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# Positive solutions of fractional $p$ -Laplacian equations with integral boundary value and two parameters

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## Abstract

We consider a class of Caputo fractional  $p$ -Laplacian differential equations with integral boundary conditions which involve two parameters. By using the Avery–Peterson fixed point theorem, we obtain the existence of positive solutions for the boundary value problem. As an application, we present an example to illustrate our main result.

**Keywords:** Caputo fractional derivative; Positive solutions;  $p$ -Laplacian operator; Integral boundary conditions; Parameter

## 1 Introduction

In this paper, we investigate the following integral boundary value problem (short for BVP) of Caputo fractional differential equations with  $p$ -Laplacian operator and parameters:

$$\begin{cases} D_{0+}^{\beta} \varphi_p(D_{0+}^{\alpha} x(t)) + f(t, x(t), D_{0+}^{\beta} x(t)) = 0, & t \in (0, 1), \\ (\varphi_p(D_{0+}^{\alpha} x(0)))^{(i)} = \varphi_p(D_{0+}^{\alpha} x(1)) = 0, & i = 1, 2, \dots, m-1, \\ x(0) + x'(0) = \int_0^1 g_0(s)x(s) ds + a, \\ x(1) + x'(1) = \int_0^1 g_1(s)x(s) ds + b, \\ x^{(j)}(0) = 0, & j = 2, 3, \dots, n-1, \end{cases} \quad (1)$$

where  $1 < n-1 < \alpha < n$ ,  $1 < m-1 < \beta < m$ ,  $\alpha - \beta > 1$ ,  $D_{0+}^{\alpha}$  and  $D_{0+}^{\beta}$  are the Caputo fractional derivatives.  $\varphi_p$  is the  $p$ -Laplacian operator,  $\varphi_p(s) = |s|^{p-2}s$ ,  $p > 1$ ,  $\varphi_p^{-1} = \varphi_q$ ,  $1/p + 1/q = 1$ .  $g_0, g_1 \in C([0, 1], [0, +\infty))$ ,  $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$  are given functions.  $a, b > 0$  are disturbance parameters.

As we all know, fractional differential equation theory is becoming more and more perfect because of its extensive application, and many significant achievements have been made; see [1–12]. As one of many applications, turbulence problem can be well characterized by the  $p$ -Laplacian operator; see [13]. Fractional  $p$ -Laplacian equations are becoming more and more important, they can be used to describe a class of diffusion phenomena, which have been widely used in the fields of fluid mechanics, material memory, biology, plasma physics, finance and chemistry. Many important results related to the boundary

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value problems of fractional differential equations with  $p$ -Laplacian operator have been obtained; see [14–24]. But in practical problems, disturbance is objective. As a boundary value problem with disturbance parameter can describe real problems better, many scholars turn their attention to it.

In [6], Jia et al. consider the fractional-order differential equation integral boundary value problem with disturbance parameters

$$\begin{cases} -{}^C D^\delta u(t) = f(t, u(t)), & t \in J, \\ m_1 u(0) - n_1 u'(0) = 0, \\ m_2 u(1) - n_2 u'(1) = \int_0^1 g(s)u(s)ds + a, \end{cases}$$

where  $J = [0, 1]$ ,  $1 < \delta \leq 2$ ,  $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ ,  $m_i \geq 0$ ,  $n_i \geq 0$ ,  $m_i^2 + n_i^2 > 0$ ,  $i = 1, 2$ ,  $g \in C([0, 1], [0, +\infty))$ , disturbance parameter  $a > 0$ , and  ${}^C D^\delta$  is the Caputo fractional derivative of order  $\delta$ . By using an upper and lower solution method, the fixed point index theorem and the Schauder fixed point theorem, sufficient conditions are obtained for the problem to have at least one positive solution, two positive solutions and no solution.

In [25], Wang et al. consider a class of fractional differential equations with integral boundary conditions which involve two disturbance parameters. By using the Guo–Krasnoselskii fixed point theorem, new results on the existence and nonexistence of positive solutions for the boundary value problem are obtained. The problem is given by

$$\begin{cases} D_{0+}^\alpha x(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = x'(0) = 0, \\ x(1) = \int_0^1 g_1(s)x(s)ds + a, \\ x'(1) = \int_0^1 g_2(s)x(s)ds - b, \end{cases}$$

where  $D_{0+}^\alpha$  is the standard Riemann–Liouville fractional derivative with  $3 < \alpha \leq 4$ ,  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is a continuous function,  $g_1, g_2 \in L^1[0, 1]$  and  $a, b \geq 0$ .

