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A sharp upper bound on the spectral radius of a nonnegative k-uniform tensor and its applications to (directed) hypergraphs

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Abstract

In this paper, we obtain a sharp upper bound on the spectral radius of a nonnegative k-uniform tensor and characterize when this bound is achieved. Furthermore, this result deduces the main result in [X. Duan and B. Zhou, Sharp bounds on the spectral radius of a nonnegative matrix, Linear Algebra Appl. 439:2961–2970, 2013] for nonnegative matrices; improves the adjacency spectral radius and signless Laplacian spectral radius of a uniform hypergraph for some known results in [D.M. Chen, Z.B. Chen and X.D. Zhang, Spectral radius of uniform hypergraphs and degree sequences, Front. Math. China 6:1279–1288, 2017]; and presents some new sharp upper bounds for the adjacency spectral radius and signless Laplacian spectral radius of a uniform directed hypergraph. Moreover, a characterization of a strongly connected k-uniform directed hypergraph is obtained.

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Keywords: Uniform tensors; Uniform (directed) hypergraphs; Spectral radius; Adjacency; Signless Laplacian

1 Introduction

Let k, n be two positive integers. As in [17, 21], an order k dimension n tensor $\mathbb{A} = (a_{i_1\cdots i_k})$ over the real field $\mathbb{R}$ is a multidimensional array with n^k entries $a_{i_1\cdots i_k} \in \mathbb{R}$, where $i_j \in [n] = \{1,2,\ldots,n\}$, $j \in [k] = \{1,2,\ldots,k\}$. Obviously, a vector is an order 1 tensor and a square matrix is an order 2 tensor.

Furthermore, we call a tensor $\mathbb{A}$ nonnegative (positive), denoted by $\mathbb{A} \geq 0$ ($\mathbb{A} > 0$), if every entry has $a_{i_1\cdots i_k} \geq 0$ ($a_{i_1\cdots i_k} > 0$). The tensor $\mathbb{A} = (a_{i_1\cdots i_k})$ is called symmetric if $a_{i_1\cdots i_k} = a_{\sigma(i_1)\cdots\sigma(i_k)}$, where σ is any permutation of the indices.

Let $\mathbb{A}$ be an order k dimension n tensor. If there is a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ such that

$$\mathbb{A}x^{k-1} = \lambda x^{[k-1]},$$

then λ is called an eigenvalue of $\mathbb{A}$ and x an eigenvector of $\mathbb{A}$ corresponding to the eigenvalue λ [17, 18, 21]. Here $\mathbb{A}x^{k-1}$ and $x^{[k-1]}$ are vectors, whose ith entries are

$$\left(\mathbb{A}x^{k-1}\right)_i = \sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2}\cdots x_{i_k}$$

and $(x^{[k-1]})_i = x_i^{k-1}$, respectively. Moreover, the spectral radius $\rho(\mathbb{A})$ of a tensor $\mathbb{A}$ is defined as

$$\rho(\mathbb{A}) = \max\bigl\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbb{A}\bigr\}.$$

Some properties of the spectral radius of a nonnegative tensor can be found in [3, 9, 14, 16–18, 21, 25–27].

Definition 1.1 ([22]) Let $\mathbb{A}$ and $\mathbb{B}$ be two tensors with order $m \geq 2$ and $k \geq 1$ dimension n, respectively. The general product $\mathbb{A}\mathbb{B}$ of $\mathbb{A}$ and $\mathbb{B}$ is the following tensor $\mathbb{C}$ with order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m} b_{i_2\alpha_1}\cdots b_{i_m\alpha_{m-1}} \quad \bigl(i \in [n], \alpha_1,\dots,\alpha_{m-1} \in [n]^{k-1}\bigr).$$

Definition 1.2 ([22]) Let $\mathbb{A} = (a_{i_1i_2\cdots i_k})$ and $\mathbb{B} = (b_{i_1i_2\cdots i_k})$ be two order k dimension n tensors. We say that $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar if there exists some invertible diagonal matrix $D = (d_{11}, d_{22}, \dots, d_{nn})$ of order n such that $\mathbb{B} = D^{-(k-1)}\mathbb{A}D$ with entries

$$b_{i_1i_2\cdots i_k} = d_{i_1i_1}^{-(k-1)} a_{i_1i_2\cdots i_k} d_{i_2i_2}\cdots d_{i_ki_k}.$$

Theorem 1.3 ([22]) *If the two order k dimension n tensors $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar, then they have the same eigenvalues including multiplicity and same spectral radius.*

Definition 1.4 ([9, 26]) Let $\mathbb{A}$ be an order k dimensional n tensor (not necessarily nonnegative). If there exists a nonempty proper subset I of the set $[n]$, such that

$$a_{i_1i_2\dots i_k} = 0 \quad \text{for all } i_1 \in I \text{ and some } i_j \notin I \text{ where } j \in \{2,\dots,k\},$$

then $\mathbb{A}$ is called weakly reducible (or sometimes I-weakly reducible). If $\mathbb{A}$ is not weakly reducible, then $\mathbb{A}$ is called weakly irreducible.

The ith slice of a tensor $\mathbb{A}$ with order $k \geq 2$ and dimension n, denoted by $\mathbb{A}_i$ in [23], is the subtensor of $\mathbb{A}$ with order $k-1$ and dimension n such that $(\mathbb{A}_i)_{i_2\cdots i_k} = a_{ii_2\cdots i_k}$. Then the ith slice sum (also called "the ith row sum") of $\mathbb{A}$ is defined as

$$r_i(\mathbb{A}) = \sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} \quad \bigl(i \in [n]\bigr).$$

Lemma 1.5 ([13, 25]) *Let $\mathbb{A}$ be a nonnegative tensor with order $k \geq 2$ and dimension n. Then we have*

$$\min_{1\leq i\leq n} r_i(\mathbb{A}) \leq \rho(\mathbb{A}) \leq \max_{1\leq i\leq n} r_i(\mathbb{A}). \tag{1.1}$$

Moreover, if $\mathbb{A}$ is weakly irreducible, then one of the equalities in (1.1) *holds if and only if* $r_1(\mathbb{A}) = r_2(\mathbb{A}) = \cdots = r_n(\mathbb{A})$.

We denote by $\binom{n}{r}$ the number of r-combinations of an n-element set, and let $\binom{n}{r} = 0$ if $r > n$ or $r < 0$. Clearly, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ when $0 \leq r \leq n$.

Lemma 1.6 ([2]) *Let n, k, and m be positive integers. Then*
(1) $\sum_{r=0}^{k} \binom{n}{r}\binom{m}{k-r} = \binom{n+m}{k}$ $(n + m \geq k)$;
(2) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ $(n \geq k \geq 1)$.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be an n-element set, noting that $s_i \neq s_j$ if $i \neq j$.

Definition 1.7 Let $n \geq 2$, $k \geq 2$, $\mathbb{A}$ be an order k dimension n tensor, we call $\mathbb{A}$ a k-uniform tensor if its entries are defined as follows: $a_{i_1 i_2 \cdots i_k} \in \mathbb{R}$ if $\{i_1, i_2, \ldots, i_k\}$ is a k-element set or $i_1 = i_2 = \cdots = i_k$, otherwise, $a_{i_1 i_2 \cdots i_k} = 0$.

