ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 4, pp. 572–584. *© Pleiades Publishing, Inc., 2019.*
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THE PARTIAL CLONE OF LINEAR FORMULAS

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UDC 512.57

Abstract: A term t is linear if no variable occurs more than once in t. An identity $s \approx t$ is said to be linear if s and t are linear terms. Identities are particular formulas. As for terms superposition operations can be defined for formulas too. We define the arbitrary linear formulas and seek for a condition for the set of all linear formulas to be closed under superposition. This will be used to define the partial superposition operations on the set of linear formulas and a partial many-sorted algebra $\mathrm{Formclone}_{\mathrm{lin}}(\tau, \tau')$. This algebra has similar properties with the partial many-sorted clone of all linear terms. We extend the concept of a hypersubstitution of type τ to the linear hypersubstitutions of type (τ, τ') for algebraic systems. The extensions of linear hypersubstitutions of type (τ, τ') send linear formulas to linear formulas, presenting weak endomorphisms of $\mathrm{Formclone}_{\mathrm{lin}}(\tau, \tau')$.

DOI: 10.1134/S0037446619040037

Keywords: term, formula, superposition, linear term, linear formula, clone, partial clone, linear hypersubstitution

1. Preliminaries

Let $X_n = \{x_1, x_2, \dots, x_n\}$ be a finite set of variables and let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be countably infinite. Let $(f_i)_{i \in I}$ be an indexed set of operation symbols, where f_i is n_i-ary and $n_i \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ for all $i \in I$, $I \neq \varnothing$. Let $(\gamma_j)_{j \in J}$ be an indexed set of relation symbols where γ_j is n_j-ary with $n_j \in \mathbb{N}^+$, $j \in J$. Let $\tau := (n_i)_{i \in I}$ and $\tau' := (n_j)_{j \in J}$, $J \neq \varnothing$. The set $W_\tau(X_n)$ of all n-ary terms of type τ is defined by

(i) each $x_k \in X_n$ is an n-ary term of type τ,

(ii) if $t_1, \dots, t_{n_i}$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X) = \bigcup_{n \geq 1} W_\tau(X_n)$ be the set of all terms of type τ. We use the same notation for the many-sorted set $W_\tau(X) := (W_\tau(X_n))_{n \in \mathbb{N}^+}$.

Note that the set of all terms of type τ is the formal language of the equational logic of universal algebra. Here the models of sets of equations $s \approx t$, consisting of the terms $s, t \in W_\tau(X)$, are universal algebras $\mathscr{A} = (A; (f_i^{\mathscr{A}})_{i \in I})$ of type τ, where $f_i^{\mathscr{A}} : A^{n_i} \to A$ are n_i-ary operations for each $i \in I$. Equations are special kinds of formulas. To describe the properties of the algebraic systems $\mathscr{A} = (A; (f_i^{\mathscr{A}})_{i \in I}, (\gamma_j^{\mathscr{A}})_{j \in J})$, where $(\gamma_j^{\mathscr{A}})_{j \in J}$ is an indexed set of relations on A, we need more general formulas. The arbitrary n-ary formulas of type (τ, τ') are defined as follows (see, e.g., [1, 2]):

Definition 1.1. Let $n \in \mathbb{N}^+$. An n-ary formula of type (τ, τ') is defined inductively:

(i) If t_1 and t_2 are n-ary terms of type τ, then $t_1 \approx t_2$ is an n-ary formula of type (τ, τ'). All variables in $t_1 \approx t_2$ are free. Note that x_i is free in a formula F provided that the quantifier $\exists$ does not occur in front of x_i in F.

(ii) If $t_1, \dots, t_{n_j}$ are n-ary terms of type τ and if γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \dots, t_{n_j})$ is an n-ary formula of type (τ, τ'). All variables in such a formula are free.

(iii) If F is an n-ary formula of type (τ, τ'), then $\neg F$ is an n-ary formula of type (τ, τ'). All free variables in F are free in $\neg F$ too. All bound variables in F are bound in $\neg F$ too.

(iv) If F_1 and F_2 are n-ary formulas of type (τ, τ') such that variables occurring simultaneously in both formulas are free in each of them, then $F_1 \vee F_2$ is an n-ary formula of type (τ, τ'). The variables

Original article submitted February 16, 2018; revised February 16, 2018; accepted May 23, 2018.

free in at least one of the formulas F_1 or F_2 are also free in $F_1 \vee F_2$. The variables bound in either F_1 or F_2 are also bound in $F_1 \vee F_2$.

(v) If F is an n-ary formula of type (τ, τ') and $x_i \in X_n$ is free in a formula F, then $\exists x_i(F)$ is an n-ary formula of type (τ, τ'). The variable x_i is bound in $\exists x_i(F)$, and all other free or bound variables in F are of the same nature in $\exists x_i(F)$.

Let $\mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary formulas of type (τ, τ') and let

$$\mathscr{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n\geq 1} \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$$

be the set of all formulas of type (τ, τ'). We will also use

$$\mathscr{F}_{(\tau,\tau')}(W_\tau(X)) := (\mathscr{F}_{(\tau,\tau')}(W_\tau(X_n)))_{n\in\mathbb{N}^+}.$$

This definition follows Malcev [1] with the difference that Malcev additionally considers the logical connectives $\wedge$ and $\to$ and the quantifier $\forall$ which can be derived from $\vee$, $\neg$, and $\exists$. We remark that the equation symbol in (i) does not belong to the relation symbols from $\{\gamma_j \mid j \in J\}$.

It is well known that the superposition operations $S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$ can inductively be defined on the many-sorted set $W_\tau(X)$ for $m, n \in \mathbb{N}^+$ by

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m),$$

and

(i) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$ for $1 \leq j \leq n$ and

(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) = f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then a many-sorted algebra

$$\operatorname{clone} \tau := ((W_\tau(X_n))_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}),$$

the clone of all terms of type τ, can be defined which satisfies the following three equalities:

(C1) $\widetilde{S}^p_m(\widetilde{Z}, \widetilde{S}^n_m(\widetilde{Y_1}, \widetilde{X_1}, \ldots, \widetilde{X_n}), \ldots, \widetilde{S}^n_m(\widetilde{Y_p}, \widetilde{X_1}, \ldots, \widetilde{X_n})) \approx \widetilde{S}^n_m(\widetilde{S}^p_n(\widetilde{Z}, \widetilde{Y_1}, \ldots, \widetilde{Y_p}), \widetilde{X_1}, \ldots, \widetilde{X_n})$ $(m, n, p \in \mathbb{N}^+)$,

(C2) $\widetilde{S}^n_m(\lambda_j, \widetilde{X_1}, \ldots, \widetilde{X_n}) \approx \widetilde{X_j}$ $(m = 1, 2, \ldots,\ 1 \leq j \leq n,\ n \in \mathbb{N}^+)$,

(C3) $\widetilde{S}^n_n(\widetilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \widetilde{Y}$ $(n \in \mathbb{N}^+)$,

where $\widetilde{Z}, \widetilde{Y_1}, \ldots, \widetilde{Y_p}, \widetilde{X_1}, \ldots, \widetilde{X_n}$ are variables for terms, $\widetilde{S}^n_m$ are operation symbols, and λ_i, $i = 1, \ldots, n$, are nullary operation symbols.

(C1) generalizes the associative law. Indeed, for $m = n = p = 1$ we have $\widetilde{S}^1_1(\widetilde{Z}, \widetilde{S}^1_1(\widetilde{Y_1}, \widetilde{X_1})) \approx \widetilde{S}^1_1(\widetilde{S}^1_1(\widetilde{Z}, \widetilde{Y_1}), \widetilde{X_1})$. Therefore, (C1) is called the *superassociative law*.

The concept of clone is one of the leading algebraic concepts. Each model of the axioms (C1)–(C3) is said to be an *abstract clone*. Let A with $|A| > 1$ be an arbitrary nonempty set, let $O^n(A)$ be the set of all n-ary operations on A, and let $O(A) := \bigcup_{n\geq 1} O^n(A)$ be the set of all operations on A. On the many-sorted set $(O^n(A))_{n\in\mathbb{N}^+}$ the superposition operations $S^{n,A}_m : O^n(A) \times (O^m(A))^n \to O^m(A)$ can be defined by

$$S^{n,A}_m(f, g_1, \ldots, g_n)(a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $a_1, \ldots, a_m \in A$. Together with the projection operations $e^{n,A}_i$ defined by $e^{n,A}_i(a_1, \ldots a_n) := a_i$ for all $a_1, \ldots, a_n \in A$ and $1 \leq i \leq n$, $n \in \mathbb{N}^+$, we obtain the many-sorted algebra

$$((O^n(A))_{n\in\mathbb{N}^+}; (S^{n,A}_m)_{m,n\in\mathbb{N}^+}, (e^{n,A}_i)_{1\leq i\leq n, n\in\mathbb{N}^+}).$$

This algebra satisfies (C1)–(C3); i.e., it is a clone. Such clones of operations are called *concrete clones*. It is well known that every abstract clone is isomorphic to a concrete one. A clone can also be regarded as a category. The duals of those categories are the so-called Lawvere Theories (see [3]).