In [26] Hao et al. consider the existence of positive solutions of higher order fractional integral boundary value problem with a parameter

$$\begin{cases} -D_{0+}^{\eta-2}(u''(t)) + \lambda f(t, u(t)) = 0, & t \in J, \\ u''(0) = u'''(0) = \dots = u^{n-2}(0) = 0, \\ {}^C D_{0+}^{k-2}(u''(t))|_{t=1} = 0, \\ \alpha u(0) - \beta u'(0) = \int_0^1 u(s)dA(s), \\ \gamma u(1) + \delta u'(1) = \int_0^1 u(s)dB(s), \end{cases}$$

where  $D_{0+}^{\eta-2}$ ,  $D_{0+}^{k-2}$  are the standard Riemann–Liouville fractional derivative,  $n-1 < \eta \leq n$ ,  $\eta \geq 4$ ,  $2 \leq k \leq n-2$ ,  $\alpha, \beta, \gamma, \delta > 0$ .  $\int_0^1 u(s)dA(s)$  and  $\int_0^1 u(s)dB(s)$  denote the Riemann–Stieltjes integrals of  $u$  with respect to  $A$  and  $B$ .  $A(t)$ ,  $B(t)$  are nondecreasing on  $[0, 1]$ ,  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous,  $\lambda > 0$  is a parameter. By using the Guo–Krasnoselskii fixed point theorem on cones, under different conditions of nonlinearity, existence and nonexistence, results for positive solutions are derived in terms of different parameter intervals.

The purpose of this paper is to establish conditions ensuring the existence of three positive solutions of BVP (1) and give an estimate of these solutions by using the Avery–

Peterson fixed point theorem. Our supposed problem is different from the problems studied before and mentioned above. Our result is new and our work extends the application of the theorem.

In this paper, a positive solution  $x = x(t)$  of BVP (1) means a solution of (1) satisfying  $x(t) > 0$ ,  $t \in [0, 1]$ .

Throughout this paper, we always assume that the following condition is satisfied:

$$(L_0) \quad 0 < a < b < 2a < +\infty, \quad 0 \leq g_0(t) \leq g_1(t) \leq 2g_0(t), \quad 0 \leq \int_0^1 g_0(s) ds, \int_0^1 g_1(s) ds < 1.$$

## 2 Preliminaries and lemmas

The basic theory of fractional-order differential equation and boundary value problem can be obtained from many places in the literature, which will not be repeated here; see [1–9]. Here we present some necessary basic results that will be used.

**Lemma 2.1** (see [2]) *The Caputo fractional derivative of order  $n - 1 < \alpha < n$  for  $t^\beta$  is given by*

$$D_{0+}^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta \in N \text{ and } \beta \geq n \text{ or } \beta \notin N \text{ and } \beta > n-1, \\ 0, & \beta \in \{0, 1, \dots, n-1\}. \end{cases}$$

**Lemma 2.2** (see [19]) *Let  $h \in C[0, 1]$  and  $1 < m - 1 < \beta < m$ . Then the BVP*

$$\begin{cases} D_{0+}^\beta u(t) = h(t), & 0 < t < 1, \\ u(1) = u^{(i)}(0) = 0, & i = 1, 2, \dots, m-1, \end{cases}$$

*has an unique solution*

$$u(t) = - \int_0^1 H(t, s) h(s) ds,$$

*where*

$$H(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} (1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2)$$

Denote

$$\begin{aligned} M_1 &= \int_0^1 g_0(s) ds, & M_2 &= \int_0^1 s g_0(s) ds, \\ N_1 &= \int_0^1 g_1(s) ds, & N_2 &= \int_0^1 s g_1(s) ds, \\ \delta^{-1} &= 1 + M_2 + N_1 - M_2 N_1 + M_1(N_2 - 2) - N_2, \\ \omega(t) &= \delta(b(M_2 - 1 + t - M_1 t) + a(2 - N_2 - t + N_1 t)). \end{aligned} \quad (3)$$

From  $(L_0)$ , we know, for  $t \in (0, 1)$ ,

$$1 > N_1 > N_2 > M_2 > 0, \quad 1 > N_1 > M_1 > M_2 > 0, \quad 2M_1 > N_1, 2M_2 > N_2. \quad (4)$$

Thus

$$\begin{aligned}
 \delta^{-1} &= 1 + M_2 + N_1 - M_2 N_1 + M_1(N_2 - 2) - N_2 \\
 &= 1 + M_2 + N_1 - M_2 N_1 + M_1 N_2 - 2M_1 - N_2 \\
 &= (1 + M_1 N_2 - 2M_1) + M_2 + N_1 - M_2 N_1 - N_2 \\
 &= (1 - M_1) + (M_1 N_2 - N_2) + (M_2 - M_2 N_1) + (N_1 - M_1) \\
 &= (1 - M_1) + N_2(M_1 - 1) + M_2(1 - N_1) + (N_1 - M_1) \\
 &= (1 - M_1)(1 - N_2) + M_2(1 - N_1) + (N_1 - M_1) \\
 &> 0.
 \end{aligned}$$

Thus, the following lemma holds.

**Lemma 2.3** *Let  $(L_0)$  hold,  $y \in C[0, 1]$  and  $1 < n - 1 < \alpha < n$ ,  $1 < m - 1 < \beta < m$ , then the following boundary value problem:*

$$\begin{cases} D_{0+}^{\alpha} x(t) = y(t), & 0 < t < 1, \\ x^{(j)}(0) = 0, & j = 2, 3, \dots, n-1, \\ x(0) + x'(0) = \int_0^1 g_0(s)x(s) ds + a, \\ x(1) + x'(1) = \int_0^1 g_1(s)x(s) ds + b, \end{cases} \quad (5)$$

has an unique solution

$$x(t) = \int_0^1 G(t, s)y(s) ds + \omega(t) \quad (6)$$

and

$$D_{0+}^{\beta} x(t) = \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} y(s) ds, \quad (7)$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \quad (8)$$

