k. Adding to the presentation of G the new generator z and the defining relation $z = y^k$, we obtain the presentation $G = \langle x, y, z;\ z = y^k,\ x^{-1}zx = z\rangle$, from which we see that G is simultaneously the free product of the same group Y and the free abelian group $\langle x, z;\ [z, x] = 1\rangle$ with amalgamated subgroups that are generated by y^k and z. Therefore, as in the proof of Proposition 9, necessity in (2) follows from Proposition 5.

The natural homomorphism σ of G onto its quotient group by the normal closure of x acts injectively on the base group Y. According to the structure theorem for subgroups of HNN-extensions [28], this means that the kernel of σ is a free group. Since the image of G under σ is an infinite cyclic group, by Proposition 7 this immediately implies (1).

The verification of sufficiency in (2) repeats almost verbatim the similar arguments of the proof of Proposition 9. Take some $g \in G \setminus 1$, denote by H the cyclic subgroup generated by y^k, and find a homomorphism of G onto a residually $\mathscr{K}$-group carrying g into a nontrivial element. Consider the two cases: $g \in H$ and $g \notin H$. If $g \in H$ then the homomorphism σ defined above works because it acts injectively on H and sends G onto an infinite cyclic group. If $g \notin H$ then observe that H is central in G and consider the natural homomorphism ρ of G onto the quotient group

$$G/H = \langle x, y;\ x^{-1}y^k x = y^k,\ y^k = 1\rangle = \langle x, y;\ y^k = 1\rangle,$$

which amounts to the free product of the infinite cyclic group $X = \langle x\rangle$ and the finite cyclic group $Y_k = \langle y;\ y^k = 1\rangle$. Note that X is a residually $\mathscr{K}$-group by Proposition 6 and Y_k belongs to $\mathscr{K}$ in view of Proposition 8 because $k \in \mathscr{Z}(\mathscr{K})$. Consequently, Proposition 6 once again shows that G/H is a residually $\mathscr{K}$-group. Since $g\rho = gH \neq 1$, the homomorphism ρ works. $\square$

Proposition 11. *Consider a root class $\mathscr{K}$ of groups and the two-generator Artin group $G = \langle a, b;\ \langle ab\rangle^m = \langle ba\rangle^m\rangle$ for some integer $m > 1$.*

(1) *If $\mathscr{K}$ contains at least one nonperiodic group then G is a residually $\mathscr{K}$-group.*

(2) *Assume that $\mathscr{K}$ consists of periodic groups.*

(i) *If m is even then G is a residually $\mathscr{K}$-group if and only if $m/2 \in \mathscr{Z}(\mathscr{K})$.*

(ii) *If m is odd then G is a residually $\mathscr{K}$-group if and only if $m \in \mathscr{Z}(\mathscr{K})$ and $2 \in \mathscr{Z}(\mathscr{K})$.*

Proof. It is known [29, Lemma 9] that G is isomorphic to $\langle x, y;\ x^m = y^2\rangle$ if m is odd and to $\langle x, y;\ x^{-1}y^k x = y^k\rangle$, where $k = m/2$, if m is even. The first isomorphism is given by the mapping $a \mapsto x^{l+1}y^{-1}$ and $b \mapsto yx^{-l}$, where $l = (m-1)/2$; and the second, by the mapping $a \mapsto x^{-1}y$ and $b \mapsto x$. Thus, the claim follows from Propositions 9 and 10. $\square$

Proposition 12. *Consider a root class $\mathscr{K}$ of groups and the two-generator Coxeter group $G = \langle a, b;\ a^2 = 1,\ b^2 = 1,\ (ab)^m = 1\rangle$ for some integer $m > 1$ Then G is a residually $\mathscr{K}$-group if and only if $m \in \mathscr{Z}(\mathscr{K})$ and $2 \in \mathscr{Z}(\mathscr{K})$.*

Proof. Add to the presentation of G the new generators x and y and the defining relations $x = a$ and $y = ab$. Then $b = xy$. Replacing in all relations of G the generators a and b by their expressions in terms of x and y, eliminate from the presentation of G the generators a and b and the relations $a = x$ and $b = xy$. This yields $G = \langle x, y;\ x^2 = 1,\ y^m = 1,\ (xy)^2 = 1\rangle$.