In [2] the authors introduced the superposition operations on sets of formulas in the following way (see also [4]):

$$R^n_m : \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n)) \times (W_\tau(X_m))^n \to \mathscr{F}_{(\tau,\tau')}(W_\tau(X_m))$$

with

(i) if $t_1 \approx t_2 \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1, \dots, s_n \in W_\tau(X_m)$, then $R^n_m(t_1 \approx t_2, s_1, \dots, s_n)$ is the formula $S^n_m(t_1, s_1, \dots, s_n) \approx S^n_m(t_2, s_1, \dots, s_n)$;

(ii) if $\gamma_j(t_1, \dots, t_{n_j}) \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1, \dots, s_n \in W_\tau(X_m)$, then $R^n_m(\gamma_j(t_1, \dots, t_{n_j}), s_1, \dots, s_n)$ is the formula

$$\gamma_j(S^n_m(t_1, s_1, \dots, s_n), \dots, S^n_m(t_{n_j}, s_1, \dots, s_n));$$

(iii) if $\neg F \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1, \dots, s_n \in W_\tau(X_m)$, then $R^n_m(\neg F, s_1, \dots, s_n)$ is the formula $\neg R^n_m(F, s_1, \dots, s_n)$;

(iv) if $F_1 \vee F_2 \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1, \dots, s_n \in W_\tau(X_m)$, then $R^n_m(F_1 \vee F_2, s_1, \dots, s_n)$ is the formula $R^n_m(F_1, s_1, \dots, s_n) \vee R^n_m(F_2, s_1, \dots, s_n)$;

(v) if $\exists x_i(F) \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1, \dots, s_n \in W_\tau(X_m)$, then $R^n_m(\exists x_i(F), s_1, \dots, s_n)$ is the formula $\exists x_i(R^n_m(F, s_1, \dots, s_n))$.

The definition of R^n_m needs the two kinds of sets:

$$(\mathscr{F}_{(\tau,\tau')}(W_{(\tau)}(X_n)))_{n\in\mathbb{N}^+} \quad \text{and} \quad (W_{(\tau)}(X_n))_{n\in\mathbb{N}^+}.$$

We may consider the following many-sorted algebra:

$$\mathrm{Formclone}(\tau,\tau') := ((\mathscr{F}_{(\tau,\tau')}(W_\tau(X_n)))_{n\in\mathbb{N}^+}, (W_\tau(X_n))_{n\in\mathbb{N}^+};$$
$$(R^n_m)_{m,n\in\mathbb{N}^+}, (S^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n, n\in\mathbb{N}^+}).$$

As proved in [2] this many-sorted algebra satisfies the following identities:

Theorem 1.2. *The multi-based algebra* Formclone(τ,τ') *satisfies* (FC1) and (FC2), where

(FC1) $\widetilde{R}^n_m(\widetilde{R}^p_n(\widetilde{F}, \widetilde{X}_1, \dots, \widetilde{X}_p), \widetilde{Y}_1, \dots, \widetilde{Y}_n) \approx \widetilde{R}^p_m(\widetilde{F}, \widetilde{S}^n_m(\widetilde{X}_1, \widetilde{Y}_1, \dots \widetilde{Y}_n), \dots, \widetilde{S}^n_m(\widetilde{X}_p, \widetilde{Y}_1, \dots, \widetilde{Y}_p))$ with $\widetilde{X}_1, \dots, \widetilde{X}_p, \widetilde{Y}_1, \dots, \widetilde{Y}_n$ variables for terms, while $\widetilde{R}^n_m$, $\widetilde{R}^p_n$, and $\widetilde{S}^n_m$ are operation symbols, and $\widetilde{F}$ is a variable for a formula.

(FC2) $\widetilde{R}^n_n(\widetilde{F}, \lambda_1, \dots, \lambda_n) \approx \widetilde{F}$. Here $\lambda_i, i = 1, \dots, n$, are nullary operation symbols.

Moreover, (C1)–(C3) are also satisfied.

2. Linear Formulas

Linear terms over a universal algebra generalize linear expressions over a vector space. A term in which variables occur at most once is said to be *linear*. For a formal definition of n-ary linear terms we replace (ii) in the definition of terms by a slightly different condition.

Definition 2.1. An n-ary linear term of type τ is defined inductively:

(i) $x_j \in X_n$ is an n-ary linear term (of type τ) for all $j \in \{1, \dots, n\}$.

(ii) If $t_1, \dots, t_{n_i}$ are n-ary linear terms and if $\mathrm{var}(t_j) \cap \mathrm{var}(t_k) = \varnothing$ for all $1 \le j < k \le n_i$, then $f_i(t_1 \dots, t_{n_i})$ is an n-ary linear term (here $\mathrm{var}(t_j)$ is the set of all variables in t_j).

(iii) The set $W^{\mathrm{lin}}_\tau(X_n)$ of all n-ary linear terms is the smallest set that contains $x_1, \dots, x_n$ and is closed under finite application of (ii).

The set of all linear terms of type τ over the countably infinite alphabet X is defined by $W^{\mathrm{lin}}_\tau(X) := \bigcup_{n\ge 1} W^{\mathrm{lin}}_\tau(X_n)$. We also use $W^{\mathrm{lin}}_\tau(X) := (W^{\mathrm{lin}}_\tau(X_n))_{n\in\mathbb{N}^+}$.

The many-sorted set $(W^{\mathrm{lin}}_\tau(X_n))_{n\in\mathbb{N}^+}$ of linear terms is not closed under S^n_m. As an example we consider $\tau = (2)$ with the binary operation symbol f. Then $S^2_2(f(x_2, x_1), x_1, f(x_2, x_1)) = f(f(x_2, x_1), x_1)$ is not linear, although $f(x_2, x_1)$ and x_1 are linear. We ask under which condition $(W^{\mathrm{lin}}_\tau(X_n))_{n\in\mathbb{N}^+}$ is closed under S^n_m.

In [5], the following was proved:

Proposition 2.2 [3]. *If $f_i(t_1,\dots,t_{n_i}) \in W_\tau^{\mathrm{lin}}(X_n)$, $s_1,\dots,s_n \in W_\tau^{\mathrm{lin}}(X_m)$, and $\mathrm{var}(s_j)\cap\mathrm{var}(s_k)=\varnothing$ for $1\le j<k\le n$, then $S_m^n(f_i(t_1,\dots,t_{n_i}),s_1,\dots,s_n)\in W_\tau^{\mathrm{lin}}(X_m)$.*

Proof. Because of

$$S_m^n(f_i(t_1,\dots,t_{n_i}),s_1,\dots,s_n) = f_i(S_m^n(t_1,s_1,\dots,s_n),\dots,S_m^n(t_{n_i},s_1,\dots,s_n))$$

we have to show that

(1) $S_m^n(t_j,s_1,\dots,s_n)\in W_\tau^{\mathrm{lin}}(X_m)$ for all $1\le j\le n_i$,

(2) $\mathrm{var}(S_m^n(t_j,s_1,\dots,s_n))\cap\mathrm{var}(S_m^n(t_k,s_1,\dots,s_n))=\varnothing$ for all $1\le j<k\le n_i$.

(1) Since t_j is linear, the variables from X_n occur only once in $t_j, 1\le j\le n_i$. For these variables we substitute terms from $\{s_1,\dots,s_n\}$. But by presumption each variable from X_m occurs only once in the set $\{s_1,\dots,s_n\}$; i.e., the variables from X_m occur only once in $S_m^n(t_j,s_1,\dots,s_n)$ and so $S_m^n(t_j,s_1,\dots,s_n)$ is linear.

(2) Since $f_i(t_1,\dots,t_{n_i})\in W_\tau^{\mathrm{lin}}(X_n)$; therefore, $\mathrm{var}(t_j)\cap\mathrm{var}(t_k)=\varnothing$ for $1\le j<k\le n_i$, i.e. in $S_m^n(t_j,s_1,\dots,s_n)$ and in $S_m^n(t_k,s_1,\dots,s_n)$ for the different variables some different terms from $s_1,\dots,s_n$ are substituted that contain different variables because of $\mathrm{var}(s_j)\cap\mathrm{var}(s_k)=\varnothing$ for $1\le j<k\le n_i$. $\square$

The previous result leads to the partial many-sorted mapping

$$\overline{S}_m^n : W_\tau^{\mathrm{lin}}(X_n)\times(W_\tau^{\mathrm{lin}}(X_m))^n \longrightarrow\!\!\!\!\!\!\circ\;\; W_\tau^{\mathrm{lin}}(X_m)$$

defined by

$$\overline{S}_m^n(t,s_1,\dots,s_n) := \begin{cases} S_m^n(t,s_1\dots,s_n) & \text{if } \mathrm{var}(s_j)\cap\mathrm{var}(s_k)=\varnothing \text{ for all } 1\le j<k\le n,\\ \text{not defined} & \text{otherwise}\end{cases}$$

for $m,n\in\mathbb{N}^+$ and a partial many-sorted algebra

$$\mathrm{clone}_{\mathrm{lin}}\,\tau := ((W_\tau^{\mathrm{lin}}(X_n))_{n\in\mathbb{N}^+}, (\overline{S}_m^n)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n,n\in\mathbb{N}^+}).$$

An equation $s\approx t$ of terms over the partial many-sorted algebra $\mathscr{A}$ is said to be a weak identity in $\mathscr{A}$ if after evaluation both sides are defined and equal. The partial many-sorted algebra $\mathrm{clone}_{\mathrm{lin}}\tau$ satisfies (C1)–(C3) as weak identities [5].

Our main goal is to define linear formulas and then to check whether the results for linear terms can be transferred to linear formulas. Given a formula F, let $\mathrm{var}(F)$ be the set of variables in F.

Definition 2.3. Let $n\ge 1$. Let $(f_i)_{i\in I}$ be an indexed set of operation symbols of type τ where f_i is n_i-ary and let $(\gamma_j)_{j\in J}$ be an indexed set of relation symbols of type τ', where γ_j is n_j-ary and $n_i,n_j\in\mathbb{N}^+$. An *n-ary linear formula of type (τ,τ')* is defined inductively:

(i) If t_1 and t_2 are n-ary linear terms of type τ, then $t_1\approx t_2$ is an n-ary linear formula of type (τ,τ'). All variables in $t_1\approx t_2$ are free.