Obviously, a 2-uniform tensor is an ordinary matrix. Let $\mathbb{A}$ be a k-uniform tensor with order k dimension n. Then $a_{i_1 i_2 \cdots i_k} \neq 0$ implies $\{i_1, i_2, \ldots, i_k\}$ is a k-element set or $i_1 = i_2 = \cdots = i_k$.

In this paper, we obtain a sharp upper bound on the spectral radius of a nonnegative k-uniform tensor in Sect. 2. By applying the bound to a nonnegative matrix, we can obtain the main result in [7]. In Sect. 3, we apply the bound to the adjacency spectral radius and signless Laplacian spectral radius of a uniform hypergraph and improve some known results in [4]. Furthermore, we give a characterization of a strongly connected k-uniform directed hypergraph and obtain some new results by applying the bound to the adjacency spectral radius and the signless Laplacian spectral radius of a uniform directed hypergraph in Sect. 4.

2 Main results

In this section, we obtain a sharp upper bound on the spectral radius of a nonnegative k-uniform tensor and characterize when this bound is achieved. Furthermore, this bound deduces the main result in [7] for a nonnegative matrix.

Theorem 2.1 *Let $n \geq 2$, $k \geq 2$, $\mathbb{A} = (a_{i_1 i_2 \cdots i_k})$ be a nonnegative k-uniform tensor with order k dimension n, $r_i = r_i(\mathbb{A}) = \sum_{i_2,\ldots,i_k=1}^{n} a_{i i_2 \cdots i_k}$ for $i \in [n]$ with $r_1 \geq r_2 \geq \cdots \geq r_n$. Let M be the largest diagonal element and N (> 0) be the largest non-diagonal element of tensor $\mathbb{A}$, $N_1 = N(k-2)!\binom{n-2}{k-2}$, $\phi_1 = r_1$, and*

$$\phi_s = \frac{1}{2}\left\{ r_s + M - N_1 + \sqrt{(r_s - M + N_1)^2 + 4N_1 \sum_{t=1}^{s-1}(r_t - r_s)} \right\} \tag{2.1}$$

for $2 \le s \le n$. *Then*

$$\rho(\mathbb{A}) \le \min_{1\le s\le n} \phi_s.$$

Let $\phi_s = \min_{1\le l\le n} \phi_l$. *If* $\mathbb{A}$ *is weakly irreducible, then*

(1) *when* $k = 2$, $\rho(\mathbb{A}) = \phi_s$ *if and only if* $r_1 = r_2 = \cdots = r_n$ *or for some* t $(2 \le t \le s)$, $\mathbb{A}$ *satisfies the following conditions*:
  (i) $a_{ii} = M$ *for* $1 \le i \le t-1$;
  (ii) $a_{ii_2} = N$ *for* $1 \le i \le n$, $1 \le i_2 \le t-1$, *and* $i \ne i_2$;
  (iii) $r_t = r_{t+1} = \cdots = r_n$;

(2) *when* $k \ge 3$, $\rho(\mathbb{A}) = \phi_s$ *if and only if* $r_1 = r_2 = \cdots = r_n$.

Proof Firstly, we show $\rho(\mathbb{A}) \le \phi_s$ for $1 \le s \le n$.

If $s = 1$, then by Lemma 1.5 we have $\rho(\mathbb{A}) \le r_1 = \phi_1$. Now we only consider the cases of $2 \le s \le n$.

Let

$$U = \operatorname{diag}(x_1, \dots, x_{s-1}, x_s, \dots, x_n),$$

where $x_i > 0$ for $1 \le i \le n$, $x_i^{k-1} = 1 + \frac{r_i - r_s}{\phi_s + N_1 - M}$ for $1 \le i \le s-1$, and $x_s = \cdots = x_n = 1$.

Now we show $x_i \ge 1$ for $1 \le i \le s-1$. By $r_1 \ge r_2 \ge \cdots \ge r_n$, we only need to show $\phi_s + N_1 - M > 0$.

If $\sum_{t=1}^{s-1}(r_t - r_s) > 0$, then by (2.1) we have

$$\phi_s > \frac{1}{2}\bigl(r_s + M - N_1 + |r_s - M + N_1|\bigr) \ge \frac{1}{2}\bigl(r_s + M - N_1 - (r_s - M + N_1)\bigr) = M - N_1,$$

and thus $\phi_s - M + N_1 > 0$.

If $\sum_{t=1}^{s-1}(r_t - r_s) = 0$, then $r_1 = r_2 = \cdots = r_s$. Thus $\phi_s - M + N_1 > 0$ by $r_1 \ge M$ and $\phi_s = r_s$ from (2.1).

Combining the above arguments, we know $x_i \ge 1$, and then U is an invertible diagonal matrix. Let $\mathbb{B} = U^{-(k-1)}\mathbb{A}U = (b_{i_1\cdots i_k})$. By Theorem 1.3, we have

$$\rho(\mathbb{A}) = \rho(\mathbb{B}). \tag{2.2}$$

By (2.1), it is easy to see that

$$\phi_s^2 - (r_s + M - N_1)\phi_s + (M - N_1)r_s - N_1\sum_{t=1}^{s-1}(r_t - r_s) = 0.$$

Then

$$\begin{aligned}(\phi_s - M + N_1)(\phi_s - r_s) &= N_1\sum_{t=1}^{s-1}(r_t - r_s) = N_1\sum_{t=1}^{s-1}(\phi_s - M + N_1)\bigl(x_t^{k-1} - 1\bigr)\\ &= N_1(\phi_s - M + N_1)\left(\sum_{t=1}^{s-1} x_t^{k-1} - (s-1)\right).\end{aligned}$$

Therefore, $\phi_s = r_s + N_1 \sum_{t=1}^{s-1} x_t^{k-1} - N_1(s-1)$ and thus

$$\sum_{t=1}^{s-1} x_t^{k-1} = \frac{\phi_s - r_s + (s-1)N_1}{N_1}. \tag{2.3}$$

In the following we show $r_i(\mathbb{B}) \le \phi_s$ for any $i \in [n] = \{1, 2, \dots, n\}$.