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t - s)^{\alpha - 1} + (1 - t)(1 - s)^{\alpha - 2}(\alpha - s), & 0 \leq s \leq t \leq 1, \\ (1 - t)(1 - s)^{\alpha - 2}(\alpha - s), & 0 \leq t \leq s \leq 1, \end{cases} \quad (9)$$

$$G_2(t, s) = \delta \int_0^1 ((M_2 - 1 + t - M_1 t)g_1(\tau) + (2 - N_2 - t + N_1 t)g_0(\tau)) G_1(\tau, s) d\tau. \quad (10)$$

*Proof* Consider BVP (5), we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds + C_0 + C_1 t + C_2 t^2 + \dots + C_{n-1} t^{n-1}.$$

In view of  $x^{(j)}(0) = 0$  ( $j = 2, 3, \dots, n-1$ ), we know  $C_2 = C_3 = \dots = C_{n-1} = 0$ , and

$$\begin{aligned}x(0) &= C_0, & x'(0) &= C_1, & x(1) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + C_0 + C_1, \\x'(1) &= \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds + C_1,\end{aligned}$$

so that

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + C_0 + C_1 t. \quad (11)$$

From the boundary condition of BVP (5), by methods similar to Lemma 2.4 in [19], through traditional analytical calculation and integration techniques, we have

$$\begin{aligned}x(t) &= \int_0^1 G_1(t, s) y(s) ds + \int_0^1 G_2(t, s) y(s) ds + \omega(t) \\&= \int_0^1 G(t, s) y(s) ds + \omega(t),\end{aligned}$$

where  $\omega(t)$ ,  $G_1(t, s)$  and  $G_2(t, s)$  are given by (3), (9) and (10).

On the other hand, in view of (11), because  $1 < m-1 < \beta < \alpha-1 < n-1$ , by Lemma 2.1, we have

$$\begin{aligned}D_{0+}^\beta x(t) &= D_{0+}^\beta \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + C_0 + C_1 t \right) \\&= D_{0+}^\beta (I_{0+}^\alpha y(t) + C_0 + C_1 t) \\&= D_{0+}^\beta I_{0+}^\alpha y(t) + D_{0+}^\beta (C_0) + D_{0+}^\beta (C_1 t) \\&= D_{0+}^\beta I_{0+}^\alpha y(t) \\&= I_{0+}^{\alpha-\beta} y(t) \\&= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds.\end{aligned} \quad (12)$$

□

**Lemma 2.4** *The BVP (1) is equivalent to the following integral equation:*

$$x(t) = \int_0^1 G(t, s) \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^\beta x(\tau)) d\tau \right) ds + \omega(t) \quad (13)$$

and

$$D_{0+}^\beta x(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^\beta x(\tau)) d\tau \right) ds, \quad (14)$$

where  $H(t, s)$ ,  $\omega(t)$  and  $G(t, s)$  are given by (2), (3) and (8).

*Proof* From Lemma 2.2 and Lemma 2.3, let  $y(t) = \varphi_q(u(t))$ ,  $h(t) = -f(t, x(t), D_{0+}^\beta x(t))$ , we have

$$\begin{aligned} y(t) &= \varphi_q(u(t)) = \varphi_q\left(\int_0^1 H(t, s)f(s, x(s), D_{0+}^\beta x(s)) ds\right) \\ &= \varphi_q\left(\int_0^1 H(t, s)f(s, x(s), D_{0+}^\beta x(s)) ds\right). \end{aligned}$$

Immediately we obtain

$$\begin{aligned} x(t) &= \int_0^1 G(t, s)y(s) ds + \omega(t) \\ &= \int_0^1 G(t, s)\varphi_q\left(\int_0^1 H(s, \tau)f(\tau, x(\tau), D_{0+}^\beta x(\tau)) d\tau\right) ds + \omega(t). \end{aligned}$$

From (12), (14) holds.

On the other hand, if  $x(t)$  satisfies (13), we can easily prove that  $x(t)$  satisfies BVP (1).  $\square$

**Lemma 2.5** Assume  $(L_0)$  hold, then the function  $H(t, s)$  defined by (2), the function  $G(t, s)$  defined by (8), and then the function  $\omega(t)$  defined by (3) satisfies

- (1)  $H(t, s) \geq 0$  is continuous for all  $t, s \in [0, 1]$ ;
- (2)  $H(t, s) \leq H(s, s)$  for all  $t, s \in [0, 1]$ ;
- (3)  $\int_0^1 H(t, s) ds = \frac{1-t^\beta}{\Gamma(\beta+1)} \leq \frac{1}{\Gamma(\beta+1)}$  for all  $t \in [0, 1]$ ;
- (4)  $G(t, s) \geq 0$  is continuous for all  $t, s \in [0, 1]$ ;
- (5)  $\omega(t) > 0$  for all  $t \in [0, 1]$ .

*Proof* (1) and (2) are proved in [19], we omit the proofs.