(ii) If $t_1,\dots,t_{n_j}$ are n-ary linear terms of type τ, if $\mathrm{var}(t_k)\cap\mathrm{var}(t_l)=\varnothing$ for $1\le k<l\le n_j$ and if γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1,\dots,t_{n_j})$ is an n-ary linear formula of type (τ,τ'). All variables in such a formula are free.

(iii) If F is an n-ary linear formula of type (τ,τ'), then $\neg F$ is an n-ary linear formula of type (τ,τ'). All free variables in F are also free in $\neg F$. All bound variables in F are also bound in $\neg F$.

(iv) If F_1 and F_2 are n-ary linear formulas of type (τ,τ') and $\mathrm{var}(F_1)\cap\mathrm{var}(F_2)=\varnothing$, then $F_1\vee F_2$ is an n-ary linear formula of type (τ,τ'). The bound variables in either F_1 or F_2 are bound in $F_1\vee F_2$ as well. (We notice that in this case $F_1\vee F_2$ is a formula since there are no variables that occur in both formulas, in F_1 and F_2.)

(v) If F is an n-ary linear formula of type (τ,τ') and $x_i\in X_n$ is free in F, then $\exists x_i(F)$ is an n-ary linear formula of type (τ,τ'). The variable x_i is bound in the formula $\exists x_i(F)$ and all other free or bound variables in F are of the same nature in $\exists x_i(F)$.

(i) corresponds to the usual definition of linear identities, e.g., see [6] or [7]. In [7] the authors showed the importance of linear identities for power algebras.

Let $\mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_{\tau}(X_n))$ be the set of all n-ary linear formulas of type (τ,τ') and let

$$\mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_{\tau}(X)) := \bigcup_{n\geq 1} \mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_{\tau}(X_n))$$

be the set of all linear formulas of type (τ,τ'). We will also use

$$\mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_{\tau}(X)) := (\mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_{\tau}(X_n)))_{n\in\mathbb{N}^+}.$$

If we use instead of (i) the condition

(i′) If t_1 and t_2 are n-ary linear terms of type τ with $\mathrm{var}(t_1)\cap\mathrm{var}(t_2)=\varnothing$ then the equation $t_1\approx t_2$ is an n-ary linear formula of type (τ,τ'). All variables in $t_1\approx t_2$ are free, then we speak of an n-ary strong linear formula. Clearly, any strong linear formula is linear.

As an example we consider type $(2,2)$. Let f be a binary operation symbol and let γ be a binary relation symbol. To determine $\mathscr{F}^{\mathrm{lin}}_{(2,2)}(W^{\mathrm{lin}}_{(2)}(X_2))$ we begin with steps (i) and (ii) of the definition. The formulas defined by (i) and (ii) are called atomic formulas. The atomic linear formulas are: $x_1\approx x_1$, $x_1\approx x_2$, $x_2\approx x_1$, $x_2\approx x_2$, $f(x_1,x_2)\approx x_1$, $f(x_1,x_2)\approx x_2$, $x_1\approx f(x_1,x_2)$, $x_2\approx f(x_1,x_2)$, $f(x_1,x_2)\approx f(x_1,x_2)$, $f(x_1,x_2)\approx f(x_2,x_1)$, $f(x_2,x_1)\approx f(x_1,x_2)$, $f(x_2,x_1)\approx f(x_2,x_1)$, $\gamma(x_1,x_2)$, $\gamma(x_2,x_1)$.

From the atomic linear formulas we obtain all other linear formulas using $\neg$, $\exists$, and $\vee$.

The many-sorted set $(\mathscr{F}^{\mathrm{lin}}_{\tau}(W^{\mathrm{lin}}_{\tau}(X_n)))_{n\in\mathbb{N}^+}$ is not closed under the superposition operations R^n_m. As an example we consider the type $\tau=(2,2)$, the linear formula $f(x_1,x_2)\approx f(x_2,x_1)$, and the linear terms $s_1=f(x_1,x_2)$ and $s_2=x_2$. Then

$$\begin{aligned}
&R^2_2(f(x_1,x_2)\approx f(x_2,x_1), f(x_1,x_2), x_2)\\
&= S^2_2(f(x_1,x_2), f(x_1,x_2), x_2)\approx S^2_2(f(x_2,x_1), f(x_1,x_2), x_2)\\
&= f(f(x_1,x_2),x_2)\approx f(x_2, f(x_1,x_2))
\end{aligned}$$

which is not linear.

Our aim is to define the condition under which the superposition of linear formulas is linear. For the proof of this condition we need

Lemma 2.4. *If F_1 and F_2 are linear n-ary formulas with $\mathrm{var}(F_1)\cap\mathrm{var}(F_2)=\varnothing$ and if $s_1,\dots,s_n$ are m-ary terms with $\mathrm{var}(s_j)\cap\mathrm{var}(s_k)=\varnothing$ for $1\leq j<k\leq n$, then*

$$\mathrm{var}(R^n_m(F_1,s_1,\dots,s_n))\cap\mathrm{var}(R^n_m(F_2,s_1,\dots,s_n))=\varnothing.$$

Proof. At first we consider atomic formulas.

(1) $F_1: s\approx t$ and $F_2: u\approx v$;

(2) $F_1: s\approx t$ and $F_2: \gamma_j(t_1,\dots,t_{n_j})$;

(3) $F_1: \gamma_k(t_1,\dots,t_{n_k})$ and $F_2: \gamma_j(t_1,\dots,t_{n_j})$.

(Because of the commutativity of $\cap$ we can skip the case $F_1: \gamma_j(t_1,\dots,t_{n_j})$ and $F_2: s\approx t$.)

(1) Since

$$\mathrm{var}(R^n_m(s\approx t, s_1,\dots,s_n)) = \mathrm{var}(S^m_n(s,s_1,\dots,s_n)\approx S^m_n(t,s_1,\dots,s_n))$$

and

$$\mathrm{var}(R^n_m(u\approx v, s_1,\dots,s_n)) = \mathrm{var}(S^m_n(u,s_1,\dots,s_n)\approx S^m_n(v,s_1,\dots,s_n))$$

because of

$$\mathrm{var}(s\approx t)\cap\mathrm{var}(u\approx v)=\varnothing,$$

$$\mathrm{var}(s_j)\cap\mathrm{var}(s_k)=\varnothing \text{ for } 1\leq j<k\leq n$$

we obtain

$$\operatorname{var}(S_n^m(s, s_1, \ldots, s_n) \approx S_n^m(t, s_1, \ldots, s_n)) \cap \operatorname{var}(S_n^m(u, s_1, \ldots, s_n) \approx S_n^m(v, s_1, \ldots, s_n)) = \varnothing.$$

(2) Since

$$\operatorname{var}(R_m^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n)) = \operatorname{var}(\gamma_j(S_n^m(t_1, s_1, \ldots, s_n)), \ldots, S_n^m(t_{n_j}, s_1, \ldots, s_n)))$$

because of

$$\operatorname{var}(s \approx t) \cap \operatorname{var}(\gamma_j(t_1, \ldots, t_{n_j}) = \varnothing,$$
$$\operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing \text{ for } 1 \leq j < k \leq n$$

the intersection

$$\operatorname{var}(S_n^m(s, s_1, \ldots, s_n) \approx S_n^m(t, s_1, \ldots, s_n)) \cap \operatorname{var}(\gamma_j(S_n^m(t_1, s_1, \ldots, s_n)), \ldots, S_n^m(t_{n_j}, s_1, \ldots, s_n)))$$

is empty.

(3) Since

$$\operatorname{var}(\gamma_k(t_1, \ldots, t_{n_k})) \cap \operatorname{var}(\gamma_j(l_1, \ldots, l_{n_j})) = \varnothing$$

because of

$$\begin{aligned}\operatorname{var}(R_m^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n)) &\cap \operatorname{var}(R_m^n(\gamma_k(l_1, \ldots, l_{n_k}), s_1, \ldots, s_n)) \\ &= \operatorname{var}(\gamma_j(S_n^m(t_1, s_1, \ldots, s_n)), \ldots, S_n^m(t_{n_j}, s_1, \ldots, s_n))) \\ &\cap \operatorname{var}(\gamma_k(S_n^m(l_1, s_1, \ldots, s_n)), \ldots, S_n^m(l_{n_k}, s_1, \ldots, s_n)))\end{aligned}$$

and $\operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing$ for $1 \leq j < k \leq n$, the intersection is empty.

Suppose now that for F_1 and F_2 the implication

$$\begin{aligned}(\operatorname{var}(F_1) \cap \operatorname{var}(F_2) = \varnothing \text{ and } \operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing \text{ for } 1 \leq j < k \leq n) \\ \Rightarrow \operatorname{var}(R_m^n(F_1, s_1, \ldots, s_n)) \cap \operatorname{var}(R_m^n(F_2, s_1, \ldots, s_n)) = \varnothing\end{aligned}$$

is satisfied. We have to prove that after applying steps (iii)–(v) to F_1 and F_2 the implication remains true for the resulting formulas. Because of $\operatorname{var}(F) = \operatorname{var}(\neg F) = \operatorname{var}(\exists x_i(F))$, $R_m^n(\neg F, s_1, \ldots, s_n) = \neg R_m^n(F, s_1, \ldots, s_n)$, and $R_m^n(\exists x_i(F), s_1, \ldots, s_n) = \exists x_i(R_m^n(F, s_1, \ldots, s_n))$ if we apply only (iii) or (v) the lemma is proved.