Let $S(\mathbb{A}) = \{\{i, i_2, \dots, i_k\} | a_{ii_2\cdots i_k} \ne 0\}$. Since M is the largest diagonal element and $N > 0$ is the largest non-diagonal element of tensor $\mathbb{A}$, by Definition 1.2, we have

$$\begin{aligned}
r_i(\mathbb{B}) &= r_i\bigl(U^{-(k-1)}\mathbb{A}U\bigr) \\
&= \sum_{i_2,\dots,i_k=1}^{n} \bigl(U^{-(k-1)}\bigr)_{ii} a_{ii_2\cdots i_k} U_{i_2i_2} \cdots U_{i_ki_k} \\
&= \frac{1}{x_i^{k-1}} \sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k} \\
&= \frac{1}{x_i^{k-1}} \left\{ r_i + \sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} (x_{i_2} \cdots x_{i_k} - 1) \right\} \\
&= \frac{1}{x_i^{k-1}} \Biggl\{ r_i + a_{i\cdots i}\bigl(x_i^{k-1} - 1\bigr) \\
&\quad + \sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} (x_{i_2} \cdots x_{i_k} - 1) - a_{i\cdots i}\bigl(x_i^{k-1} - 1\bigr) \Biggr\} \\
&\le \frac{1}{x_i^{k-1}} \Biggl\{ r_i + M\bigl(x_i^{k-1} - 1\bigr) \\
&\quad + \sum_{i_2,\dots,i_k=1}^{n} a_{ii_2\cdots i_k} (x_{i_2} \cdots x_{i_k} - 1) - a_{i\cdots i}\bigl(x_i^{k-1} - 1\bigr) \Biggr\} \\
&\le \frac{1}{x_i^{k-1}} \left\{ r_i + M\bigl(x_i^{k-1} - 1\bigr) + N(k-1)! \sum_{\{i,i_2,\dots,i_k\} \in S(\mathbb{A})} (x_{i_2} \cdots x_{i_k} - 1) \right\} \\
&\le \frac{1}{x_i^{k-1}} \Biggl\{ r_i + M\bigl(x_i^{k-1} - 1\bigr) \\
&\quad + N(k-1)! \sum_{\{i,i_2,\dots,i_k\} \in S(\mathbb{A})} \left(\frac{x_{i_2}^{k-1} + \cdots + x_{i_k}^{k-1}}{k-1} - 1 \right) \Biggr\} \\
&\le \frac{1}{x_i^{k-1}} \Biggl\{ r_i + M\bigl(x_i^{k-1} - 1\bigr) \\
&\quad + N(k-1)! \sum_{r=0}^{k-1} \sum_{\{i_2,\dots,i_k\} \in N_r^s} \left(\frac{x_{i_2}^{k-1} + \cdots + x_{i_k}^{k-1}}{k-1} - 1 \right) \Biggr\} \\
&= \frac{1}{x_i^{k-1}} \Biggl\{ r_i + M\bigl(x_i^{k-1} - 1\bigr) \\
&\quad + N(k-1)! \sum_{r=0}^{k-2} \sum_{\{i_2,\dots,i_k\} \in N_r^s} \left(\frac{x_{i_2}^{k-1} + \cdots + x_{i_k}^{k-1}}{k-1} - 1 \right) \Biggr\},
\end{aligned}$$

where $N_r^s = \{\{i_2,\dots,i_k\} \mid i_2,\dots,i_k \in \{1,2,\dots,n\} \setminus \{i\}$, and there are exactly r elements in $\{i_2,\dots,i_k\}$ such that they are not less than $s\}$ for $0 \le r \le k-1$. Obviously, the family of all $(k-1)$-element subsets of $\{1,2,\dots,n\} \setminus \{i\}$ is just equal to $\bigcup_{r=0}^{k-1} N_r^s$. Thus we have

$$r_i(\mathbb{B}) \le M + \frac{1}{x_i^{k-1}}\left\{r_i - M + N(k-1)!\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s}\left(\frac{x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}}{k-1}-1\right)\right\}, \tag{2.4}$$

and the equality holds in (2.4) if and only if (a), (b), (c), and (d) hold:

(a) $x_i^{k-1} = 1$ or $a_{i\cdots i} = M$ for $x_i > 1$;

(b) for any $\{i,i_2,\dots,i_k\} \in S(\mathbb{A})$, $x_{i_2}\cdots x_{i_k} = 1$ or $a_{ii_2\cdots i_k} = N$ for $x_{i_2}\cdots x_{i_k} > 1$;

(c) $x_{i_2} = \cdots = x_{i_k}$ for any $\{i,i_2,\dots,i_k\} \in S(\mathbb{A})$;

(d) $\sum_{\{i,i_2,\dots,i_k\}\in S(\mathbb{A})}(\frac{x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}}{k-1}-1) = \sum_{r=0}^{k-1}\sum_{\{i_2,\dots,i_k\}\in N_r^s}(\frac{x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}}{k-1}-1)$.

Case 1: $s \le i \le n$.

Clearly, $\{i_2,\dots,i_k\} \in N_r^s$ implies that we should choose r elements from the set $\{s,\dots,n\} \setminus \{i\}$ and choose $k-1-r$ elements from the set $\{1,2,\dots,s-1\}$, then we have

$$\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s} 1 = \sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s}{r}. \tag{2.5}$$

Similarly, we have

$$\begin{aligned}
&\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s}\left(x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}\right)\\
&\quad= \sum_{r=0}^{k-2}\binom{s-2}{k-2-r}\binom{n-s}{r}\left(\sum_{t=1}^{s-1}x_t^{k-1}\right)\\
&\qquad+\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s-1}{r-1}\left(\sum_{t=s}^{n}x_t^{k-1}-x_i^{k-1}\right).
\end{aligned} \tag{2.6}$$

We note $x_s = \cdots = x_n = 1$ and $r_1 \ge \cdots \ge r_s \ge \cdots \ge r_i \ge \cdots \ge r_n$, then by (2.3), (2.4), (2.5), and (2.6), we have

$$\begin{aligned}
r_i(\mathbb{B}) &\le r_i + N(k-1)!\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s}\left(\frac{x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}}{k-1}-1\right)\\
&\le r_s + N(k-2)!\sum_{r=0}^{k-2}\binom{s-2}{k-2-r}\binom{n-s}{r}\left(\sum_{t=1}^{s-1}x_t^{k-1}\right)\\
&\quad+N(k-2)!\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s-1}{r-1}\left(\sum_{t=s}^{n}x_t^{k-1}-x_i^{k-1}\right)\\
&\quad-N(k-1)!\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s}{r}\\
&= r_s + N(k-2)!\binom{n-2}{k-2}\sum_{t=1}^{s-1}x_t^{k-1} + N(k-2)!\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s-1}{r-1}(n-s)
\end{aligned}$$

$$
\begin{aligned}
&\quad - N(k-1)!\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s}{r}\\
&= r_s + N_1\sum_{t=1}^{s-1}x_t^{k-1}\\
&\quad + N(k-2)!\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\left[\binom{n-s-1}{r-1}(n-s)-(k-1)\binom{n-s}{r}\right]\\
&= r_s + N_1\sum_{t=1}^{s-1}x_t^{k-1} - N(k-2)!\sum_{r=0}^{k-2}\binom{s-1}{k-1-r}\binom{n-s}{r}(k-1-r)\\
&= r_s + N_1\sum_{t=1}^{s-1}x_t^{k-1} - N(k-2)!\sum_{r=0}^{k-2}(s-1)\binom{s-2}{k-2-r}\binom{n-s}{r}\\
&= r_s + N_1\sum_{t=1}^{s-1}x_t^{k-1} - N(k-2)!(s-1)\binom{n-2}{k-2}\\
&= r_s + N_1\sum_{t=1}^{s-1}x_t^{k-1} - (s-1)N_1\\
&= \phi_s,
\end{aligned}
$$

where equality holds if and only if the following condition (e) holds: (e) $r_i = r_s$.

