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Fixed point theorems for sum operator with parameter

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Abstract

This paper develops some new existence and uniqueness theorems of a fixed point for a class of sum operator equations with parameter $\lambda_1 A(x,x) + \lambda_2 B(x,x) + \lambda_3 Cx + \lambda_4 Dx = x$, where A, B are two mixed monotone operators, C is an increasing operator, D is a decreasing operator. In the case of positive parameters, the results obtained in this paper extend many existing conclusions in the field of study. Furthermore, by using the properties of Green's function and the above fixed point theorems of sum operator, the unique positive solution a class of fractional differential equations with integro-differential boundary value conditions is given. Application of the results to the study of fractional differential equations is also given in the article.

Keywords: Mixed monotone operators; Concave-convex operator; Operator equation; Positive solution; Fixed point theorem; Fractional differential equation

1 Introduction

This paper discusses the existence and uniqueness of solution for a class of operator equation

$$\lambda_1 A(x,x) + \lambda_2 B(x,x) + \lambda_3 Cx + \lambda_4 Dx = x,$$

where A, B are two mixed monotone operators, C is an increasing operator, D is a decreasing operator, $\lambda_i > 0$ $(i = 1,2,3,4)$ and satisfies the following conditions:

Situation 1:

1. $A(\lambda x, \lambda^{-1} y) \geq \varphi(\lambda) A(x,y)$, $\varphi(\lambda) \in (\lambda, 1]$, $\forall \lambda \in (0,1)$, $x, y \in P$;
2. for any fixed $y \in P$, $B(\cdot, y) : P \to P$ is concave; for any $x \in P$, $B(x, \cdot) : P \to P$ is convex;
3. $C : P \to P$ is increasing sub-homogeneous.
4. $D(\lambda^{-1} y) \geq \lambda Dy$, $\forall \lambda \in (0,1)$, $y \in P$.

Situation 2:

1. $A(\lambda x, \lambda^{-1} y) \geq \lambda A(x,y)$, $\forall \lambda \in (0,1)$, $x, y \in P$;
2. for any fixed $y \in P$, $B(\cdot, y) : P \to P$ is concave; for any $x \in P$, $B(x, \cdot) : P \to P$ is convex;
3. $C(\lambda x) \geq \varphi(\lambda) Cx$, $\varphi(\lambda) \in (\lambda, 1]$, $\forall \lambda \in (0,1)$, $x, y \in P$;
4. $D(\lambda^{-1} y) \geq \lambda Dy$, $\forall \lambda \in (0,1)$, $y \in P$.

In fact, the operator equation $\lambda_1 A(x,x) + \lambda_2 B(x,x) + \lambda_3 Cx + \lambda_4 Dx = x$ generalizes and improves many articles.

When the parameter $\lambda_3 = 1$, $\lambda_i = 0$ $(i = 1,2,4)$, situation 2 can be reduced to the case that an increasing operator C meets $C(tx) \geq t^{\alpha(t)} Cx$, $\alpha(t) \in (0,1)$, $\forall t \in (0,1)$, $x \in P_h$. Obviously, it is the result of the paper [1].

When the parameter $\lambda_1 = 1$, $\lambda_i = 0$ $(i = 2,3,4)$, there is $A(\lambda x, \lambda^{-1} y) \geq \varphi(\lambda) A(x,y)$, $\varphi(\lambda) \in (\lambda, 1]$, $\forall \lambda \in (0,1)$, $x, y \in P$ in situation 1. In [2], we can see that the mixed monotone operator A meets the same properties.

When the parameter $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_4 = 0$, we deduce from situation 1 that Theorem 2.1 of [3] is established and from situation 2 that Theorem 2.4 of [3] holds.

When the parameter $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_4 = 1$, we derive Theorem 3.1 of [4] from situation 1 and Theorem 3.8 of [4] from situation 1.

When the parameter $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 0$, the conditions of situation 1 are the same as those of Theorem 3.1 in [5], thus they can get same conclusions.