In the case (iv) and if $F_1 = F_{11} \vee F_{12}$, from $\operatorname{var}(F_{11} \vee F_{12}) = \operatorname{var}(F_{11}) \cup \operatorname{var}(F_{12})$ our presumption has the form $(\operatorname{var}(F_{11}) \cup \operatorname{var}(F_{12})) \cap \operatorname{var}(F_2) = \varnothing$. This gives

$$(\operatorname{var}(F_{11}) \cap \operatorname{var}(F_2)) \cup (\operatorname{var}(F_{12}) \cap \operatorname{var}(F_2)) = \varnothing$$

and then $\operatorname{var}(F_{11}) \cap \operatorname{var}(F_2) = \varnothing$ and $\operatorname{var}(F_{12}) \cap \operatorname{var}(F_2) = \varnothing$. By assumption,

$$\operatorname{var}(R_m^n(F_{11}, s_1, \ldots, s_n)) \cap \operatorname{var}(R_m^n(F_2, s_1, \ldots, s_n)) = \varnothing,$$

$$\operatorname{var}(R_m^n(F_{12}, s_1, \ldots, s_n)) \cap \operatorname{var}(R_m^n(F_2, s_1, \ldots, s_n)) = \varnothing.$$

Then the union of the left-hand sides of these equations is the empty set and then

$$(\operatorname{var}(R_m^n(F_{11}, s_1, \ldots, s_n)) \cup \operatorname{var}(R_m^n(F_{12}, s_1, \ldots, s_n))) \cap \operatorname{var}(R_m^n(F_2, s_1, \ldots, s_n)) = \varnothing$$

and by the first remark

$$\operatorname{var}(R_m^n(F_{11}, s_1, \ldots, s_n) \vee R_m^n(F_{12}, s_1, \ldots, s_n)) \cap \operatorname{var}(R_m^n(F_2, s_1, \ldots, s_n)) = \varnothing$$

and

$$\operatorname{var}(R_m^n(F_{11} \vee F_{12}, s_1, \ldots, s_n)) \cap \operatorname{var}(R_m^n(F_2, s_1, \ldots, s_n)) = \varnothing.$$

If F_2 has the form $F_2 = F_{21} \vee F_{22}$ then proceed in a similar way. If F_1 has the form $F_1 = F_{11} \vee F_{12}$ and F_2 has the form $F_2 = F_{21} \vee F_{22}$, then $\mathrm{var}(F_1) \cap \mathrm{var}(F_2) = \varnothing$ gives

$$\mathrm{var}(F_{11} \vee F_{12}) \cap \mathrm{var}(F_{21} \vee F_{22}) = (\mathrm{var}(F_{11}) \cup \mathrm{var}(F_{12})) \cap (\mathrm{var}(F_{21}) \cup \mathrm{var}(F_{22})) = \varnothing$$

and then

$$(\mathrm{var}(F_{11}) \cap (\mathrm{var}(F_{21}) \cup \mathrm{var}(F_{22}))) \cup (\mathrm{var}(F_{12}) \cap (\mathrm{var}(F_{21}) \cup \mathrm{var}(F_{22}))) = \varnothing.$$

This means

$$\mathrm{var}(F_{11}) \cap (\mathrm{var}(F_{21}) \cup \mathrm{var}(F_{22})) = \varnothing \quad \text{and} \quad \mathrm{var}(F_{12}) \cap (\mathrm{var}(F_{21}) \cup \mathrm{var}(F_{22})) = \varnothing.$$

Moreover, $\mathrm{var}(F_{11}) \cap \mathrm{var}(F_{21}) = \varnothing$, $\mathrm{var}(F_{11}) \cap \mathrm{var}(F_{22}) = \varnothing$, $\mathrm{var}(F_{12}) \cap \mathrm{var}(F_{21}) = \varnothing$, and $\mathrm{var}(F_{12}) \cap \mathrm{var}(F_{22}) = \varnothing$. Since, with respect to the use of $\vee$; F_{11} and F_{12} are of lower complexity as F_1, and F_{21} and F_{22} are of lower complexity as F_2, we have

$$\begin{aligned}
\mathrm{var}(R^n_m(F_{11}, s_1, \dots, s_n)) \cap \mathrm{var}(R^n_m(F_{21}, s_1, \dots, s_n)) = \varnothing,\\
\mathrm{var}(R^n_m(F_{11}, s_1, \dots, s_n)) \cap \mathrm{var}(R^n_m(F_{22}, s_1, \dots, s_n)) = \varnothing,\\
\mathrm{var}(R^n_m(F_{12}, s_1, \dots, s_n)) \cap \mathrm{var}(R^n_m(F_{21}, s_1, \dots, s_n)) = \varnothing,\\
\mathrm{var}(R^n_m(F_{12}, s_1, \dots, s_n)) \cap \mathrm{var}(R^n_m(F_{22}, s_1, \dots, s_n)) = \varnothing;
\end{aligned}$$

and then the union of the left-hand sides gives the empty set. Hence,

$$\begin{aligned}
&\mathrm{var}(R^n_m(F_{11}, s_1, \dots, s_n)) \vee \mathrm{var}(R^n_m(F_{12}, s_1, \dots, s_n))\\
&\cap \mathrm{var}(R^n_m(F_{21}, s_1, \dots, s_n)) \vee \mathrm{var}(R^n_m(F_{22}, s_1, \dots, s_n))\\
&= \mathrm{var}(R^n_m(F_{11} \vee F_{12}, s_1, \dots, s_n)) \cap \mathrm{var}(R^n_m(F_{21} \vee F_{22}, s_1, \dots, s_n)) = \varnothing,
\end{aligned}$$

which is what we wanted to prove. □

Theorem 2.5. *If F is an n-ary linear formula of type (τ, τ') and if $s_1, \dots, s_n$ are m-ary linear terms of type τ such that $\mathrm{var}(s_j) \cap \mathrm{var}(s_k) = \varnothing$ for $1 \le j < k \le n$, then $R^n_m(F, s_1, \dots, s_n)$ is an m-ary linear formula of type (τ, τ').*

Proof. We will give a proof by induction on the complexity of the definition of an n-ary linear formula of type (τ, τ') and start with atomic formulas.

1. Let F be an equation $t_1 \approx t_2$ with $t_1, t_2 \in W^{\mathrm{lin}}_\tau(X)$. Then $R^n_m(t_1 \approx t_2, s_1, \dots, s_n)$ is the equation $S^n_m(t_1, s_1, \dots, s_n) \approx S^n_m(t_2, s_1, \dots, s_n)$. Since $\mathrm{var}(s_j) \cap \mathrm{var}(s_k) = \varnothing$ for $1 \le j < k \le n$, by Proposition 2.2 the terms on the left-hand and the right-hand sides are linear and thus the whole equation is linear and $R^n_m(F, s_1, \dots, s_n) \in \mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_\tau(X_m))$.

2. Let F be a formula of the form $\gamma_j(t_1, \dots, t_{n_j}) \in \mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_\tau(X_m))$. Because of

$$R^n_m(\gamma_j(t_1, \dots, t_{n_j}), s_1, \dots, s_n) = \gamma_j(S^n_m(t_1, s_1, \dots, s_n), \dots, S^n_m(t_{n_j}, s_1, \dots, s_n)),$$

we show that the right-hand side is a linear formula. By Definition 2.1 this formula is linear if

(a) $S^n_m(t_k, s_1, \dots, s_n) \in W^{\mathrm{lin}}_\tau(X_n)$ for $1 \le k \le n_j$,

(b) $\mathrm{var}(S^n_m(t_k, s_1, \dots, s_n)) \cap \mathrm{var}(S^n_m(t_j, s_1, \dots, s_n)) = \varnothing$ for $1 \le k \le n_j$.

Since $t_k \in W^{\mathrm{lin}}_\tau(X_n)$ and $\mathrm{var}(s_i) \cap \mathrm{var}(s_j) = \varnothing$ for $1 \le j < k \le n$, by Proposition 2.2 the first condition is satisfied. By Definition 2.3 (ii), $\mathrm{var}(t_k) \cap \mathrm{var}(t_l) = \varnothing$. Since $\mathrm{var}(s_i) \cap \mathrm{var}(s_j) = \varnothing$ for $1 \le i < j \le n$, the second condition is also satisfied.

3. Let F, F_1, and F_2 be n-ary linear formulas of type (τ, τ'). Inductively, we assume that $R^n_m(F, s_1, \dots, s_n)$, $R^n_m(F_1, s_1, \dots, s_n)$, and $R^n_m(F_2, s_1, \dots, s_n)$ are linear. We have to prove that $R^n_m(\neg F, s_1, \dots, s_n)$, $R^n_m(\exists x_i(F), s_1, \dots, s_n)$, and $R^n_m(F_1 \vee F_2, s_1, \dots, s_n)$ are linear if $\mathrm{var}(F_1) \cap \mathrm{var}(F_2) = \varnothing$. Since

$$R^n_m(\neg F, s_1, \dots, s_n) = \neg R^n_m(F, s_1, \dots, s_n) \quad \text{and} \quad R^n_m(\exists x_i(F), s_1, \dots, s_n) = \exists x_i(R^n_m(F, s_1, \dots, s_n))$$

by Definition 2.3,

$$R^n_m(\neg F, s_1, \dots, s_n), \quad R^n_m(\exists x_i(F), s_1, \dots, s_n) \in \mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W^{\mathrm{lin}}_\tau(X_m)).$$

4. In this case $R^n_m(F_1 \vee F_2, s_1, \dots, s_n) = R^n_m(F_1, s_1, \dots, s_n) \vee R^n_m(F_2, s_1, \dots, s_n)$. From the presumptions for F_1 and F_2 by Lemma 2.4 $\mathrm{var}(R^n_m(F_1, s_1, \dots, s_n)) \cap \mathrm{var}(R^n_m(F_2, s_1, \dots, s_n)) = \varnothing$ and so $R^n_m(F_1 \vee F_2, s_1, \dots, s_n)$ is linear by Definition 2.3 (iv).