Case 2: $1 \le i \le s-1$.

Subcase 2.1: $s \ge 3$.

Clearly, $\{i_2,\dots,i_k\} \in N_r^s$ implies that we should choose r elements from the set $\{s,\dots,n\}$ and choose $k-1-r$ elements from the set $\{1,2,\dots,s-1\} \setminus \{i\}$, then $\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s} 1 = \sum_{r=0}^{k-2}\binom{s-2}{k-1-r}\binom{n-s+1}{r}$. Similarly, we have

$$
\begin{aligned}
&\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s}\left(x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}\right)\\
&\quad = \sum_{r=0}^{k-2}\binom{s-3}{k-r-2}\binom{n-s+1}{r}\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right)\\
&\qquad + \sum_{r=0}^{k-2}\binom{s-2}{k-1-r}\binom{n-s}{r-1}\left(\sum_{t=s}^{n}x_t^{k-1}\right)\\
&\quad = \binom{n-2}{k-2}\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right) + \sum_{r=0}^{k-2}\binom{s-2}{k-1-r}\binom{n-s}{r-1}(n-s+1).
\end{aligned}
$$

Then

$$
\begin{aligned}
&N(k-1)!\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^s}\left(\frac{x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}}{k-1}-1\right)\\
&\quad = N_1\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right) + N(k-2)!\sum_{r=0}^{k-2}\binom{s-2}{k-1-r}\binom{n-s}{r-1}(n-s+1)
\end{aligned}
$$

$$
\begin{aligned}
&-N(k-1)!\sum_{r=0}^{k-2}\binom{s-2}{k-1-r}\binom{n-s+1}{r}\\
&=N_1\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right)-N(k-2)!\sum_{r=0}^{k-2}(k-1-r)\binom{s-2}{k-1-r}\binom{n-s+1}{r}\\
&=N_1\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right)-N(k-2)!\sum_{r=0}^{k-2}(s-2)\binom{s-3}{k-r-2}\binom{n-s+1}{r}\\
&=N_1\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right)-N(k-2)!(s-2)\binom{n-2}{k-2}\\
&=N_1\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right)-(s-2)N_1.
\end{aligned}
$$

Thus, by (2.3), (2.4), and the definition of x_i^{k-1} for $1\le i\le s-1$, we have

$$
\begin{aligned}
r_i(\mathbb{B})&\le M+\frac{1}{x_i^{k-1}}\left\{r_i-M+N_1\left(\sum_{t=1}^{s-1}x_t^{k-1}-x_i^{k-1}\right)-(s-2)N_1\right\}\\
&=M-N_1+\frac{1}{x_i^{k-1}}\left\{r_i-M+N_1\sum_{t=1}^{s-1}x_t^{k-1}-(s-2)N_1\right\}\\
&=\phi_s.
\end{aligned}
$$

Subcase 2.2: $s=2$.

In this case, we need to show $r_1(\mathbb{B})\le\phi_2$. Noting that $x_2=\cdots=x_n=1$, by (2.4) and the definition of N_r^2, we have

$$
\begin{aligned}
r_1(\mathbb{B})&\le M+\frac{1}{x_1^{k-1}}\left\{r_1-M+N(k-1)!\sum_{r=0}^{k-2}\sum_{\{i_2,\dots,i_k\}\in N_r^2}\left(\frac{x_{i_2}^{k-1}+\cdots+x_{i_k}^{k-1}}{k-1}-1\right)\right\}\\
&=M+\frac{1}{x_1^{k-1}}(r_1-M).
\end{aligned}
$$

By (2.3), we have $x_1^{k-1}=\frac{\phi_2-r_2+N_1}{N_1}$. Then, by (2.1) and the definition of ϕ_2, we have

$$
\begin{aligned}
&\frac{1}{x_1^{k-1}}(r_1-M)\\
&=\frac{N_1(r_1-M)}{\phi_2-r_2+N_1}\\
&=\frac{2N_1(r_1-M)}{N_1+M-r_2+\sqrt{(N_1-M+r_2)^2+4N_1(r_1-r_2)}}\\
&=\frac{2N_1(r_1-M)(N_1+M-r_2-\sqrt{(N_1-M+r_2)^2+4N_1(r_1-r_2)})}{(N_1+M-r_2)^2-((N_1-M+r_2)^2+4N_1(r_1-r_2))}\\
&=-\frac{N_1+M-r_2-\sqrt{(N_1-M+r_2)^2+4N_1(r_1-r_2)}}{2}.
\end{aligned}
$$

Thus

$$r_1(\mathbb{B}) \le M + \frac{1}{x_1^{k-1}}(r_1 - M) = \phi_2.$$

Combining Subcases 2.1 and 2.2, we have $r_i(\mathbb{B}) \le \phi_s$ for $1 \le i \le s-1$, and combining Cases 1 and 2, we have $r_i(\mathbb{B}) \le \phi_s$ for $1 \le i \le n$. Then $\rho(\mathbb{A}) = \rho(\mathbb{B}) \le \max_{1\le i\le n} r_i(\mathbb{B}) \le \phi_s$ for $2 \le s \le n$ by (2.2) and Lemma 1.5.

Therefore, we know $\rho(\mathbb{A}) \le \phi_s$ for $1 \le s \le n$ and thus $\rho(\mathbb{A}) \le \min_{1\le s\le n} \phi_s$.

Now suppose that $\mathbb{A}$ is weakly irreducible. Then $\mathbb{B}$ is also weakly irreducible by $\mathbb{B} = U^{-(k-1)}\mathbb{A}U$. Let $\phi_s = \min_{1\le l\le n} \phi_l$.

Case 1: $s = 1$.

By Lemma 1.5 and the fact $r_1 = \max_{1\le i\le n} r_i$, we have $\rho(\mathbb{A}) = \phi_1$ if and only if $r_1 = r_2 = \cdots = r_n$.

Case 2: $2 \le s \le n$.

Then $\rho(\mathbb{B}) = \max_{1\le i\le n} r_i(\mathbb{B})$ and thus $r_1(\mathbb{B}) = r_2(\mathbb{B}) = \cdots = r_n(\mathbb{B}) = \phi_s$ by $\phi_s = \rho(\mathbb{A}) = \rho(\mathbb{B}) \le \max_{1\le i\le n} r_i(\mathbb{B}) \le \phi_s$ and Lemma 1.5. Therefore, (a), (b), (c), and (d) hold for any $i \in [n]$, (e) holds for any $i \in \{s, \ldots, n\}$.

Subcase 2.1: $r_1 = r_s$.

By $r_1 \ge r_2 \ge \cdots \ge r_n$ and (e) $r_i = r_s$ for $s \le i \le n$, then we have $r_1 = r_2 = \cdots = r_n$.

Subcase 2.2: $r_1 > r_s$.

Let t be the smallest integer such that $r_t = r_s$ for $1 < t \le s$. Since $r_s = r_{s+1} = \cdots = r_n$, we have $r_t = r_{t+1} = \cdots = r_n$ and $x_i > 1$ for $i = 1, 2, \ldots, t-1$.