The theory of nonlinear operators plays a significant role in modern mathematics and there are many excellent results (see [1–7]). Inspired by the these paper, we want to make some contributions to this field. As one of the applications of operator theory, fractional differential equations have attracted much attention by many researchers as a result of a myriad of their applications in many engineering and scientific disciplines, such as mechanics, biomedicine, physics, and so on, see [8–15] and the references therein. Besides, there are some methods such as comparison theorem, the monotone iterative technique, the method of lower and upper solutions, Leray-Schauder theory, Krasnoselskii's fixed point theorems, and some other fixed point theorems in cones. They play an irreplaceable role in the existence, uniqueness, and multiplicity of positive solutions for fractional differential equations [16–22]. In recent decades, more and more fractional differential equations are solved based on the nonlinear operators theory [23–25]. In this paper, we make use of the new operator equation theory to investigate a class of new fractional differential equations.

The characteristic features of this paper are displayed as follows. Firstly, comparatively speaking, we generalize the results of the above article. Secondly, there are seldom investigated operator equations with parameters. A class of new operator equations with four operators $\lambda_1 A(x,x) + \lambda_2 B(x,x) + \lambda_3 Cx + \lambda_4 Dx = x$ is studied, and the fixed point theorem of this sum-type operator is obtained. We gain the existence and uniqueness solution of the operator equation, and construct two iterative sequences to uniformly approximate this solution in the fixed point theorem. Thirdly, by using the new results, we study a class of new fractional differential equations and get some great conclusions. Fourthly, some concrete examples are given to illustrate the main ideas.

The main body of this paper is organized as follows. In Sect. 2, we review the theory and results of fractional calculus and some definitions, notations in Banach space. In Sect. 3, a class of fixed point theorem is presented. In Sect. 4, by the theorem of Sect. 2, a kind of fractional differential equation is studied and some examples to illustrative our work are presented.

2 Preliminaries

In this section, since all the work is in the Banach space, a brief review about the Banach space and relevant contents is given for the reader's convenience; it includes some definitions, lemmas, and basic results. These will be used in the following proofs of our theorem. For more details, we refer the reader to [8–12].

Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if the following hold:

(i) If $x \in P$, $\lambda \geq 0$, then $\lambda x \in P$;

(ii) If $x \in P$ and $-x \in P$, then $x = \theta$,

where θ is the zero element of E.

There is partially ordered by a cone $P \subset E$, $x \leq y$, $x, y \in E \Leftrightarrow y - x \in P$, in which, if $x \leq y$ and $x \neq y$, we denote $x < y$ or $y > x$.

If $\dot{P} = \{x \in P | x \text{ is an interior point of } P\}$ is nonempty, cone P is said to be solid. Then, we call P normal if there exists a constant $N > 0$ such that, for $\forall x, y \in E$, $\theta \leq x \leq y$, there is $\|x\| \leq N\|y\|$, where N is called the normality constant of P. For $\forall x, y \in E$, the denotation $x \sim y$ means that there exist $\lambda, \mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$, and denoting the set $P_h = \{x \in E \mid x \sim h\}$, there is $P_h \subset P$.

Definition 2.1 ([8]) $A : P \times P \to P$ is called a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ implies $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.2 ([9, 10]) An operator $A : E \to E$ is said to be homogeneous if it satisfies

$$A(tx) = tAx, \quad \forall t > 0, x \in E.$$

An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tAx, \quad \forall t \in (0, 1), x \in P.$$

Definition 2.3 ([9, 10]) Let $D = P$ or $D = \dot{P}$ and a real number $\alpha \in [0, 1)$. An operator $A : D \to D$ is said to be α-concave if it satisfies

$$A(tx) \geq t^{\alpha} Ax, \quad \forall t \in (0, 1), x \in D.$$

Definition 2.4 ([11]) Let D be a convex subset in E. An operator $A : D \to E$ is called a convex operator if, for $\forall x, y \in D$ with $y \leq x$ and every $t \in [0, 1]$,

$$A\big(tx + (1 - t)y\big) \leq tAx + (1 - t)Ay,$$

$A : D \to E$ is called a concave operator if

$$A\big(tx + (1 - t)y\big) \geq tAx + (1 - t)Ay.$$

3 Main results

In this section, we use the definitions, notations of section Preliminaries to investigate some new fixed point theorem. Furthermore, we can obtain the following sufficient conditions of existence and uniqueness of positive solutions for the operator equation $\lambda_1 A(x, x) + \lambda_2 B(x, x) + \lambda_3 Cx + \lambda_4 Dx = x$.