(3) For  $t \in [0, 1]$ , by a simple integral operation, we can obtain

$$\begin{aligned} \int_0^1 H(t, s) ds &= \int_0^t ((1-s)^{\beta-1} - (t-s)^{\beta-1}) ds + \int_t^1 (1-s)^{\beta-1} ds \\ &= \frac{1-t^\beta}{\Gamma(\beta+1)} \leq \frac{1}{\Gamma(\beta+1)}. \end{aligned}$$

(4) From (9), we know  $G_1(t, s) \geq 0$ ,  $t, s \in [0, 1]$ , and  $G_1(t, s) > 0$ ,  $t, s \in (0, 1)$ . Combined with  $(L_0)$ , for  $t \in [0, 1]$ , we have

$$\begin{aligned} \frac{\partial G_2(t, s)}{\partial t} &= \delta \int_0^1 ((1-M_1)g_1(\tau) + (-1+N_1)g_0(\tau))G_1(\tau, s) d\tau \\ &= \delta \int_0^1 (g_1(\tau) - M_1g_1(\tau) - g_0(\tau) + N_1g_0(\tau))G_1(\tau, s) d\tau \\ &= \delta \int_0^1 (g_1(\tau) - g_0(\tau) - M_1g_1(\tau) + N_1g_0(\tau))G_1(\tau, s) d\tau \\ &> \delta \int_0^1 (g_1(\tau) - g_0(\tau) - M_1g_1(\tau) + M_1g_0(\tau))G_1(\tau, s) d\tau \\ &= \delta \int_0^1 (g_1(\tau) - g_0(\tau) - M_1(g_1(\tau) - g_0(\tau)))G_1(\tau, s) d\tau \end{aligned}$$

$$\begin{aligned}
&= \delta \int_0^1 (g_1(\tau) - g_0(\tau))(1 - M_1)G_1(\tau, s) d\tau \\
&> 0,
\end{aligned}$$

so that  $G_2(t, s)$  monotonically increase with respect to  $t$ .

As a consequence, from (10), we get

$$\begin{aligned}
G_2(t, s) &\geq G_2(0, s) = \delta \int_0^1 ((M_2 - 1)g_1(\tau) + (2 - N_2)g_0(\tau))G_1(\tau, s) d\tau \\
&\geq \delta \int_0^1 \left( (M_2 - 1)g_1(\tau) + \frac{1}{2}(2 - N_2)g_1(\tau) \right) G_1(\tau, s) d\tau \\
&= \delta \int_0^1 \left( M_2 - \frac{1}{2}N_2 \right) g_1(\tau) G_1(\tau, s) d\tau \\
&\geq 0.
\end{aligned}$$

Hence,  $G(t, s) \geq 0$ .

(5) From (3), for  $t \in [0, 1]$ , we know

$$\begin{aligned}
\omega'(t) &= b(1 - M_1) + a(N_1 - 1) \\
&> a(1 - M_1) + a(N_1 - 1) \\
&> a(1 - N_1) + a(N_1 - 1) \\
&= 0,
\end{aligned}$$

so that

$$\begin{aligned}
\omega(t) &\geq \omega(0) = b(M_2 - 1) + a(2 - N_2) \\
&> b(M_2 - 1) + a(2 - 2M_2) \\
&= (1 - M_2)(2a - b) \\
&> 0, \quad t \in [0, 1].
\end{aligned}$$

□

**Lemma 2.6** Let  $\eta \in (0, \frac{1}{2})$ , then

$$\begin{aligned}
\max_{t \in [0, 1]} G(t, s) &\leq G_1(0, s) + G_2(1, s), \\
\min_{t \in [0, \eta]} G(t, s) &\geq \rho(G_1(0, s) + G_2(1, s)),
\end{aligned}$$

where

$$\rho = \frac{M_2 - \frac{1}{2}N_2}{1 - M_1 + M_2 + N_1 - N_2}. \quad (15)$$

*Proof Step 1:* We prove

$$\min_{t \in [0, \eta]} G_1(t, s) \geq (1 - \eta)G_1(0, s) > \frac{1}{2} \max_{t \in [0, 1]} G_1(t, s). \quad (16)$$

For  $0 \leq s < t \leq 1$  and  $t \in [0, \eta]$ ,

$$\Gamma(\alpha) \frac{\partial G_1(t, s)}{\partial t} = (\alpha - 1)(t - s)^{\alpha-2} - (\alpha - s)(1 - s)^{\alpha-2} \leq (\alpha - s)(t - s)^{\alpha-2} - (\alpha - s)(1 - s)^{\alpha-2} < 0,$$

so that

$$G_1(t, s) \leq G_1(s, s) = (\alpha - s)(1 - s)^{\alpha-1} < (\alpha - s)(1 - s)^{\alpha-2} = G_1(0, s)$$

and

$$\begin{aligned} \frac{G_1(t, s)}{G_1(0, s)} &= \frac{(t - s)^{\alpha-1} + (\alpha - s)(1 - t)(1 - s)^{\alpha-2}}{(\alpha - s)(1 - s)^{\alpha-2}} \\ &= \frac{(t - s)^{\alpha-1}}{(\alpha - s)(1 - s)^{\alpha-2}} + 1 - t \\ &\geq 1 - t \\ &\geq 1 - \eta > \frac{1}{2}. \end{aligned}$$

For  $s \geq t$  and  $t \in [0, \eta]$ ,

$$\Gamma(\alpha) \frac{\partial G_1(t, s)}{\partial t} = -(\alpha - s)(1 - s)^{\alpha-2} < 0$$

so that

$$G_1(t, s) \leq G_1(0, s) = (\alpha - s)(1 - s)^{\alpha-2}$$

and

$$\frac{G_1(t, s)}{G_1(0, s)} = \frac{(\alpha - s)(1 - t)(1 - s)^{\alpha-2}}{(\alpha - s)(1 - s)^{\alpha-2}} = 1 - t \geq 1 - \eta > \frac{1}{2}.$$

Therefore, (16) holds.