When $k \ge 3$, (c) and (d) cannot hold at the same time. Because there are r elements in $\{i_2, \ldots, i_k\}$ chosen from $\{s, \ldots, n\}$ and $k-1-r$ elements in $\{i_2, \ldots, i_k\}$ chosen from $\{1, \ldots, s-1\}$, and then $x_{i_2} = \cdots = x_{i_k}$ cannot hold when $1 \le r \le k-2$. Thus we only consider the case of $k = 2$.

In the case of $k = 2$, (d) implies

$$\sum_{\{i,i_2\}\in S(\mathbb{A})} (x_{i_2} - 1) = \sum_{r=0}^{1} \sum_{\{i_2\}\in N_r^s} (x_{i_2} - 1) = \sum_{\substack{i_2=1\\ i_2\ne i}}^{t-1} (x_{i_2} - 1).$$

Then (i)–(iii) follow from (a), (b), (c), (d) for $1 \le i \le n$, and (e) for $s \le i \le n$, and thus (1) and (2) hold.

Conversely, if $r_1 = r_2 = \cdots = r_n$, then by Lemma 1.5, $\rho(\mathbb{A}) = \phi_1 = r_1$. If $k = 2$ and (i)–(iii) hold, then (a), (b), (c), and (d) hold for $1 \le i \le n$, (e) holds for $s \le i \le n$. Then we have $r_i(\mathbb{B}) = \phi_s$ for $1 \le i \le n$. Therefore, by Lemma 1.5, we have $\rho(\mathbb{A}) = \rho(\mathbb{B}) = \max_{1\le i\le n} r_i(\mathbb{B}) = \phi_s$ for $s = 2, \ldots, n$. □

Let $k = 2$. Then $\mathbb{A}$ is a matrix, weak irreducibility for tensors corresponds to irreducibility for matrices, and slice sum for tensors corresponds to row sum for matrices. The following result follows immediately.

Corollary 2.2 ([7], Theorem 2.1) *Let A be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \ldots, r_n$, where $r_1 \ge r_2 \ge \cdots \ge r_n$. Let M be the largest diagonal element and N be the* *largest non-diagonal element of* A. *Suppose that* $N > 0$. *Let* $\phi_1 = r_1$ *and, for* $2 \le s \le n$,

$$\phi_s = \frac{1}{2}\left(r_s + M - N + \sqrt{(r_s - M + N)^2 + 4N\sum_{t=1}^{s-1}(r_t - r_s)}\right). \tag{2.7}$$

Then $\rho(A) \le \min_{1\le s\le n} \phi_s$.

Let $\phi_s = \min_{1\le l\le n} \phi_l$. *If* A *is irreducible, then* $\rho(A) = \phi_s$ *if and only if* $r_1 = r_2 = \cdots = r_n$ *or for some* t $(2 \le t \le s)$, A *satisfies the following conditions*:

(i) $a_{ii} = M$ *for* $1 \le i \le t-1$;

(ii) $a_{ii_2} = N$ *for* $1 \le i \le s-1$ *and* $1 \le i_2 \le t-1$ *with* $i \ne i_2$;

(iii) $r_t = \cdots = r_n$;

(iv) $a_{ii_2} = N$ *for* $s \le i \le n$ *and* $1 \le i_2 \le t-1$.

3 Applications to a k-uniform hypergraph

A hypergraph is a natural generalization of an ordinary graph [1].

A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ on n vertices is a set of vertices, say, $V(\mathcal{H}) = \{1, 2, \dots, n\}$ and a set of edges, say, $E(\mathcal{H}) = \{e_1, e_2, \dots, e_m\}$, where $e_i = \{i_1, i_2, \dots, i_l\}$, $i_j \in [n]$, $j = 1, 2, \dots, l$. Let $k \ge 2$, if $|e_i| = k$ for any $i = 1, 2, \dots, m$, then $\mathcal{H}$ is called a k-uniform hypergraph. When $k = 2$, then $\mathcal{H}$ is an ordinary graph. The degree d_i of vertex i is defined as $d_i = |\{e_j : i \in e_j \in E(\mathcal{H})\}|$. If $d_i = d$ for any vertex i of a hypergraph $\mathcal{H}$, then $\mathcal{H}$ is called d-regular. A *walk* W of length ℓ in $\mathcal{H}$ is a sequence of alternate vertices and edges: $v_0, e_1, v_1, e_2, \dots, e_\ell, v_\ell$, where $\{v_i, v_{i+1}\} \subseteq e_{i+1}$ for $i = 0, 1, \dots, \ell-1$. The hypergraph $\mathcal{H}$ is said to be connected if every two vertices are connected by a walk.

Definition 3.1 ([6, 18]) Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a k-uniform hypergraph on n vertices. The adjacency tensor of $\mathcal{H}$ is defined as the order k dimension n tensor $\mathbb{A}(\mathcal{H})$, whose $(i_1 i_2 \cdots i_k)$-entry is

$$\left(\mathbb{A}(\mathcal{H})\right)_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \dots, i_k\} \in E(\mathcal{H}), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbb{D}(\mathcal{H})$ be an order k dimension n diagonal tensor with its diagonal entry $\mathbb{D}_{ii\cdots i}$ being d_i, the degree of vertex i for all $i \in V(\mathcal{H}) = [n]$. Then $\mathbb{Q}(\mathcal{H}) = \mathbb{D}(\mathcal{H}) + \mathbb{A}(\mathcal{H})$ is called the signless Laplacian tensor of the hypergraph $\mathcal{H}$. Clearly, the adjacency tensor and the signless Laplacian tensor of a k-uniform hypergraph $\mathcal{H}$ are nonnegative symmetric k-uniform tensors and, for any $1 \le i \le n$,

$$r_i\left(\mathbb{A}(\mathcal{H})\right) = \sum_{i_2,\dots,i_k=1}^{n} \left(\mathbb{A}(\mathcal{H})\right)_{ii_2\cdots i_k} = d_i, r_i\left(\mathbb{Q}(\mathcal{H})\right) = \sum_{i_2,\dots,i_k=1}^{n} \left(\mathbb{Q}(\mathcal{H})\right)_{ii_2\cdots i_k} = 2d_i.$$

It was proved in [9, 20] that a k-uniform hypergraph $\mathcal{H}$ is connected if and only if its adjacency tensor $\mathbb{A}(\mathcal{H})$ (and thus the signless Laplacian tensor $\mathbb{Q}(\mathcal{H})$) is weakly irreducible.

Recently, several papers studied the spectral radii of the adjacency tensor $\mathbb{A}(\mathcal{H})$ and the signless Laplacian tensor $\mathbb{Q}(\mathcal{H})$ of a k-uniform hypergraph $\mathcal{H}$ (see [4, 6, 18, 19, 27, 28] and so on). In this section, we apply Theorem 2.1 to the adjacency tensor $\mathbb{A}(\mathcal{H})$ and the signless Laplacian tensor $\mathbb{Q}(\mathcal{H})$ of a k-uniform hypergraph $\mathcal{H}$. If $k = 2$, we obtain Theorem 3.1 and

Theorem 4.2 in [7]. If $k \geq 3$, we improve some known results about the bounds of $\rho(\mathbb{A}(\mathcal{H}))$ and $\rho(\mathbb{Q}(\mathcal{H}))$ in [4].