Theorem 3.1 *There are four operators A, B, C, D, where $A, B : P \times P \to P$ are two mixed monotone operators, $C : P \to P$ is an increasing sub-homogeneous operator, $D : P \to P$ is a decreasing operator, and if the following conditions are satisfied*:

(A_1) *There exists $\varphi(\lambda) \in (\lambda, 1]$ such that*

$$A\big(\lambda x, \lambda^{-1} y\big) \geq \varphi(\lambda) A(x, y), \qquad D\big(\lambda^{-1} y\big) \geq \lambda D y, \quad \forall \lambda \in (0,1), x, y \in P; \tag{1}$$

(A_2) *For any fixed $y \in P$, $B(\cdot, y) : P \to P$ is concave; for any $x \in P$, $B(x, \cdot) : P \to P$ is convex*;

(A_3) *There exists $\frac{1}{2} \leq \widetilde{c} \leq 1$ such that $B(\theta, lh) \geq \widetilde{c} B(lh, \theta)$, $l \geq 1$*;

(A_4) *There exists $h \in P$ with $h \neq \theta$ such that $A(h,h), B(h,h), Ch, Dh \in P_h$*;

(A_5) *There exists $\delta > 0$ such that $[\lambda_2 B(x,y) + \lambda_3 Cx + \lambda_4 Dy] \leq \delta \lambda_1 A(x,y)$, $\forall x, y \in P_h$.*

Then the following conclusions hold:

$(C1)$ $A, B : P_h \times P_h \to P_h$, $C, D : P_h \to P_h$;

$(C2)$ *There exist $u_0, v_0 \in P_h$, $r \in (0,1)$ such that $r v_0 \leq u_0 < v_0$,*

$$\begin{aligned} u_0 &\leq \lambda_1 A(u_0, v_0) + \lambda_2 B(u_0, v_0) + \lambda_3 C u_0 + \lambda_4 D v_0 \\ &\leq \lambda_1 A(v_0, u_0) + \lambda_2 B(v_0, u_0) + \lambda_3 C v_0 + \lambda_4 D u_0 \leq v_0; \end{aligned}$$

$(C3)$ *The operator equation $\lambda_1 A(x,x) + \lambda_2 B(x,x) + \lambda_3 Cx + \lambda_4 Dx = x$ has a unique solution $x^* \in P_h$*;

$(C4)$ *Constructing the iterative sequences as follows*:

$$\begin{aligned} x_n &= \lambda_1 A(x_{n-1}, y_{n-1}) + \lambda_2 B(x_{n-1}, y_{n-1}) + \lambda_3 C x_{n-1} + \lambda_4 D y_{n-1}, \\ y_n &= \lambda_1 A(y_{n-1}, x_{n-1}) + \lambda_2 B(y_{n-1}, x_{n-1}) + \lambda_3 C y_{n-1} + \lambda_4 D x_{n-1}, \quad n = 1, 2, \ldots, \end{aligned}$$

for any initial values $x_0, y_0 \in P_h$, we have

$$x_n \to x^*, \qquad y_n \to x^*, \quad \text{as } n \to \infty.$$

Proof Define the operator $T = \lambda_1 A + \lambda_2 B + \lambda_3 C + \lambda_4 D$ by

$$T(x,y) = \lambda_1 A(x,y) + \lambda_2 B(x,y) + \lambda_3 Cx + \lambda_4 Dy, \quad \forall x, y \in P.$$

Firstly, we show that $T : P_h \times P_h \to P_h$. The proof of $A, B : P_h \times P_h \to P_h$, $C, D : P_h \to P_h$ is expanded. By $A(\lambda x, \lambda^{-1} y) \geq \varphi(\lambda) A(x,y)$ of (A_1), we deduce

$$A\big(\lambda^{-1} x, \lambda y\big) \leq \varphi(\lambda)^{-1} A(x,y). \tag{2}$$