Using the equality $x = x^{-1}$, we can rearrange this presentation as

$$G = \langle x, y;\ x^2 = 1,\ y^m = 1,\ x^{-1}yx = y^{-1}\rangle.$$

It is known (see [26, § 1.2] for instance) that the order of G is finite and equals $2m$. By Proposition 4, this in particular implies that G is a residually $\mathscr{K}$-group if and only if G is in $\mathscr{K}$. Moreover, the relation $x^{-1}yx = y^{-1}$ implies that the cyclic subgroup Y generated by y is normal in G. Since $G/Y \cong \mathbb{Z}_2$; therefore, G is finite polycyclic. Proposition 8 shows that $G \in \mathscr{K}$ if and only if $m \in \mathscr{Z}(\mathscr{K})$ and $2 \in \mathscr{Z}(\mathscr{K})$. $\square$

6. Proofs of Corollaries 1 and 2

Put

$$G = \langle a_v\ (v \in V);\ \langle a_v a_w \rangle^{m_e} = \langle a_w a_v \rangle^{m_e} \text{ for } e = \{v, w\} \in E \rangle,$$

$$\overline{G} = \big\langle a_v\ (v \in V);\ a_v^2 = 1 \text{ for } v \in V,\ (a_v a_w)^{m_e} = 1 \text{ for } e = \{v, w\} \in E \big\rangle$$

and assume that $T = (V, E)$ is a tree. Construct from T the new graph $\mathscr{T}$ by splitting each edge of T in two with a new vertex. More formally, the vertex set of $\mathscr{T}$ coincides with $V \cup E$, while the edges of $\mathscr{T}$ are all possible sets of the form $\{e, v\}$ ($e \in E$, $v \in e$).

It is obvious that $\mathscr{T}$ is also a tree. Construct from it and the group G or the group $\overline{G}$ a tree product of groups as follows:

To each $v \in V$ inherited from the original tree T assign the cyclic group G_v generated by the generator a_v which is infinite for the Artin group and of order 2 for the Coxeter group. To the new vertex corresponding to the edge $e = \{v, w\} \in E$ of T assign the group G_e equal to the two-generator Artin group

$$\langle a_v, a_w;\ \langle a_v a_w \rangle^{m_e} = \langle a_w a_v \rangle^{m_e} \rangle$$

or the two-generator Coxeter group

$$\big\langle a_v, a_w;\ a_v^2 = 1,\ a_w^2 = 1,\ (a_v a_w)^{m_e} = 1 \big\rangle$$

respectively. Finally, to the edge ε joining $e \in E$ and $v \in e$ assign the group $H_\varepsilon = G_v$ and the embeddings $\varphi_{\varepsilon v} : H_\varepsilon \to G_v$ and $\varphi_{\varepsilon e} : H_\varepsilon \to G_e$ extending the identity mappings of a_v. The tree product constructed in this fashion clearly coincides with the original Artin or Coxeter group.

The natural homomorphism of G_e ($e = \{v, w\} \in E$) onto the quotient group of G_e by the normal closure of $a_v a_w^{-1}$ carries G_e onto each of the subgroups $H_\varepsilon \varphi_{\varepsilon e}$ and $H_\delta \varphi_{\delta e}$ ($\varepsilon = \{e, v\}$ and $\delta = \{e, w\}$) acting identically on these subgroups. Consequently, $H_\varepsilon \varphi_{\varepsilon e}$ and $H_\delta \varphi_{\delta e}$ are retracts in G_e. Moreover, for each edge $\varepsilon = \{e, v\}$ the subgroup $H_\varepsilon \varphi_{\varepsilon v}$ coincides with G_v and, in particular, is a retract in G_v. Thus, the claims of Corollaries 1 and 2 follow from Theorem 2 and Propositions 11 and 12. $\square$

7. Proof of Theorem 3

Given $i \in \mathbb{Z}$, denote by B_i an isomorphic copy of the group B and by $\sigma_i : B \to B_i$ the corresponding isomorphism. Define the tree product G of the groups B_i taking as the tree the infinite chain whose vertices are enumerated with successive integers and assigning to each vertex $i \in \mathbb{Z}$ the group B_i, while to the edge $\{i,\ i+1\}$ for $i \in \mathbb{Z}$ the group H and the embeddings $\alpha_i : H \to B_i$ and $\beta_i : H \to B_{i+1}$, where α_i is the restriction to H of the isomorphism σ_i and β_i is the composition of the isomorphism φ and the restriction to K of the isomorphism σ_{i+1}. Observe that then in the group G we have $h\sigma_i = h\varphi\sigma_{i+1}$ for all $h \in H$ and $i \in \mathbb{Z}$.

Since H and K are retracts in B, the images of H under α_i and β_i are retracts in B_i and B_{i+1} respectively for each $i \in \mathbb{Z}$. Consequently, the tree product G is a residually $\mathscr{K}$-group by Theorem 2.

By the Reidemeister–Schreier method (see [26, § 2.3] for instance) it is not difficult to show that the normal closure of B in X admits the same presentation as that of G; the subgroup B_i of G corresponds to the subgroup $t^i B t^{-i}$ of X. Therefore, X is an extension of a residually $\mathscr{K}$-group G by the infinite cyclic group generated by t, and so X is a residually $\mathscr{K}$-group by Proposition 7. $\square$

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