Theorem 3.2 *Let $k \geq 3$, $\mathcal{H}$ be a k-uniform hypergraph with degree sequence $d_1 \geq \cdots \geq d_n$, $\mathbb{A}(\mathcal{H})$ be the adjacency tensor of $\mathcal{H}$. Let $A_1 = \frac{1}{k-1}\binom{n-2}{k-2}$, $\phi_1 = d_1$, and*

$$\phi_s = \frac{1}{2}\left\{d_s - A_1 + \sqrt{(d_s + A_1)^2 + 4A_1\sum_{t=1}^{s-1}(d_t - d_s)}\right\} \tag{3.1}$$

for $2 \leq s \leq n$. Then

$$\rho\bigl(\mathbb{A}(\mathcal{H})\bigr) \leq \min_{1\leq s\leq n} \phi_s. \tag{3.2}$$

If $\mathcal{H}$ is connected, then the equality in (3.2) holds if and only if $\mathcal{H}$ is regular.

Proof Let $\mathbb{A} = \mathbb{A}(\mathcal{H})$. We apply Theorem 2.1 to $\mathbb{A}(\mathcal{H})$, then we have $M = 0$, $N = \frac{1}{(k-1)!}$, $r_i = d_i$ for $1 \leq i \leq n$, $A_1 = N_1$, and (3.1) is from (2.1). Thus (3.2) holds by Theorem 2.1.

If $\mathcal{H}$ is connected, then by Theorem 2.1 the equality in (3.2) holds if and only if $r_1(\mathbb{A}(\mathcal{H})) = r_2(\mathbb{A}(\mathcal{H})) = \cdots = r_n(\mathbb{A}(\mathcal{H}))$, which says exactly that $\mathcal{H}$ is regular, since $r_i(\mathbb{A}(\mathcal{H})) = d_i$ for any $1 \leq i \leq n$. □

Theorem 3.3 *Let $k \geq 3$, $\mathcal{H}$ be a k-uniform hypergraph with degree sequence $d_1 \geq \cdots \geq d_n$, $\mathbb{Q}(\mathcal{H})$ be the signless Laplacian tensor of $\mathcal{H}$. Let $A_1 = \frac{1}{k-1}\binom{n-2}{k-2}$, $\psi_1 = 2d_1$, and*

$$\psi_s = \frac{1}{2}\left\{2d_s + d_1 - A_1 + \sqrt{(2d_s - d_1 + A_1)^2 + 8A_1\sum_{t=1}^{s-1}(d_t - d_s)}\right\} \tag{3.3}$$

for $2 \leq s \leq n$. Then

$$\rho\bigl(\mathbb{Q}(\mathcal{H})\bigr) \leq \min_{1\leq s\leq n} \psi_s. \tag{3.4}$$

If $\mathcal{H}$ is connected, then the equality in (3.4) holds if and only if $\mathcal{H}$ is regular.

Proof Let $\mathbb{A} = \mathbb{Q}(\mathcal{H})$. We apply Theorem 2.1 to $\mathbb{Q}(\mathcal{H})$, then we have $M = d_1$, $N = \frac{1}{(k-1)!}$, $r_i = 2d_i$ for $1 \leq i \leq n$, $A_1 = N_1$, and (3.3) is from (2.1). Thus (3.4) holds by Theorem 2.1.

If $\mathcal{H}$ is connected, then by Theorem 2.1 the equality in (3.4) holds if and only if $r_1(\mathbb{Q}(\mathcal{H})) = r_2(\mathbb{Q}(\mathcal{H})) = \cdots = r_n(\mathbb{Q}(\mathcal{H}))$, which says exactly that $\mathcal{H}$ is regular, since $r_i(\mathbb{Q}(\mathcal{H})) = 2d_i$ for any $1 \leq i \leq n$. □

4 Applications to *k*-uniform directed hypergraph

Directed hypergraphs have found applications in imaging processing [8], optical network communications [15], computer science and combinatorial optimization [10]. However, unlike spectral theory of undirected hypergraphs, there are very few results in spectral theory of directed hypergraphs.

A directed hypergraph $\overrightarrow{\mathcal{H}}$ is a pair $(V(\overrightarrow{\mathcal{H}}), E(\overrightarrow{\mathcal{H}}))$, where $V(\overrightarrow{\mathcal{H}}) = [n]$ is the set of vertices and $E(\overrightarrow{\mathcal{H}}) = \{e_1, e_2, \ldots, e_m\}$ is the set of arcs. An arc $e \in E(\overrightarrow{\mathcal{H}})$ is a pair $e = (j_1, e(j_1))$, where

$e(j_1) = \{j_2, \ldots, j_t\}$, $j_l \in V(\overrightarrow{\mathcal{H}})$, and $j_l \neq j_h$ if $l \neq h$ for $l, h \in [t]$ and $t \in [n]$. The vertex j_1 is called the tail (or out-vertex) and every other vertex $j_2, \ldots, j_t$ is called a head (or in-vertex) of the arc e. The out-degree of a vertex $j \in V(\overrightarrow{\mathcal{H}})$ is defined as $d_j^+ = |E_j^+|$, where $E_j^+ = \{e \in E(\overrightarrow{\mathcal{H}}) : j$ is the tail of $e\}$. If for any $j \in V(\overrightarrow{\mathcal{H}})$, the degree d_j^+ has the same value d, then $\overrightarrow{\mathcal{H}}$ is called a directed d-out-regular hypergraph.

For a vertex $i \in V(\overrightarrow{\mathcal{H}})$, we denote by E_i the set of arcs containing the vertex i, i.e., $E_i = \{e \in E(\overrightarrow{\mathcal{H}}) : i \in e\}$. Two distinct vertices i and j are weak-connected if there is a sequence of arcs $(e_1, \ldots, e_t)$ such that $i \in e_1$, $j \in e_t$, and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [t-1]$. Two distinct vertices i and j are strong-connected, denoted by $i \to j$, if there is a sequence of arcs $(e_1, \ldots, e_t)$ such that i is the tail of e_1, j is a head of e_t, and a head of e_r is the tail of e_{r+1} for all $r \in [t-1]$. A directed hypergraph is called weakly connected if every pair of different vertices of $\overrightarrow{\mathcal{H}}$ is weak-connected. A directed hypergraph is called strongly connected if every pair of different vertices i and j of $\overrightarrow{\mathcal{H}}$ satisfies $i \to j$ and $j \to i$.

Similar to the definition of a k-uniform hypergraph, we define a k-uniform directed hypergraph as follows: A directed hypergraph $\overrightarrow{\mathcal{H}} = (V(\overrightarrow{\mathcal{H}}), E(\overrightarrow{\mathcal{H}}))$ is called a k-uniform directed hypergraph if $|e| = k$ for any arc $e \in E(\overrightarrow{\mathcal{H}})$. When $k = 2$, then $\overrightarrow{\mathcal{H}}$ is an ordinary digraph.