Furthermore, the condition $A(h,h) \in P_h$ of (A_4) shows that there exist positive constants $\overline{u_1}$, $\overline{v_1}$ such that

$$\overline{u_1} h \leq A(h,h) \leq \overline{v_1} h. \tag{3}$$

For $\forall x, y \in P_h$, we can find two sufficiently small numbers $c_1, c_2 \in (0,1)$ such that

$$c_1 h \leq x \leq c_1^{-1} h, \qquad c_2 h \leq y \leq c_2^{-1} h. \tag{4}$$

Let $c = \min\{c_1, c_2\}$, then by (1)–(4), we get

$$\begin{aligned}
A(x,y) &\le A\bigl(c_1^{-1}h, c_2 h\bigr) \le A\bigl(c^{-1}h, ch\bigr) \le \varphi(c)^{-1}A(h,h) \le \varphi(c)^{-1}\overline{v_1}h,\\
A(x,y) &\ge A\bigl(c_1 h, c_2^{-1}h\bigr) \ge A\bigl(ch, c^{-1}h\bigr) \ge \varphi(c)A(h,h) \ge \varphi(c)\overline{u_1}h.
\end{aligned}$$

Thus, $A : P_h \times P_h \to P_h$ is proved. Then, due to (A_2), for any $\lambda \in (0,1)$, there is

$$B(x,y) = B\bigl(x, \lambda\lambda^{-1}y + (1-\lambda)\theta\bigr) \le \lambda B\bigl(x, \lambda^{-1}y\bigr) + (1-\lambda)B(x,\theta),$$

thus, $\lambda B(x,\lambda^{-1}y) \ge B(x,y) - (1-\lambda)B(x,\theta)$. Subsequently, we can find a sufficiently large l such that $x, y, \lambda^{-1}y \le lh$. Combining with Definition 2.4 and from the condition of (A_2)–(A_3), we know that

$$\begin{aligned}
B\bigl(\lambda x, \lambda^{-1}y\bigr) &= B\bigl(\lambda x + (1-\lambda)\theta, \lambda^{-1}y\bigr)\\
&\ge \lambda B\bigl(x, \lambda^{-1}y\bigr) + (1-\lambda)B\bigl(\theta, \lambda^{-1}y\bigr)\\
&\ge B(x,y) - (1-\lambda)B(x,\theta) + (1-\lambda)B\bigl(\theta, \lambda^{-1}y\bigr)\\
&\ge B(x,y) + (1-\lambda)\bigl(B(\theta, lh) - B(lh,\theta)\bigr)\\
&\ge B(x,y) + (1-\lambda)\left[B(\theta, lh) - \frac{1}{\widetilde{c}}B(\theta, lh)\right]\\
&\ge \left[1 + (1-\lambda)\left(1 - \frac{1}{\widetilde{c}}\right)\right]B(x,y)\\
&= \left[\left(2 - \frac{1}{\widetilde{c}}\right) + \left(\frac{1}{\widetilde{c}} - 1\right)\lambda\right]B(x,y)\\
&\ge \lambda B(x,y),
\end{aligned}$$

that is,

$$B\bigl(\lambda x, \lambda^{-1}y\bigr) \ge \lambda B(x,y). \tag{5}$$

Then we gain

$$B\bigl(\lambda^{-1}x, \lambda y\bigr) \le \lambda^{-1}B(x,y), \quad \lambda \in (0,1).$$

From the condition $B(h,h) \in P_h$, there exist two positive constants $\overline{u_2}$, $\overline{v_2}$ such that $\overline{u_2}h \le B(h,h) \le \overline{v_2}h$. For any $x, y \in P_h$,

$$\begin{aligned}
B(x,y) &\le B\bigl(c_1^{-1}h, c_2 h\bigr) \le B\bigl(c^{-1}h, ch\bigr) \le c^{-1}B(h,h) \le c^{-1}\overline{v_2}h,\\
B(x,y) &\ge B\bigl(c_1 h, c_2^{-1}h\bigr) \ge B\bigl(ch, c^{-1}h\bigr) \ge cB(h,h) \ge c\overline{u_2}h.
\end{aligned}$$

Thus, $B : P_h \times P_h \to P_h$ holds. Since $Ch, Dh \in P_h$,

$$\overline{u_3}h \le Ch \le \overline{v_3}h, \qquad \overline{u_4}h \le Dh \le \overline{v_4}h, \tag{6}$$

where $\overline{u_i}$, $\overline{v_i}$ ($i = 3, 4$) are positive constants. By the properties of operators C, D, for any $\lambda \in (0,1)$, $x, y \in P$, there is

$$C(tx) \geq tCx, \qquad C\bigl(t^{-1}x\bigr) \leq t^{-1}Cx, \qquad D\bigl(t^{-1}x\bigr) \geq tDx, \qquad D(tx) \leq t^{-1}Dx.$$