3. The Partial Clone of Linear Formulas

Using Theorem 2.5, we define the partial many-sorted mappings

$$\overline{R}^n_m : \mathscr{F}^{\text{lin}}_{(\tau,\tau')}(W^{\text{lin}}_\tau(X_n)) \times (W^{\text{lin}}_\tau(X_m))^n \multimap\!\!\!\to \mathscr{F}_{(\tau,\tau')}W^{\text{lin}}_\tau(X_m)$$

by

$$\overline{R}^n_m(F, s_1, \dots, s_n) := \begin{cases} R^n_m(F, s_1 \dots, s_n) & \text{if } \operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing \text{ for all } 1 \le j < k \le n, \\ \text{not defined} & \text{otherwise} \end{cases}$$

for $m, n \in \mathbb{N}^+$, and we define the partial many-sorted algebra

$$\text{Formclone}_{\text{lin}}(\tau,\tau') := ((\mathscr{F}^{\text{lin}}_{(\tau,\tau')}(W^{\text{lin}}_\tau(X_n)))_{n\in\mathbb{N}^+}, (W^{\text{lin}}_\tau(X_n))_{n\in\mathbb{N}^+}; (\overline{R}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\le i\le n, n\in\mathbb{N}^+}).$$

Theorem 3.1. *The partial many-sorted algebra* $\text{Formclone}_{\text{lin}}(\tau,\tau')$ *satisfies* (FC1) *and* (FC2) *as weak identities.*

Proof. We replace $\widetilde{F}$ in (FC1) by an arbitrary formula $F \in \mathscr{F}^{\text{lin}}_{(\tau,\tau')}(W^{\text{lin}}_\tau(X_m))$, $\widetilde{X}_1, \dots, \widetilde{X}_p$ by arbitrary n-ary linear terms from $W^{\text{lin}}_\tau(X_n)$ and $\widetilde{Y}_1, \dots, \widetilde{Y}_n$ by arbitrary m-ary linear terms from $W^{\text{lin}}_\tau(X_m)$ and the operation symbols by the corresponding partial operations $\overline{R}^n_m$, $\overline{R}^p_n$, $\overline{R}^p_m$, $\overline{S}^n_m$ and obtain

$$\overline{R}^n_m(\overline{R}^p_n(F, t_1, \dots, t_p), s_1, \dots, s_n) = \overline{R}^p_m(F, \overline{S}^n_m(t_1, s_1, \dots, s_n), \dots, \overline{S}^n_m(t_p, s_1, \dots, s_n)).$$

If $\operatorname{var}(s_i) \cap \operatorname{var}(s_j) = \varnothing$ for $1 \le i < j \le n$ and $\operatorname{var}(t_l) \cap \operatorname{var}(t_k) = \varnothing$ for $1 \le l < k \le p$, then the left-hand side is defined and equals $R^n_m(R^p_n(F, t_1, \dots, t_p), s_1, \dots, s_n)$. Then $\overline{S}^n_m(t_1, s_1, \dots, s_n), \dots, \overline{S}^n_m(t_p, s_1, \dots, s_n)$ are also defined and equal to $S^n_m(t_1, s_1, \dots, s_n), \dots, S^n_m(t_p, s_1, \dots, s_n)$. Moreover, $\operatorname{var}(S^n_m(t_l, s_1, \dots, s_n)) \cap \operatorname{var}(S^n_m(t_k, s_1, \dots, s_n)) = \varnothing$ since $\operatorname{var}(t_l) \cap \operatorname{var}(t_k) = \varnothing$. Therefore, the right-hand side is also defined and equal to $R^p_m(F, S^n_m(t_1, s_1, \dots, s_n), \dots, S^n_m(t_p, s_1, \dots, s_n))$. In [2] it was proved (Theorem 2.2) that both sides are equal. Thus (FC1) is a weak identity in $\text{Formclone}_{\text{lin}}(\tau,\tau')$. To check (FC2), we replace $\widetilde{F}$ in (FC2) by an arbitrary formula $F \in \mathscr{F}^{\text{lin}}_{(\tau,\tau')}W^{\text{lin}}_\tau(X_n))$, $\lambda_1, \dots, \lambda_n$ by $x_1, \dots, x_n$, and the operation symbol $\widetilde{R}^n_n$ by the partial operation $\overline{R}^n_n$ and obtain $R^n_n(F, x_1, \dots x_n) = F$ since the left-hand side is defined. As proved in [2], both sides are equal. $\square$

4. Linear Hypersubstitutions of Type (τ, τ')

Hypersubstitutions are mappings that send operation symbols to terms of the same arity: $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$. Let $\text{Hyp}(\tau)$ be the set of all those hypersubstitutions of type τ. Each hypersubstitution can be extended to a mapping $\bar{\sigma} : W_\tau(X) \to W_\tau(X)$ by

(i) $\bar{\sigma}[x_l] := x_l$ for variables;
(ii) $\bar{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}_n(\sigma(f_i), \bar{\sigma}[t_1], \dots, \bar{\sigma}[t_{n_i}])$ if $t_1, \dots, t_{n_i} \in W_\tau(X_n)$.

By applying extensions of hypersubstitutions to the terms s and t of an identity $s \approx t$ the operation symbols in s and t are replaced by terms. If the resulting equations are identities in an algebra or a class of algebras of type τ, the equation $s \approx t$ is said to be a hyperidentity. Hyperidentities are particular second order formulas. See, e.g., [8] for more background on hypersubstitutions and hyperidentities.

For every term t and every hypersubstitution σ, the set of variables occurring in $\bar{\sigma}[t]$ is a subset of the set of variables in t [9].

Lemma 4.1. $\operatorname{var}(\bar{\sigma}[t]) \subseteq \operatorname{var}(t)$ *for all* $t \in W_\tau(X)$ *and* $\sigma \in \text{Hyp}(\tau)$. $\square$

On the set $\text{Hyp}(\tau)$ of all hypersubstitutions of type τ a binary associative operation $\circ_h$ can be defined by $\sigma_1 \circ_h \sigma_2 := \bar{\sigma}_1 \circ \sigma_2$, where $\circ$ denotes the usual composition of functions. Together with the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) := f_i(x_1, \dots, x_{n_i})$ for all $i \in I$ we get the monoid $(\text{Hyp}(\tau); \circ_h, \sigma_{id})$.

If the operation symbols are sent only to linear terms, we speak of a linear hypersubstitution. In [9] was proved that the extension of a linear hypersubstitution sends $W^{\text{lin}}_\tau(X)$ to $W^{\text{lin}}_\tau(X)$ and that the set $\mathrm{Hyp}^{\mathrm{lin}}(\tau)$ of all linear hypersubstitutions of type τ forms a submonoid of the monoid of all hypersubstitutions.

The concept of hypersubstitution can be extended in the following way to the hypersubstitutions for algebraic systems of type (τ, τ'):

DEFINITION 4.2. Each pair $\sigma := (\sigma_H, \sigma_R)$ of mappings $\sigma_H : \{f_i \mid i \in I\} \to W_\tau(X)$ and $\sigma_R : \{\gamma_j \mid j \in J\} \to \mathscr{F}_{(\tau,\tau')}(W_\tau(X))$, which sends n_i-ary operation symbols to n_i-ary terms and n_j-ary relation symbols to n_j-ary formulas is a *hypersubstitution* for algebraic systems of type (τ, τ'), for short, a hypersubstitution of type (τ, τ'). Let $\mathrm{Hyprel}(\tau, \tau')$ be the set of all hypersubstitutions of type (τ, τ').

Here σ_H is a usual hypersubstitution of type τ and $\bar{\sigma}_H$ is its extension.

Using σ_H and σ_R, define $\widehat{\sigma_R} : \mathscr{F}(W_\tau(X)) \to \mathscr{F}(W_\tau(X))$ as follows:

DEFINITION 4.3. (i) $\widehat{\sigma_R}[s \approx t]$ is equal to $\widehat{\sigma_H}[s] \approx \widehat{\sigma_H}[t]$.
(ii) $\widehat{\sigma_R}[\gamma_j(t_1, \ldots, t_{n_j})]$ is the formula $R_n^{n_j}(\sigma_R(\gamma_j), \widehat{\sigma_H}[t_1], \ldots, \widehat{\sigma_H}[t_{n_j}])$, $t_1, \ldots, t_{n_j} \in W_\tau(X_n)$.
(iii) $\widehat{\sigma_R}[\neg F] := \neg\widehat{\sigma_R}[F]$.
(iv) $\widehat{\sigma_R}[F_1 \vee F_2] := \widehat{\sigma_R}[F_1] \vee \widehat{\sigma_R}[F_2]$.
(v) $\widehat{\sigma_R}[\exists x_i(F)] := \exists x_i(\widehat{\sigma_R}[F])$.

Clearly, $\widehat{\sigma_R}$ sends n-ary formulas to n-ary formulas.