*Step 2:* We prove

$$\min_{t \in [0, \eta]} G_2(t, s) \geq \rho G_2(1, s) = \rho \max_{t \in [0, 1]} G_2(t, s). \quad (17)$$

From Lemma 2.5, we know that  $G_2(t, s)$  is a monotone increasing function with respect to  $t \in [0, 1]$ , so that

$$\min_{t \in [0, \eta]} G_2(t, s) = G_2(0, s), \quad \max_{t \in [0, 1]} G_2(t, s) = G_2(1, s).$$

By (L<sub>0</sub>) and (4), we have

$$\begin{aligned} \frac{G_2(0, s)}{G_2(1, s)} &= \frac{\delta \int_0^1 ((M_2 - 1)g_1(\tau) + (2 - N_2)g_0(\tau))G_1(\tau, s) d\tau}{\delta \int_0^1 ((M_2 - 1 + 1 - M_1)g_1(\tau) + (2 - N_2 - 1 + N_1)g_0(\tau))G_1(\tau, s) d\tau} \\ &\geq \frac{\int_0^1 ((M_2 - 1)g_1(\tau) + \frac{1}{2}(2 - N_2)g_1(\tau))G_1(\tau, s) d\tau}{\int_0^1 ((M_2 - M_1)g_1(\tau) + (1 - N_2 + N_1)g_1(\tau))G_1(\tau, s) d\tau} \end{aligned}$$



$$\begin{aligned}
&= \frac{\int_0^1 (M_2 - 1 + 1 - \frac{1}{2}N_2)g_1(\tau)G_1(\tau, s) d\tau}{\int_0^1 (M_2 - M_1 + 1 - N_2 + N_1)g_1(\tau)G_1(\tau, s) d\tau} \\
&= \frac{(M_2 - \frac{1}{2}N_2) \int_0^1 g_1(\tau)G_1(\tau, s) d\tau}{(1 - M_1 + M_2 + N_1 - N_2) \int_0^1 g_1(\tau)G_1(\tau, s) d\tau} \\
&= \frac{M_2 - \frac{1}{2}N_2}{1 - M_1 + M_2 + N_1 - N_2} \\
&= \rho.
\end{aligned}$$

Obviously,  $\rho > 0$  and

$$\rho - \frac{1}{2} = \frac{M_2 - \frac{1}{2}N_2}{1 - M_1 + M_2 + N_1 - N_2} - \frac{1}{2} = \frac{M_2 - 1 + M_1 - N_1}{2(1 - M_1 + M_2 + N_1 - N_2)} < 0,$$

so that  $0 < \rho < \frac{1}{2}$  and (17) hold.

Finally, from (16) and (17), we can easily show that the following results hold:

$$\begin{aligned}
\max_{t \in [0,1]} G(t, s) &= \max_{t \in [0,1]} (G_1(t, s) + G_2(t, s)) \\
&\leq \max_{t \in [0,1]} G_1(t, s) + \max_{t \in [0,1]} G_2(t, s) \\
&= G_1(0, s) + G_2(1, s)
\end{aligned}$$

and

$$\begin{aligned}
\min_{t \in [0,\eta]} G(t, s) &\geq \min_{t \in [0,\eta]} G_1(t, s) + \min_{t \in [0,\eta]} G_2(t, s) \\
&\geq \frac{1}{2} G_1(0, s) + \rho G_2(1, s) \\
&> \rho G_1(0, s) + \rho G_2(1, s) \\
&= \rho (G_1(0, s) + G_2(1, s)) \\
&\geq \rho \max_{t \in [0,1]} G(t, s).
\end{aligned}$$

□

**Lemma 2.7** Assume  $(L_0)$  hold, then the function  $\omega(t)$  satisfies the following properties:

- (1)  $\omega(t) \leq \omega(1) = \max_{t \in [0,1]} \omega(t)$ ;
- (2)  $\min_{t \in [0,\eta]} \omega(t) \geq \rho \max_{t \in [0,1]} \omega(t)$ , where  $\rho$  is given by (15).

*Proof* From Lemma 2.5 and (3), we have

$$\begin{aligned}
\min_{t \in [0,\eta]} \omega(t) &= \omega(0) = \delta(b(M_2 - 1) + a(2 - N_2)), \\
\max_{t \in [0,1]} \omega(t) &= \omega(1) = \delta(b(M_2 - M_1) + a(1 - N_2 + N_1)),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\omega(0)}{\omega(1)} &= \frac{\delta(b(M_2 - 1) + a(2 - N_2))}{\delta(b(M_2 - M_1) + a(1 - N_2 + N_1))} \\
&\geq \frac{b(M_2 - 1) + \frac{1}{2}b(2 - N_2)}{b(M_2 - M_1) + b(2 - N_2 - 1 + N_1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{M_2 - \frac{1}{2}N_2}{1 - M_1 + M_2 + N_1 - N_2} \\
&= \rho.
\end{aligned}$$

Hence,

$$\omega(t) \leq \omega(1) = \max_{t \in [0,1]} \omega(t), \quad \min_{t \in [0,\eta]} \omega(t) \geq \rho \max_{t \in [0,1]} \omega(t). \quad \square$$

To finish this section, we present the well-known Avery–Peterson fixed point theorem as follows.