The following definition for the adjacency tensor and signless Laplacian tensor of a directed hypergraph was proposed by Chen and Qi in [5].

Definition 4.1 ([5]) Let $\overrightarrow{\mathcal{H}} = (V(\overrightarrow{\mathcal{H}}), E(\overrightarrow{\mathcal{H}}))$ be a k-uniform directed hypergraph. The adjacency tensor of the directed hypergraph $\overrightarrow{\mathcal{H}}$ is defined as the order k dimension n tensor $\mathbb{A}(\overrightarrow{\mathcal{H}})$, whose $(i_1 i_2 \cdots i_k)$-entry is

$$\bigl(\mathbb{A}(\overrightarrow{\mathcal{H}})\bigr)_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } (i_1, e(i_1)) \in E(\overrightarrow{\mathcal{H}}) \text{ and } e(i_1) = (i_2, \ldots, i_k), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbb{D}(\overrightarrow{\mathcal{H}})$ be an order k dimension n diagonal tensor with its diagonal entry $d_{ii\cdots i}$ being d_i^+, the out-degree of vertex i, for all $i \in V(\overrightarrow{\mathcal{H}}) = [n]$. Then $\mathbb{Q}(\overrightarrow{\mathcal{H}}) = \mathbb{D}(\overrightarrow{\mathcal{H}}) + \mathbb{A}(\overrightarrow{\mathcal{H}})$ is the signless Laplacian tensor of the directed hypergraph $\overrightarrow{\mathcal{H}}$.

Clearly, the adjacency tensor and the signless Laplacian tensor of a k-uniform directed hypergraph $\overrightarrow{\mathcal{H}}$ are nonnegative k-uniform tensors, but not symmetric in general. For any $1 \le i \le n$, we have

$$r_i\bigl(\mathbb{A}(\overrightarrow{\mathcal{H}})\bigr) = \sum_{i_2, \ldots, i_k = 1}^{n} \bigl(\mathbb{A}(\overrightarrow{\mathcal{H}})\bigr)_{i i_2 \cdots i_k} = d_i^+$$

and

$$r_i\bigl(\mathbb{Q}(\overrightarrow{\mathcal{H}})\bigr) = \sum_{i_2, \ldots, i_k = 1}^{n} \bigl(\mathbb{Q}(\overrightarrow{\mathcal{H}})\bigr)_{i i_2 \cdots i_k} = 2 d_i^+.$$

The following statement is an alternative explanation of weak irreducibility.

Definition 4.2 ([9, 12]) Suppose that $\mathbb{A} = (a_{i_1 i_2 \ldots i_k})_{1 \le i_j \le n (j=1,\ldots,k)}$ is a nonnegative tensor of order k and dimension n. We call a nonnegative matrix $G(\mathbb{A})$ the representation associated matrix to the nonnegative tensor $\mathbb{A}$ if the (i,j)th entry of $G(\mathbb{A})$ is defined to be the summation of $a_{ii_2 \ldots i_k}$ with indices $\{i_2, \ldots, i_k\} \ni j$. We call the tensor $\mathbb{A}$ weakly reducible if its representation $G(\mathbb{A})$ is a reducible matrix.

Let $A = (a_{ij})$ be a nonnegative square matrix of order n. The associated digraph $D(A) = (V, E)$ of A (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i,j) \mid a_{ij} > 0\}$.

Now we give a characterization of a strongly connected k-uniform directed hypergraph.

Theorem 4.3 *Let $\overrightarrow{\mathcal{H}}$ be a k-uniform directed hypergraph, $\mathbb{A} = \mathbb{A}(\overrightarrow{\mathcal{H}}) = (a_{i_1 i_2 \cdots i_k})$ be the adjacency tensor of $\overrightarrow{\mathcal{H}}$, $G(\mathbb{A})$ be the representation associated matrix of $\mathbb{A}$, and $D(G(\mathbb{A}))$ be the associated directed graph of $G(\mathbb{A})$. Then the following four conditions are equivalent*:

(i) *$\mathbb{A}$ is weakly irreducible.*
(ii) *$G(\mathbb{A})$ is irreducible.*
(iii) *$D(G(\mathbb{A}))$ is strongly connected.*
(iv) *$\overrightarrow{\mathcal{H}}$ is strongly connected.*

Proof By Proposition 15 in [27] and $\mathbb{A} = \mathbb{A}(\overrightarrow{\mathcal{H}})$ is a nonnegative tensor, we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Now we show (iii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (iv): Let $D(G(\mathbb{A}))$ is strongly connected, now we show $\overrightarrow{\mathcal{H}}$ is strongly connected.

For any $i, j \in V(\overrightarrow{\mathcal{H}}) = V(D(G(\mathbb{A})))$, there exists a directed path P from i to j in $D(G(\mathbb{A}))$ by $D(G(\mathbb{A}))$ being strongly connected. We assume $P = ij_1 j_2 \cdots j_t j$, then $(i, j_1), (j_1, j_2), \ldots, (j_t, j) \in E(D(G(\mathbb{A})))$, which implies $\sum_{j_1 \in \{i_2,\ldots,i_k\}} a_{ii_2 \cdots i_k} > 0$, $\sum_{j_2 \in \{i_2,\ldots,i_k\}} a_{j_1 i_2 \cdots i_k} > 0$, $\ldots$, $\sum_{j_t \in \{i_2,\ldots,i_k\}} a_{j_{t-1} i_2 \cdots i_k} > 0$, and $\sum_{j \in \{i_2,\ldots,i_k\}} a_{j_t i_2 \cdots i_k} > 0$, thus there exists a sequence of arcs $(e_1, e_2, \ldots, e_t, e_{t+1})$, where $e_l \in \overrightarrow{\mathcal{H}}$ and $l \in [t+1]$, such that i is the tail of e_1, j_1 is a head of e_1, j_l is the tail of e_{l+1}, j_{l+1} is a head of e_{l+1} for $1 \le l \le t-1$, j_t is the tail of e_{t+1}, j is a head of e_{t+1}, say, $i \to j$ in $\mathcal{H}$. Therefore $\overrightarrow{\mathcal{H}}$ is strongly connected.

(iv) $\Rightarrow$ (iii): Let $\overrightarrow{\mathcal{H}}$ be strongly connected. Now we show that $D(G(\mathbb{A}))$ is strongly connected.