The extension $\tilde{\sigma}$ of the hypersubstitution σ of type (τ, τ') is defined to be the pair $\tilde{\sigma} := (\bar{\sigma}_H, \widehat{\sigma_R})$,

$$\tilde{\sigma} : W_\tau(X) \times \mathscr{F}_{(\tau,\tau')}(W_\tau(X)) \to W_\tau(X) \times \mathscr{F}_{(\tau,\tau')}(W_\tau(X)).$$

We define the product $\circ_r$ on $\mathrm{Hyprel}(\tau, \tau')$ by

$$\sigma_1 \circ_r \sigma_2 = (\sigma_{1_H}, \sigma_{1_R}) \circ_r (\sigma_{2_H}, \sigma_{2_R}) := (\bar{\sigma}_{1_H} \circ \sigma_{2_H}, \widehat{\sigma_{1_R}} \circ \sigma_{2_R}).$$

Since $\bar{\sigma}_{1_H} \circ \sigma_{2_H}$ sends $\{f_i \mid i \in I\}$ to $W_\tau(X)$ and $\widehat{\sigma_{1_R}} \circ \sigma_{2_R}$ sends $\{\gamma_j \mid j \in J\}$ to $\mathscr{F}_{(\tau,\tau')}(W_\tau(X))$ and since arities are preserved, $\sigma_1 \circ_r \sigma_2$ is again a hypersubstitution of type (τ, τ'). Since $\circ_r$ is defined componentwise as composition of functions, it is associative. The pair $\sigma_{id} := (\sigma_{H_{id}}, \sigma_{R_{id}})$ with $\sigma_{R_{id}} : \gamma_j \to \gamma_j(x_1, \ldots, x_{n_j})$ for every $j \in J$ is the identity hypersubstitution. Clearly, $\sigma_{H_{id}}[t] = t$ for every $t \in W_\tau(X)$ (see [8]); and for the second component we prove

Lemma 4.4. $\widehat{\sigma_{R_{id}}}[F] = [F]$ *for every formula* F.

PROOF. We follow the inductive definition of formulas.
(i) If F is $s \approx t$, then

$$\widehat{\sigma_{id}}[s \approx t] = \bar{\sigma}_{H_{id}}[s] \approx \bar{\sigma}_{H_{id}}[t] = s \approx t.$$

(ii) If F is the formula $\gamma_j(t_1, \ldots, t_{n_j})$, $j \in J$, then

$$\begin{aligned}\widehat{\sigma_{R_{id}}}[F] = \widehat{\sigma_{R_{id}}}[\gamma_j(t_1, \ldots, t_{n_j})] &= R_n^{n_j}(\sigma_{R_{id}}(\gamma_j), \bar{\sigma}_{H_{id}}[t_1], \ldots, \bar{\sigma}_{H_{id}}[t_{n_j}]) \\ &= R_n^{n_j}(\gamma_j(x_1, \ldots, x_{n_j}), t_1, \ldots, t_{n_j}) \\ = \gamma_j(S_n^{n_j}(x_1, t_1, \ldots, t_{n_j}), &\ldots, S_n^{n_j}(x_{n_j}, t_1, \ldots, t_{n_j})) = \gamma_j(t_1, \ldots, t_{n_j}).\end{aligned}$$

Assume that $\widehat{\sigma_{Rid}}[F] = F$, $\widehat{\sigma_{Rid}}[F_1] = F_1$ and $\widehat{\sigma_{Rid}}[F_2] = F_2$. Then
(iii), (v) $\widehat{\sigma_{R_{id}}}[\neg F] = \neg\widehat{\sigma_{R_{id}}}[F] = \neg F$, $\widehat{\sigma_{R_{id}}}[\exists x_i(F)] = \exists x_i(\widehat{\sigma_{R_{id}}}[F]) = \exists x_i(F)$,
(iv) $\widehat{\sigma_{R_{id}}}[F_1 \vee F_2] = \widehat{\sigma_{R_{id}}}[F_1] \vee \widehat{\sigma_{R_{id}}}[F_2] = F_1 \vee F_2$. □

Therefore, $(\mathrm{Hyprel}(\tau, \tau'); \circ_r, \sigma_{id})$ is a monoid.

Our aim is to define linear hypersubstitutions of type (τ, τ'). Do the linear hypersubstitutions of type (τ, τ') form a submonoid of the monoid of all hypersubstitutions of type (τ, τ')?

DEFINITION 4.5. Let (τ, τ') be a type of formulas. Each pair $\sigma := (\sigma_H, \sigma_R)$ of the mappings $\sigma_H : \{f_i \mid i \in I\} \to W_\tau^{\mathrm{lin}}(X)$ and $\sigma_R : \{\gamma_j \mid j \in J\} \to \mathscr{F}^{\mathrm{lin}}_{(\tau,\tau')}(W_\tau^{\mathrm{lin}}(X))$ which sends n_i-ary operation symbols to n_i-ary linear terms and n_j-ary relation symbols to n_j-ary linear formulas is said to be a *linear* *hypersubstitution* for algebraic systems of type (τ, τ'), in short, a linear hypersubstitution of type (τ, τ'). Let $\mathrm{Hyprel}^{\mathrm{lin}}(\tau, \tau')$ be the set of all linear hypersubstitutions of type (τ, τ').

We know already that the extension of a usual linear hypersubstitution sends $W_\tau^{\mathrm{lin}}(X)$ to $W_\tau^{\mathrm{lin}}(X)$. We want to prove that the extension of a linear hypersubstitution of type (τ, τ') sends linear terms to linear terms and linear formulas to linear formulas. But at first we prove the useful lemma:

Lemma 4.6. *Let* $t_l, t_k \in W_\tau^{\mathrm{lin}}(X)$ *with* $\mathrm{var}(t_l) \cap \mathrm{var}(t_k) = \varnothing$. *Let* $\sigma_H \in \mathrm{Hyp}^{\mathrm{lin}}(\tau)$. *Then*

$$\mathrm{var}(\bar{\sigma}_H[t_l]) \cap \mathrm{var}(\bar{\sigma}_H[t_k]) = \varnothing.$$

Proof. By Lemma 4.1, $\mathrm{var}(t_l) \supseteq \mathrm{var}(\bar{\sigma}_H[t_l])$ and $\mathrm{var}(t_k) \supseteq \mathrm{var}(\bar{\sigma}_H[t_k])$ and so

$$\varnothing = \mathrm{var}(t_l) \cap \mathrm{var}(t_k) \supseteq \mathrm{var}(\bar{\sigma}_H[t_l]) \cap \mathrm{var}(\sigma_H[t_k]). \quad \square$$

Lemma 4.7. *Let* $\sigma = (\sigma_H, \sigma_R)$ *be a linear hypersubstitution of type* (τ, τ'). *Then* $\widehat{\sigma_R}$ *sends linear formulas to linear formulas.*

Proof. We follow the inductive definition of linear formulas.

(i) If F has the form $t_1 \approx t_2, t_1, t_2 \in W_\tau^{\mathrm{lin}}(X)$; then $\widehat{\sigma_R}(t_1 \approx t_2)$ is the equation $\bar{\sigma}_H[t_1] \approx \bar{\sigma}_H[t_2]$. By [9, Lemma 2.4], $\bar{\sigma}_H$ sends linear terms to linear terms since σ_H is a linear hypersubstitution of type τ. Therefore, $\widehat{\sigma_R}(t_1 \approx t_2)$ is a linear formula.

(ii) If F has the form $\gamma_j(t_1, \ldots, t_{n_j}), t_1, \ldots, t_{n_j} \in W_\tau^{\mathrm{lin}}(X)$ and $\mathrm{var}(t_l) \cap \mathrm{var}(t_k) = \varnothing$ for $1 \le l < k \le n_j$, then $\bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_{n_j}]$ are linear and by Lemma 4.6 $\mathrm{var}(\bar{\sigma}_H[t_l]) \cap \mathrm{var}(\bar{\sigma}_H[t_k] = \varnothing$ for $1 \le l < k \le n_j$ and by Definition 4.3 (ii) and Theorem 2.5 $\widehat{\sigma_R}[\gamma_j(t_1, \ldots, t_{n_j})]$ is linear.

Inductively, we assume that $\widehat{\sigma_R}[F]$, $\widehat{\sigma_R}[F_1]$, and $\widehat{\sigma_R}[F_2]$ are linear and $\mathrm{var}(\widehat{\sigma_R}[F_1]) \cap \mathrm{var}(\widehat{\sigma_R}[F_2]) = \varnothing$. We want to show that $\widehat{\sigma_R}[\neg F]$, $\widehat{\sigma_R}[\exists x_i(F)]$ and $\widehat{\sigma_R}[F_1 \vee F_2]$ are linear. Because of $\widehat{\sigma_R}[\neg F] = \neg \widehat{\sigma_R}[F]$ and $\widehat{\sigma_R}[\exists x_i(F)] = \exists x_i(\widehat{\sigma_R}[F])$ by Definition 2.3 (iii), (v); therefore, $\widehat{\sigma_R}[\neg F]$ and $\widehat{\sigma_R}[\exists x_i(F)]$ are linear. Since $\widehat{\sigma_R}[F_1]$ and $\widehat{\sigma_R}[F_2]$ are linear and $\mathrm{var}(\widehat{\sigma_R}[F_1]) \cap \mathrm{var}(\widehat{\sigma_R}[F_2]) = \varnothing$ by Definition 2.3 (iv); therefore, $\widehat{\sigma_R}[F_1] \vee \widehat{\sigma_R}[F_2]$ is linear. Then $\widehat{\sigma_R}[F_1 \vee F_2]$ is linear by Definition 4.3 (iv). $\square$

Theorem 4.8. *The set* $\mathrm{Hyp}^{\mathrm{lin}}(\tau, \tau')$ *of all linear hypersubstitutions of type* (τ, τ') *forms a submonoid of the monoid* $(\mathrm{Hyp}(\tau, \tau'); \circ_r, \sigma_{id})$ *of all hypersubstitutions of type* (τ, τ').