Let  $\gamma$  and  $\theta$  be nonnegative continuous convex functionals on  $P$ ,  $\varphi$  be a nonnegative continuous concave functional on  $P$ , and  $\psi$  be a nonnegative continuous functional on  $P$ . For  $A, B, C, D > 0$ , we define the following convex set:

$$\begin{aligned}
P(\gamma; D) &= \{x \in P : \gamma(x) < D\}, \\
P(\gamma, \varphi; B, D) &= \{x \in P : B \leq \varphi(x), \gamma(x) \leq D\}, \\
P(\gamma, \theta, \varphi; B, C, D) &= \{x \in P : B \leq \varphi(x), \theta(x) \leq C, \gamma(x) \leq D\},
\end{aligned}$$

and a closed set

$$P(\gamma, \psi; A, D) = \{x \in P : A \leq \psi(x), \gamma(x) \leq D\}.$$

**Lemma 2.8** (see [27]) *Let  $P$  be a cone in a real Banach space  $E$ . Let  $\gamma$  and  $\theta$  be nonnegative continuous convex functionals on  $P$ ,  $\varphi$  be a nonnegative continuous concave functional on  $P$ , and  $\psi$  be a nonnegative continuous functional on  $P$  satisfying  $\psi(\lambda x) \leq \lambda \psi(x)$  for  $0 \leq \lambda \leq 1$ , such that, for some positive numbers  $M$  and  $D$ ,  $\varphi(x) \leq \psi(x)$ ,  $\|x\| \leq M\gamma(x)$  for all  $x \in \overline{P(\gamma; D)}$ . Suppose*

$$T : \overline{P(\gamma; D)} \rightarrow \overline{P(\gamma; D)}$$

*is completely continuous and there exist positive numbers  $A, B$ , and  $C$  with  $A < B$  such that*

*(H1)  $\{x \in P(\gamma, \theta, \varphi; B, C, D) : \varphi(x) > B\} \neq \emptyset$ , and  $\varphi(x) > B$  for  $x \in P(\gamma, \theta, \varphi; B, C, D)$ ;*

*(H2)  $\varphi(Tx) > B$  for  $x \in P(\gamma, \varphi; B, D)$  with  $\theta(Tx) > A$ ;*

*(H3)  $0 \notin P(\gamma, \psi; A, D)$  and  $\psi(Tx) < A$  for  $x \in P(\gamma, \psi; A, D)$  with  $\psi(x) = A$ .*

*Then  $T$  has at least three fixed point  $x_1, x_2, x_3 \in \overline{P(\gamma; D)}$  such that*

$$\gamma(x_i) \leq D, \quad i = 1, 2, 3; \quad \varphi(x_1) > B, \quad A < \varphi(x_2), \quad \psi(x_2) < B; \quad \psi(x_3) < A.$$

### 3 Main results

In this section, we prove the existence of positive solution of BVP (1) by applying the following Avery–Peterson fixed point theorem.

We consider the Banach space  $E = \{x \in C[0, 1] : D_{0+}^\beta x \in C[0, 1]\}$  with the norm

$$\|x\| = \max \left\{ \max_{t \in [0,1]} |x(t)|, \max_{t \in [0,1]} |D_{0+}^\beta x(t)| \right\}.$$

Let

$$P = \left\{ x \in E : x(t) \geq 0, D_{0+}^{\beta} x(t) \geq 0, \min_{t \in [0, \eta]} x(t) \geq \rho \max_{t \in [0, 1]} x(t) \right\},$$

then  $P$  is a cone in  $E$ .

Define the operator  $T : P \rightarrow E$  by

$$Tx(t) := \int_0^1 G(t, s) \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^{\beta} x(\tau)) d\tau \right) ds + \omega(t).$$

**Lemma 3.1** Assume  $(L_0)$  hold, then  $T : P \rightarrow P$  is a completely continuous operator.

*Proof* For  $x \in P$ , it is easy to see that  $T$  is continuous operator and  $Tx(t) \geq 0$ . By (14), we have

$$D_{0+}^{\beta} Tx(t) = \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^{\beta} x(\tau)) d\tau \right) ds \geq 0.$$

From Lemma 2.5 and Lemma 2.6 and Lemma 2.7, similar to Lemma 3.1 in [19], we can easily prove that  $T$  is a completely continuous operator.  $\square$

Define continuous nonnegative convex functionals as

$$\gamma(x) = \|x\|, \quad \theta(x) = \psi(x) = \max_{t \in [0, 1]} |x(t)|.$$

Define continuous nonnegative concave functionals as

$$\varphi(x) = \min_{t \in [0, \eta]} |x(t)|.$$

Thus

$$\rho \theta(x) \leq \varphi(x) \leq \theta(x) = \psi(x), \quad \|x\| \leq M \gamma(x),$$

where  $M = 1$ .