Using similar processes, we have

$$\begin{aligned}
&Cx \leq C\bigl(c_1^{-1}h\bigr) \leq C\bigl(c^{-1}h\bigr) \leq c^{-1}Ch \leq c^{-1}\overline{v_3}h,\\
&Cx \geq C(c_1h) \geq C(ch) \geq cCh \geq c\overline{u_3}h,\\
&Dy \leq D(c_2h) \leq D(ch) \leq c^{-1}Dh \leq c^{-1}\overline{v_4}h,\\
&Dy \geq D\bigl(c_2^{-1}h\bigr) \geq D\bigl(c^{-1}h\bigr) \geq cDh \geq c\overline{u_4}h,
\end{aligned}$$

which shows that $C, D : P_h \to P_h$. From the above deduction, we have

$$\begin{aligned}
&\bigl(\lambda_1\varphi(c)\overline{u_1} + \lambda_2 c\overline{u_2} + \lambda_3 c\overline{u_3} + \lambda_4 c\overline{u_4}\bigr)h\\
&\quad \leq T(x,y) = \lambda_1 A(x,y) + \lambda_2 B(x,y) + \lambda_3 Cx + \lambda_4 Dy\\
&\quad \leq \bigl(\lambda_1\varphi(c)^{-1}\overline{v_1} + \lambda_2 c^{-1}\overline{v_2} + \lambda_3 c^{-1}\overline{v_3} + \lambda_4 c^{-1}\overline{v_4}\bigr)h,
\end{aligned}$$

i.e., $T : P_h \times P_h \to P_h$, and then we can get $T(h,h) \in P_h$.

Secondly, we demonstrate that there exists $\eta(t,x,y) \in (t,1]$ such that

$$T\bigl(tx, t^{-1}y\bigr) \geq \eta(t,x,y)T(x,y), \quad \forall t \in (0,1), x, y \in P_h.$$

Combining with (1), (5) and the properties of operator C, D, we have

$$\begin{aligned}
T\bigl(tx, t^{-1}y\bigr) &= \lambda_1 A\bigl(tx, t^{-1}y\bigr) + \lambda_2 B\bigl(tx, t^{-1}y\bigr) + \lambda_3 C(tx) + \lambda_4 D\bigl(t^{-1}y\bigr)\\
&\geq \lambda_1\varphi(t)A(x,y) + \lambda_2 tB(x,y) + \lambda_3 tCx + \lambda_4 tDy\\
&= \varphi(t)\lambda_1 A(x,y) + t\bigl[\lambda_2 B(x,y) + \lambda_3 Cx + \lambda_4 Dy\bigr].
\end{aligned}$$

Owing to $A, B : P_h \times P_h \to P_h$, $C, D : P_h \to P_h$, there is $\lambda_1 A(x,y) \sim \lambda_2 B(x,y) + \lambda_3 Cx + \lambda_4 Dy$. Define $K\{\frac{x}{y}\} = \inf\{k \in R | x \leq ky\}$, and let

$$J(x,y) = K\left(\frac{\lambda_2 B(x,y) + \lambda_3 Cx + \lambda_4 Dy}{\lambda_1 A(x,y)}\right),$$

we can easily get $J(x,y) \leq \delta$ by (A_5). Next, considering a function $\overline{h}(s) = \frac{\varphi(t)+J(x,y)t}{(J(x,y)+1)s}$, it shows that h is continuous and strictly decreasing about s. Due to $\varphi(t) > t$, there is

$$\overline{h}\left(\frac{\delta t + \varphi(t)}{\delta + 1}\right) = \frac{\varphi(t) + J(x,y)t}{(J(x,y)+1)\frac{\delta t+\varphi(t)}{\delta+1}} > 1, \qquad \overline{h}\bigl(\varphi(t)\bigr) = \frac{\varphi(t) + J(x,y)t}{(J(x,y)+1)\varphi(t)} < 1.$$