Proof. Suppose that $\sigma_1, \sigma_2 \in \mathrm{Hyp}^{\mathrm{lin}}(\tau, \tau')$. We have to show that $\sigma_1 \circ_r \sigma_2 \in \mathrm{Hyp}^{\mathrm{lin}}(\tau, \tau')$ and $\sigma_{id} \in \mathrm{Hyp}^{\mathrm{lin}}(\tau, \tau')$. Note that

$$\sigma_1 \circ_r \sigma_2 = (\sigma_{1_H}, \sigma_{1_R}) \circ_r (\sigma_{2_H}, \sigma_{2_R}) = (\bar{\sigma}_{1_H} \circ \sigma_{2_H}, \widehat{\sigma_{1_R}} \circ \sigma_{2_R}).$$

Also, $\bar{\sigma}_{1_H} \circ \sigma_{2_H} \in \mathrm{Hyp}^{\mathrm{lin}}(\tau, \tau')$ as proved in [9]. By Lemma 4.7 $\widehat{\sigma_{1_R}} \circ \sigma_{2_R} = \widehat{\sigma_{1_R}}[\sigma_{2_R}(\gamma_j)]$ is an n-ary linear formula. Since $f_i(x_1, \ldots, x_{n_i})$ is a linear term and $\gamma_j(x_1, \ldots, x_{n_i})$ is a linear formula, σ_{id} is a linear hypersubstitution of type (τ, τ'). $\square$

It is well known (see [8, 10] and elsewhere) that the extension $\bar{\sigma} : W_\tau(X) \to W_\tau(X)$ of each hypersubstitution of type τ is permutable with the superposition operations $S_m^n : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$, $m, n \in \mathbb{N}^+$; i.e.,

$$\bar{\sigma}[S_m^n(s, t_1, \ldots, t_n)] = S_m^n(\bar{\sigma}[s], \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]) \text{ for all } s \in W_\tau(X_n), t_1, \ldots, t_n \in W_\tau(X_m). \qquad (*)$$

Indeed, if $x_j \in X_n$, then $\bar{\sigma}[x_j] = x_j$, $S_m^n(x_j, t_1, \ldots, t_n) = t_j$, and $S_m^n(\bar{\sigma}[x_j], \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]) = \bar{\sigma}[S_m^n(x_j, t_1, \ldots, t_n)]$. Assume that $s = f_i(s_1, \ldots, s_{n_i}) \in W_\tau(X_n)$ and the equation is satisfied for $s_1, \ldots, s_{n_i}$. Then

$$\begin{aligned}
&\bar{\sigma}[S_m^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n)] \\
&= \bar{\sigma}[f_i(S_m^n(s_1, t_1, \ldots, t_n), \ldots, S_m^n(s_n t_1, \ldots, t_n)] \text{ by the definition of } S_m^n \\
&= S_m^{n_i}(\sigma(f_i), \bar{\sigma}[S_m^n(s_1, t_1, \ldots, t_n)], \ldots, \bar{\sigma}[S_m^n(s_{n_i}, t_1, \ldots, t_n)]) \text{ by the definition of } \bar{\sigma} \\
&= S_m^{n_i}(\sigma(f_i), S_m^n(\bar{\sigma}[s_1], \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]), \ldots, S_m^n(\bar{\sigma}[s_{n_i}], \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n])) \text{ by hypothesis} \\
&= S_m^n(S_n^{n_i}(\sigma(f_i), \bar{\sigma}[s_1], \ldots, \bar{\sigma}[s_{n_i}]), \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]) \text{ by (C1)} \\
&= S_m^n(\bar{\sigma}[f_i(s_1, \ldots, s_{n_i})], \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]) \text{ by the definition of } \bar{\sigma} \\
&= S_m^n(\bar{\sigma}[s], \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]).
\end{aligned}$$

We ask whether this remains true for the superposition operations $R^n_m : \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n)) \times W_\tau(X_m)^n \to \mathscr{F}_{(\tau,\tau')}(W_\tau(X_m))$, for the extension $\bar{\sigma}_H$ of the first component and the mapping $\hat{\sigma}$ of a hypersubstitution $\sigma = (\sigma_H, \sigma_R)$ of type (τ, τ').

Lemma 4.9. *Let $\sigma = (\sigma_H, \sigma_R)$ be a hypersubstitution of type (τ, τ'). Then*

$$\widehat{\sigma_R}[R^n_m(F, s_1, \ldots, s_n)] = R^n_m(\widehat{\sigma_R}[F], \bar{\sigma}_H[s_1], \ldots, \bar{\sigma}_H[s_n]) \tag{$**$}$$

for all $F \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1, \ldots, s_n \in W_\tau(X_n)$.

Proof. We follow the inductive definition of formulas of type (τ, τ').

(i) If F has the form $s \approx t$, then $\widehat{\sigma_R}[s \approx t]$ is $\bar{\sigma}_H[s] \approx \bar{\sigma}_H[t]$ and

$$\begin{aligned}
&\widehat{\sigma_R}[R^n_m(s \approx t, t_1, \ldots, t_n)]\\
&= \widehat{\sigma_R}[S^n_m(s, t_1, \ldots, t_n) \approx S^n_m(t, t_1, \ldots, t_n)] \text{ by the definition of } R^n_m\\
&= \bar{\sigma}_H[S^n_m(s, t_1, \ldots, t_n)] \approx \bar{\sigma}_H[S^n_m(t, t_1, \ldots, t_n)] \text{ by the definition of } \widehat{\sigma_R}\\
&= S^n_m(\bar{\sigma}_H[s], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \approx S^n_m(\bar{\sigma}_H[t], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by } (*)\\
&= R^n_m(\bar{\sigma}_H[s] \approx \bar{\sigma}_H[t], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } R^n_m\\
&= R^n_m(\widehat{\sigma_R}[s \approx t], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } \widehat{\sigma_R}.
\end{aligned}$$

(ii) Assume that F has the form $\gamma_j(s_1, \ldots, s_{n_j})$ for an n_j-ary relation symbol γ_j and n-ary terms $t_1, \ldots, t_n$. Then

$$\begin{aligned}
&\widehat{\sigma_R}[R^n_m(\gamma_j(s_1, \ldots, s_{n_j}), t_1, \ldots, t_n)]\\
&= \widehat{\sigma_R}[\gamma_j(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_j}, t_1, \ldots, t_n))] \text{ by the definition of } R^n_m\\
&= R^{n_j}_m(\sigma_R(\gamma_j), \bar{\sigma}_H[S^n_m(s_1, t_1, \ldots, t_n)], \ldots, \bar{\sigma}_H[S^n_m(s_{n_j}, t_1, \ldots, t_n)]) \text{ by the definition of } \widehat{\sigma_R}\\
&= R^{n_j}_m(\sigma_R(\gamma_j), S^n_m(\bar{\sigma}_H[s_1], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]), \ldots, S^n_m(\bar{\sigma}_H[s_{n_j}], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n])) \text{ by } (*)\\
&= R^n_m(R^{n_j}_n(\sigma_R(\gamma_j), \bar{\sigma}_H[s_1], \ldots, \bar{\sigma}_H[s_{n_j}]), \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by (FC1)}\\
&= R^n_m(\widehat{\sigma_R}(\gamma_j(s_1, \ldots, s_{n_j})), \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } \widehat{\sigma_R}.
\end{aligned}$$

(iii), (v) Assume that $(**)$ is satisfied for a formula F. Then

$$\begin{aligned}
&\widehat{\sigma_R}[R^n_m(\neg F, t_1, \ldots, t_n)]\\
&= \widehat{\sigma_R}[\neg R^n_m(F, t_1, \ldots, t_n)] \text{ by the definition of } R^n_m\\
&= \neg(\widehat{\sigma_R}[R^n_m(F, t_1, \ldots, t_n)]) \text{ by the definition of } \widehat{\sigma_R}\\
&= \neg R^n_m(\widehat{\sigma_R}[F], , \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]), \text{ by } (**)\\
&= R^n_m(\neg\widehat{\sigma_R}(F), \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } R^n_m\\
&= R^n_m(\widehat{\sigma_R}(\neg F), \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } \widehat{\sigma_R}.
\end{aligned}$$

We argue similarly for $\exists x_i(F)$.