Let

$$J_1 = \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^1 H(s, \tau) d\tau \right) ds,$$

$$J_2 = \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha - \beta - 1} \varphi_q \left( \int_0^1 H(s, \tau) d\tau \right) ds,$$

and

$$J_3 = \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^{\eta} H(s, \tau) d\tau \right) ds.$$

**Theorem 3.1** Suppose  $(L_0)$  hold, and there exist constants  $A, B, D \geq \omega(1)$  with  $A < B < \rho D \min\{\frac{J_3}{J_1}, \frac{J_3}{J_2}\}$  and  $C = \frac{B}{\rho}$ , such that

- ( $L_1$ )  $f(t, x, y) \leq \min\{\varphi_p(\frac{D-\omega(1)}{J_1}), \varphi_p(\frac{D}{J_2})\}$ ,  $(t, x, y) \in [0, 1] \times [0, D] \times [0, D]$ ;  
 ( $L_2$ )  $f(t, x, y) > \varphi_p(\frac{B-\rho\omega(1)}{\rho J_3})$ ,  $(t, x, y) \in [0, \eta] \times [B, \frac{B}{\rho}] \times [0, D]$ ;  
 ( $L_3$ )  $f(t, x, y) < \varphi_p(\frac{A-\omega(1)}{J_1})$ ,  $(t, x, y) \in [0, 1] \times [0, A] \times [0, D]$ .

Then BVP (1) has at least three positive solutions  $x_1, x_2, x_3$ , satisfying

$$\|x_i\| \leq D \quad (i = 1, 2, 3), \quad (18)$$

$$\min_{t \in [0, \eta]} |x_1(t)| > B, \quad A < \min_{t \in [0, \eta]} |x_2(t)|, \quad \max_{t \in [0, 1]} |x_2(t)| < B, \quad \max_{t \in [0, 1]} |x_3(t)| < A. \quad (19)$$

*Proof* Obviously, the function  $x$  is a positive solution of BVP (1) if and only if  $x$  is a fixed point of the operator  $T$  in  $P$ .

For  $x \in \overline{P(\gamma; D)}$ , we get

$$\max_{t \in [0, 1]} |x(t)| \leq D, \quad \max_{t \in [0, 1]} |D_{0+}^\beta x(t)| \leq D,$$

this implies

$$0 \leq x(t), \quad D_{0+}^\beta x(t) \leq D.$$

From ( $L_1$ ), we get

$$\begin{aligned} \max_{t \in [0, 1]} |Tx(t)| &\leq \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^\beta x(\tau)) d\tau \right) ds + \omega(1) \\ &\leq \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \varphi_p \left( \frac{D - \omega(1)}{J_1} \right) \int_0^1 H(s, \tau) d\tau \right) ds + \omega(1) \\ &= \frac{D - \omega(1)}{J_1} \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^1 H(s, \tau) d\tau \right) ds + \omega(1) \\ &= D \end{aligned}$$

and

$$\begin{aligned} \max_{t \in [0, 1]} |D_{0+}^\beta Tx(t)| &= \max_{t \in [0, 1]} \left| \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^\beta x(\tau)) d\tau \right) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha - \beta - 1} \varphi_q \left( \varphi_p \left( \frac{D}{J_2} \right) \int_0^1 H(s, \tau) d\tau \right) ds \\ &= \frac{D}{J_2} \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha - \beta - 1} \varphi_q \left( \int_0^1 H(s, \tau) d\tau \right) ds \\ &= D, \end{aligned}$$

so that

$$\gamma(Tx) = \|Tx\| = \max \left\{ \max_{t \in [0, 1]} |Tx(t)|, \max_{t \in [0, 1]} |D_{0+}^\beta Tx(t)| \right\} \leq D.$$

Therefore  $T : \overline{P(\gamma; D)} \rightarrow \overline{P(\gamma; D)}$ .

From  $\frac{B}{\rho} \in P(\gamma, \theta, \varphi; B, C, D)$  and  $\varphi(\frac{B}{\rho}) > B$ , we have

$$\{x \in P(\gamma, \theta, \varphi; B, C, D) : \varphi(x) > B\} \neq \emptyset.$$

For  $x \in P(\gamma, \theta, \varphi; B, C, D)$ , we know that  $B \leq x(t) \leq C = \frac{B}{\rho}$  for  $t \in [0, \eta]$  and  $0 \leq D_{0+}^{\beta} x(t) \leq D$ .

By  $(L_2)$ ,

$$\begin{aligned} \varphi(Tx) &= \min_{t \in [0, \eta]} |Tx(t)| \\ &\geq \int_0^1 \rho(G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^{\eta} H(s, \tau) \varphi_p \left( \frac{B - \rho\omega(1)}{\rho J_3} \right) d\tau \right) ds + \rho\omega(1) \\ &= \rho \frac{B - \rho\omega(1)}{\rho J_3} \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^{\eta} H(s, \tau) d\tau \right) ds + \rho\omega(1) \\ &= B. \end{aligned}$$

So  $\varphi(Tx) > B$  for all  $x \in P(\gamma, \theta, \varphi; B, C, D)$ . Hence, the condition  $(H1)$  of Lemma 2.8 is satisfied.