For any $i, j \in V(D(G(\mathbb{A}))) = V(\overrightarrow{\mathcal{H}})$, $i \to j$ in $\overrightarrow{\mathcal{H}}$ by $\overrightarrow{\mathcal{H}}$ being strongly connected, say, there exists a sequence of arcs $(e_1, e_2, \ldots, e_t, e_{t+1})$, where $e_l \in \overrightarrow{\mathcal{H}}$ for $l \in [t+1]$, such that i is the tail of e_1, j is a head of e_{t+1}, and a head of e_r is the tail of e_{r+1} for all $r \in [t]$. We assume that j_r is the tail of e_{r+1} and a head of e_r for all $r \in [t]$, then $\sum_{j_1 \in \{i_2,\ldots,i_k\}} a_{ii_2 \cdots i_k} > 0$, $\sum_{j_{r+1} \in \{i_2,\ldots,i_k\}} a_{j_r i_2 \cdots i_k} > 0$ for $1 \le r \le t-1$, and $\sum_{j \in \{i_2,\ldots,i_k\}} a_{j_t i_2 \cdots i_k} > 0$. Thus $(i, j_1) \in E(D(G(\mathbb{A})))$, $(j_r, j_{r+1}) \in E(D(G(\mathbb{A})))$ for $1 \le r \le t-1$ and $(j_t, j) \in E(D(G(\mathbb{A})))$, which implies that there exists a walk $ij_1 j_2 \cdots j_t j$ in $D(G(\mathbb{A}))$. Therefore $D(G(\mathbb{A}))$ is strongly connected. □

Recently, several papers studied the spectral radii of the adjacency tensor $\mathbb{A}(\overrightarrow{\mathcal{H}})$ and the signless Laplacian tensor $\mathbb{Q}(\overrightarrow{\mathcal{H}})$ of a k-uniform directed hypergraph $\overrightarrow{\mathcal{H}}$ (see [5, 24] and so on).

Let $\overrightarrow{\mathcal{H}}$ be a k-uniform directed hypergraph. If $\overrightarrow{\mathcal{H}}$ is strongly connected, then by Theorem 4.3 and the above definitions, $\mathbb{A}(\overrightarrow{\mathcal{H}})$ and thus $\mathbb{Q}(\overrightarrow{\mathcal{H}})$ are weakly irreducible. Thus we can apply Theorem 2.1 to the adjacency tensor $\mathbb{A}(\overrightarrow{\mathcal{H}})$ and the signless Laplacian tensor $\mathbb{Q}(\overrightarrow{\mathcal{H}})$ of a (strongly connected) k-uniform directed hypergraph $\overrightarrow{\mathcal{H}}$. If $k = 2$, we obtain Theorem 2.7 in [11]. If $k \geq 3$, we obtain some new results about the bounds of $\rho(\mathbb{A}(\overrightarrow{\mathcal{H}}))$ and $\rho(\mathbb{Q}(\overrightarrow{\mathcal{H}}))$ as follows.

Theorem 4.4 *Let $k \geq 3$, $\overrightarrow{\mathcal{H}}$ be a k-uniform directed hypergraph with out-degree sequence $d_1^+ \geq \cdots \geq d_n^+$, $\mathbb{A}(\overrightarrow{\mathcal{H}})$ be the adjacency tensor of $\overrightarrow{\mathcal{H}}$. Let $A_1 = \frac{1}{k-1}\binom{n-2}{k-2}$, $\phi_1 = d_1^+$, and*

$$\phi_s = \frac{1}{2}\left\{d_s^+ - A_1 + \sqrt{\left(d_s^+ + A_1\right)^2 + 4A_1\sum_{t=1}^{s-1}\left(d_t^+ - d_s^+\right)}\right\} \tag{4.1}$$

for $2 \leq s \leq n$. Then

$$\rho\left(\mathbb{A}(\overrightarrow{\mathcal{H}})\right) \leq \min_{1\leq s\leq n} \phi_s. \tag{4.2}$$

Moreover, if $\overrightarrow{\mathcal{H}}$ is a strongly connected k-uniform directed hypergraph, then the equality in (4.2) holds if and only if $d_1^+ = d_2^+ = \cdots = d_n^+$.

Proof Let $\mathbb{A} = \mathbb{A}(\overrightarrow{\mathcal{H}})$. We apply Theorem 2.1 to $\mathbb{A}(\overrightarrow{\mathcal{H}})$, then we have $M = 0$, $N = \frac{1}{(k-1)!}$, $r_i = d_i^+$ for $1 \leq i \leq n$, $A_1 = N_1$, and (4.1) is from (2.1). Thus (4.2) holds by Theorem 2.1, and the equality in (4.2) holds if and only if $d_1^+ = d_2^+ = \cdots = d_n^+$ by Theorem 2.1 and Theorem 4.3. □

Theorem 4.5 *Let $k \geq 3$, $\overrightarrow{\mathcal{H}}$ be a k-uniform directed hypergraph with out-degree sequence $d_1^+ \geq \cdots \geq d_n^+$, $\mathbb{Q}(\overrightarrow{\mathcal{H}})$ be the signless Laplacian tensor of $\overrightarrow{\mathcal{H}}$. Let $A_1 = \frac{1}{k-1}\binom{n-2}{k-2}$, $\psi_1 = 2d_1^+$, and*

$$\psi_s = \frac{1}{2}\left\{2d_s^+ + d_1^+ - A_1 + \sqrt{\left(2d_s^+ - d_1^+ + A_1\right)^2 + 8A_1\sum_{t=1}^{s-1}\left(d_t^+ - d_s^+\right)}\right\} \tag{4.3}$$

for $2 \leq s \leq n$. Then

$$\rho\left(\mathbb{Q}(\overrightarrow{\mathcal{H}})\right) \leq \min_{1\leq s\leq n} \psi_s. \tag{4.4}$$

Moreover, if $\overrightarrow{\mathcal{H}}$ is a strongly connected k-uniform directed hypergraph, then the equality in (4.4) holds if and only if $d_1^+ = d_2^+ = \cdots = d_n^+$.

Proof Let $\mathbb{A} = \mathbb{Q}(\overrightarrow{\mathcal{H}})$. We apply Theorem 2.1 to $\mathbb{Q}(\overrightarrow{\mathcal{H}})$, then we have $M = d_1^+$, $N = \frac{1}{(k-1)!}$, $r_i = 2d_i^+$ for $1 \leq i \leq n$, $A_1 = N_1$, and (4.3) is from (2.1). Thus (4.4) holds by Theorem 2.1, and the equality in (4.4) holds if and only if $d_1^+ = d_2^+ = \cdots = d_n^+$ by Theorem 2.1 and Theorem 4.3. □

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Abbreviations
P.R. China, People's Republic of China; MOE-LSC, Key Laboratory of Scientific and Engineering Computing (Ministry of Education); SHL-MAC, Shanghai municipal education commission key laboratory of multi-physics modeling analysis and computation; Grant Nos, Grant Numbers; Grant No, Grant Number; i.e., id est; Commun. Math. Sci., Communications in Mathematical Sciences; Front. Math. China, Frontiers of Mathematics in China; J. Ind. Manag. Optim., Journal of Industrial and Management Optimization; Linear Algebra Appl., Linear Algebra and Its Applications; Discrete Appl. Math., Discrete Applied Mathematics; Sci. China Math., Science China-Mathematics; Inform. Process. Lett., Information Processing Letters; Numer. Math., Numerische Mathematik; IEEE,: Institute of Electrical and Electronics Engineers; CAMSAP, Computational Advances in Multi-Sensor Adaptive Processing; Appl. Math. Comput., Applied Mathematics and Computation; Graphs Combin.,: Graphs and Combinatorics; J. Symbolic Comput.,: Journal of Symbolic Computation; SIAM J. Matrix Anal. Appl.,: SIAM Journal on Matrix Analysis and Applications..

Availability of data and materials
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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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