Assume now that $(**)$ is satisfied for F_1 and F_2 and that $\operatorname{var}(F_1) \cap \operatorname{var}(F_2) = \varnothing$. Then $F_1 \vee F_2 \in \mathscr{F}_{(\tau,\tau')}(W_\tau(X))$ and

$$\begin{aligned}
&\widehat{\sigma_R}[R^n_m(F_1 \vee F_2, t_1, \ldots, t_n)]\\
&= \widehat{\sigma_R}[R^n_m(F_1, t_1, \ldots, t_n)] \vee \widehat{\sigma_R}[R^n_m(F_2, t_1, \ldots, t_n)] \text{ by the definition of } R^n_m \text{ and } \widehat{\sigma_R}\\
&= R^n_m(\widehat{\sigma_R}[F_1], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \vee R^n_m(\widehat{\sigma_R}[F_2], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by } (**)\\
&= R^n_m(\widehat{\sigma_R}[F_1] \vee \widehat{\sigma_R}[F_2], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } R^n_m\\
&= R^n_m(\widehat{\sigma_R}[F_1 \vee F_2], \bar{\sigma}_H[t_1], \ldots, \bar{\sigma}_H[t_n]) \text{ by the definition of } \widehat{\sigma_R}. \quad \square
\end{aligned}$$

The hypersubstitutions of type τ can be regarded as many-sorted functions $(\sigma_n)_{n\in\mathbb{N}^+}$ with $\sigma_n : F^n_\tau \to W_\tau(X_n)$, where $F^n_\tau := \{f_j \mid j \in I_n\}$ and $I_n \subseteq I$ is the set of all indices such that f_j is n-ary. Since the extension $\bar{\sigma}_n$ preserves the arity, it sends $W_\tau(X_n)$ to $W_\tau(X_n)$ and $(\sigma_n)_{n\in\mathbb{N}^+}$ sends $(W_\tau(X_n))_{n\in\mathbb{N}^+}$, as a many-sorted mapping, to $(W_\tau(X_n))_{n\in\mathbb{N}^+}$. Equation $(*)$ becomes the sequence of equations

$$\bar{\sigma}_m[S^n_m(s, t_1, \ldots, t_n)] = S^n_m(\bar{\sigma}_n[s], \bar{\sigma}_m[t_1], \ldots, \bar{\sigma}_m[t_n])$$

for all $m, n \in \mathbb{N}^+$, $s \in W_\tau(X_n)$, $t_1, \ldots, t_n \in W_\tau(X_m)$. This means that $(\bar{\sigma}_n)_{n\in\mathbb{N}^+}$ is a many-sorted endomorphism of clone $\tau = ((W_\tau(X_n))_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (x^n_i)_{i\le n, n\in\mathbb{N}^+})$.

In a similar way, the extension of a hypersubstitution of type (τ, τ') can be regarded as a pair of sequences $((\bar{\sigma}_{H_n})_{n\in\mathbb{N}^+}, (\widehat{\sigma_{R_n}})_{n\in\mathbb{N}^+})$.

Equation $(**)$ becomes the sequence of equations

$$\widehat{\sigma_{R_n}}[R^n_m(F, t_1, \ldots, t_n)] = R^n_m(\widehat{\sigma_{R_n}}[F], \bar{\sigma}_{H_m}[t_1], \ldots, \bar{\sigma}_{H_m}[t_n])$$

for all $m, n \in \mathbb{N}^+$, $F \in \mathscr{F}_{(\tau\tau')}(W_\tau(X_m))$, $t_1, \ldots, t_n \in W_\tau(X_m)$. This means that $((\bar{\sigma}_{H_n})_{n\in\mathbb{N}^+}, (\widehat{\sigma_{R_n}})_{n\in\mathbb{N}^+})$ is a many-sorted endomorphism of the many-sorted algebra

$$\text{Formclone}(\tau, \tau') := ((\mathscr{F}_{(\tau,\tau')}(W_\tau(X_n)))_{n\in\mathbb{N}^+}, (W_\tau(X_n))_{n\in\mathbb{N}^+};$$
$$(R^n_m)_{m,n\in\mathbb{N}^+}, (S^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n, n\in\mathbb{N}^+}).$$

In Section 2 we mentioned that the linear terms form the partial many-sorted algebra

$$\text{clone}_{\text{lin}}\, \tau := ((W^{\text{lin}}_\tau(X_n))_{n\in\mathbb{N}^+}, (\overline{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n, n\in\mathbb{N}^+})$$

and in Section 3 we proved that the linear formulas and the linear terms form a partial many-sorted algebra

$$\text{Formclone}_{\text{lin}}(\tau, \tau') := ((\mathscr{F}^{\text{lin}}_{(\tau,\tau')}(W^{\text{lin}}_\tau(X_n)))_{n\in\mathbb{N}^+}, (W^{\text{lin}}_\tau(X_n))_{n\in\mathbb{N}^+};$$
$$(\overline{R}^n_m)_{m,n\in\mathbb{N}^+}, (\overline{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\le i\le n, n\in\mathbb{N}^+}).$$

If $\mathscr{A} = (A; (f^A_i)_{i\in I})$ is a partial algebra with partial operations $f^A_i : A^n \multimap A$, then $h : A \to A$ is said to be a weak endomorphism if for all $i \in I$ we have

$$\text{if } (a_1, \ldots, a_{n_i}) \in \operatorname{dom} f^A_i, \text{ then } (h(a_1), \ldots, h(a_{n_i})) \in \operatorname{dom} f^A_i \text{ and}$$
$$h(f^A_i(a_1, \ldots, a_{n_i})) = f^A_i(h(a_1), \ldots, h(a_{n_i})).$$

In [5] it was proved that the extension of any linear hypersubstitution is a weak endomorphism of

$$\text{clone}_{\text{lin}}\tau := ((W^{\text{lin}}_\tau(X_n))_{n\in\mathbb{N}^+}, (\overline{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n, n\in\mathbb{N}^+}).$$

We prove now that this is true for linear hypersubstitutions of type (τ, τ') and $\text{Formclone}_{\text{lin}}(\tau\tau')$.

Theorem 4.10. *The extension of each linear hypersubstitution of type (τ, τ') is a weak endomorphism of*

$$\text{Formclone}_{\text{lin}}(\tau, \tau') := ((\mathscr{F}^{\text{lin}}_{(\tau,\tau')}(W^{\text{lin}}_\tau(X_n)))_{n\in\mathbb{N}^+}, (W^{\text{lin}}_\tau(X_n))_{n\in\mathbb{N}^+};$$
$$(\overline{R}^n_m)_{m,n\in\mathbb{N}^+}, (\overline{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\le i\le n, n\in\mathbb{N}^+}).$$

Proof. We show that the many-sorted mapping $\tilde{\sigma} = ((\bar{\sigma}_{H_n})_{n\in\mathbb{N}^+}, (\widehat{\sigma_{R_n}})_{n\in\mathbb{N}^+})$ with

$$\widehat{\sigma_{R_n}} : \mathscr{F}^{\text{lin}}_{(\tau,\tau')}(W^{\text{lin}}_\tau(X_n)) \to \mathscr{F}^{\text{lin}}_{(\tau,\tau')}\big(W^{\text{lin}}_\tau(X_n)\big)$$

and $\bar{\sigma}_{H_n} : W^{\text{lin}}_\tau(X_n) \to W^{\text{lin}}_\tau(X_n)$ is a weak endomorphism of $\text{Formclone}_{\text{lin}}(\tau, \tau')$, i.e. we have to show that if $(F, s_1, \ldots, s_n) \in \operatorname{dom} \overline{R}^n_m$, then $(\widehat{\sigma_{R_m}}[F], \bar{\sigma}_{H_m}[s_1], \ldots, \bar{\sigma}_{H_m}[s_n]) \in \operatorname{dom} \overline{R}^n_m$ and then

$$\widehat{\sigma_{R_n}}\big[\overline{R}^n_m(F, s_1, \ldots, s_n)\big] = \overline{R}^n_m(\widehat{\sigma_{R_n}}[F], \bar{\sigma}_{H_m}[t_1], \ldots, \bar{\sigma}_{H_m}[t_n])$$

for all $m, n \in \mathbb{N}^+$, $F \in \mathscr{F}^{\text{lin}}_{(\tau\tau')}\big(W^{\text{lin}}_\tau(X_m)\big)$, and $t_1, \ldots, t_n \in W^{\text{lin}}_\tau(X_m)$. If $(F, s_1, \ldots, s_n) \in \operatorname{dom} \overline{R}^n_m$, then F is a linear formula, $s_1, \ldots, s_n$ are linear terms and $\operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing$ for all $1 \le j < k \le n$. By Lemma 4.6 we have $\operatorname{var}(\bar{\sigma}_{H_n})[s_j] \cap \operatorname{var}(\bar{\sigma}_{H_n})[s_k] = \varnothing$ and since by Lemma 4.7, $\widehat{\sigma_{R_m}}[F]$ is linear, by Theorem 2.5 $(\widehat{\sigma_{R_m}}[F], \bar{\sigma}_{H_m}[s_1], \ldots, \bar{\sigma}_{H_m}[s_n]) \in \operatorname{dom} \overline{R}^n_m$. By the definition of $\overline{R}^n_m$ and $(**)$ both sides are equal. $\square$

5. Final Remarks and Conclusions

Starting with the concept of linear term we defined linear formulas. We gave an answer to the question under which condition the set of all linear formulas is closed under superposition of formulas. This leads to the partial many-sorted algebra $\mathrm{Formclone}_{\mathrm{lin}}(\tau, \tau')$. This algebra satisfies (FC1) and (FC2) and moreover (C1)–(C3) as weak identities. In Section 4 we introduced the concept of linear hypersubstitution of type (τ, τ') and their extensions, proved that they form a monoid under composition and that their extensions are weak endomorphisms of $\mathrm{Formclone}_{\mathrm{lin}}(\tau, \tau')$.

There are several open problems and suggestions for the further research:

1. Is each weak endomorphism of $\mathrm{Formclone}_{\mathrm{lin}}(\tau, \tau')$ an extension of a linear hypersubstitution of type (τ, τ')?

2. Study the semigroup-theoretical properties of monoids of linear hypersubstitutions of type (τ, τ'). What is the order of elements for particular types (τ, τ')? What about regular elements and regular submonoids? Study the Green's relations:

3. Try another definition of linear formulas. Study strong linear formulas. Under which conditions is the set of all strong linear formulas closed under superposition of formulas? Define the partial many-sorted algebra $\mathrm{Formclone}_{\mathrm{stronglin}}(\tau, \tau')$ and study its properties. Do the strong linear hypersubstitutions of type (τ, τ') form a monoid?

4. Use some different definitions of terms, for instance generalized terms, to define formulas and linear formulas. Which are the differences?

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