For all  $x \in P(\gamma, \varphi; B, D)$  with  $\theta(Tx) > C = \frac{B}{\rho}$ , we have

$$\varphi(Tx) \geq \rho\theta(Tx) > \rho C = \rho \frac{B}{\rho} = B.$$

Thus, the condition  $(H2)$  of Lemma 2.8 holds.

Because of  $\psi(0) = 0 < A$ , then  $0 \notin P(\gamma, \psi; A, D)$ . For  $x \in P(\gamma, \psi; A, D)$  with  $\psi(x) = A$ , we know  $\gamma(x) \leq D$ . It means that  $\max_{t \in [0, 1]} x(t) = A$  and  $0 \leq D_{0+}^{\beta} x(t) \leq D$ .

From  $(L_3)$ , we can obtain

$$\begin{aligned} \psi(Tx) &= \max_{t \in [0, 1]} |Tx(t)| \\ &\leq \max_{t \in [0, 1]} \int_0^1 G(t, s) \varphi_q \left( \int_0^1 H(s, \tau) f(\tau, x(\tau), D_{0+}^{\beta} x(\tau)) d\tau \right) ds + \omega(1) \\ &< \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^1 H(s, \tau) \varphi_p \left( \frac{A - \omega(1)}{J_1} \right) d\tau \right) ds + \omega(1) \\ &= \frac{A - \omega(1)}{J_1} \int_0^1 (G_1(0, s) + G_2(1, s)) \varphi_q \left( \int_0^1 H(s, \tau) d\tau \right) ds + \omega(1) \\ &= A. \end{aligned}$$

Therefore, the condition  $(H3)$  of Lemma 2.8 holds.

To sum up, the conditions of Lemma 2.8 are all verified and we notice that  $x_i(t) \geq \omega(0) > 0$ . Hence, BVP (1) has at least three positive solutions  $x_1, x_2, x_3$  satisfying (18) and (19).  $\square$

#### 4 Example

Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{7}{3}} \varphi_{\frac{3}{2}}(D_{0+}^{\frac{11}{3}} x(t)) = f(t, x(t), D_{0+}^{\frac{7}{3}} x(t)), & t \in (0, 1), \\ (\varphi_{\frac{3}{2}}(D_{0+}^{\frac{11}{3}} x(0)))' = \varphi_{\frac{3}{2}}(D_{0+}^{\frac{11}{3}} x(1)) = 0, \\ x(0) + x'(0) = \int_0^1 s x(s) ds + \frac{5}{3}, \\ x(1) + x'(1) = \int_0^1 (s^2 + s) x(s) ds + \frac{7}{3}, \\ x''(0) = x^{(3)} = 0, \end{cases} \quad (20)$$

where  $\alpha = \frac{11}{3}$ ,  $\beta = \frac{7}{3}$ ,  $p = \frac{3}{2}$ ,  $g_0(t) = t$ ,  $g_1(t) = t^2 + t$ , and

$$f(t, x, y) = \begin{cases} \tan(\frac{t}{100}) + 2^{x/10} x^2 + \cos(y), & 0 \leq x \leq 65, \\ \tan(\frac{t}{100}) + \cos(y) + 160\sqrt{5}, & 65 < x \leq 10,000. \end{cases}$$

Choose  $A = 3$ ,  $B = 65$ ,  $D = 25,000$ ,  $\eta = \frac{1}{4}$ . By simple computation, we have

$$\begin{aligned} \rho &= 0.0384615, & M &= 1, & \omega(1) &= 2.83721, \\ M_1 &= \frac{1}{2}, & M_2 &= \frac{1}{3}, & N_1 &= \frac{5}{6}, & N_2 &= \frac{7}{12}, & \delta &= \frac{72}{43}, \\ J_1 &= 0.147109, & J_2 &= 0.0722385, & J_3 &= 0.0310138. \end{aligned}$$

We can check that the nonlinear term  $f(t, x, y)$  satisfies

$$(L_1) \quad \max f(t, x, y) \approx 383.413 \leq \min\{\varphi_p(\frac{D-\omega(1)}{J_1}), \varphi_p(\frac{D}{J_2})\} \approx 412.216, \quad (t, x, y) \in [0, 1] \times [0, 25,000] \times [0, 25,000];$$

$$(L_2) \quad \min f(t, x, y) \approx 381.413 > \varphi_p(\frac{B-\rho\omega(1)}{\rho J_3}) \approx 228.283, \quad (t, x, y) \in [0, \frac{1}{3}] \times [65, \frac{B}{\rho}] \times [0, 25,000];$$

$$(L_3) \quad \max f(t, x, y) \approx 1.02108 < \varphi_p(\frac{A-\omega(1)}{J_1}) \approx 1.05195, \quad (t, x, y) \in [0, 1] \times [0, 3] \times [0, 25,000].$$

Thus, from Theorem 3.1, we know that BVP (20) has at least three positive solutions  $x_1$ ,  $x_2$ ,  $x_3$ , satisfying

$$\|x_i\| \leq 25,000 \quad (i = 1, 2, 3),$$

$$\min_{t \in [0, \eta]} |x_1| > 65, \quad 3 < \min_{t \in [0, \eta]} |x_2|, \quad \max_{t \in [0, 1]} |x_2| < 65, \quad \max_{t \in [0, 1]} |x_3| < 3.$$

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