Because of the monotone decreasing of $\overline{h}$, we can get that there exist $\eta(t,x,y) \in (\frac{\delta t+\varphi(t)}{\delta+1}, \varphi(t)) \subset (t,1]$,

$$\overline{h}\bigl(\eta(t,x,y)\bigr) = \frac{\varphi(t) + J(x,y)t}{(J(x,y)+1)\eta(t,x,y)} = 1,$$

thus, $J(x,y)=\frac{\varphi(t)-\eta(t,x,y)}{\eta(t,x,y)-t}$. By the expression of K, J, there is

$$\lambda_2 B(x,y)+\lambda_3 Cx+\lambda_4 Dy \le \frac{\varphi(t)-\eta(t,x,y)}{\eta(t,x,y)-t}\lambda_1 A(x,y), \quad \forall t\in(0,1), x,y\in P_h.$$

Thus, we obtain

$$\begin{aligned}
T\bigl(tx,t^{-1}y\bigr) &\ge \varphi(t)\lambda_1 A(x,y)+t\bigl[\lambda_2 B(x,y)+\lambda_3 Cx+\lambda_4 Dy\bigr] \\
&= \eta(t,x,y)\lambda_1 A(x,y)+\bigl(\varphi(t)-\eta(t,x,y)\bigr)\lambda_1 A(x,y) \\
&\quad + t\bigl[\lambda_2 B(x,y)+\lambda_3 Cx+\lambda_4 Dy\bigr] \\
&\ge \eta(t,x,y)\lambda_1 A(x,y)+\bigl(\varphi(t)-\eta(t,x,y)\bigr)\frac{\eta(t,x,y)-t}{\varphi(t)-\eta(t,x,y)} \\
&\quad \times \bigl[\lambda_2 B(x,y)+\lambda_3 Cx+\lambda_4 Dy\bigr] \\
&\quad + t\bigl[\lambda_2 B(x,y)+\lambda_3 Cx+\lambda_4 Dy\bigr] \\
&= \eta(t,x,y)\bigl[\lambda_1 A(x,y)+\lambda_2 B(x,y)+\lambda_3 Cx+\lambda_4 Dy\bigr] \\
&= \eta(t,x,y)T(x,y), \quad \forall t\in(0,1), x,y\in P_h.
\end{aligned}$$

Therefore, there exists $\eta(t,x,y)\in(t,1]$ such that

$$T\bigl(tx,t^{-1}y\bigr)\ge \eta(t,x,y)T(x,y), \quad \forall t\in(0,1), x,y\in P_h. \tag{7}$$

Thirdly, we prove conclusion $(C2)$. Owing to $T(h,h)\in P_h$, we have $t_0 h\le T(h,h)\le t_0^{-1}h$, where $t_0\in(0,1)$ is a small constant we choose. As a result of $\eta(t,x,y)\in(t,1]$, we have $1<\frac{\eta(t_0,x,y)}{t_0}\le\frac{1}{t_0}$. By the Archimedes principle, we can take a positive integer k such that $(\frac{\eta(t_0,x,y)}{t_0})^k\ge\frac{1}{t_0}$. This inequality can be rewritten as

$$\eta(t_0,x,y)\ge\left(\frac{1}{t_0}\right)^{\frac{1}{k}}\cdot t_0, \qquad \frac{1}{\eta(t_0,x,y)}\le\frac{1}{t_0}\cdot t_0^{\frac{1}{k}}.$$

Set $u_0=t_0^k h$, $v_0=t_0^{-k}h$. We can get $u_0,v_0\in P_h$, $u_0=t_0^{2k}v_0<v_0$. Consequently, there exists $r\in(0,t^{2k}]$ such that $r\in(0,1)$, $u_0\ge rv_0$. In addition, thanks to the mixed monotone properties of A, B, the increasing properties of C, and the decreasing properties of D, T is a mixed monotone operator, and then $T(u_0,v_0)\le T(v_0,u_0)$. From (7), there is

$$\begin{aligned}
T(u_0,v_0) &= T\bigl(t_0^k h,t_0^{-k}h\bigr) \\
&= T\bigl(t_0\cdot t_0^{k-1}h,t_0^{-1}\cdot t_0^{-k+1}h\bigr) \\
&\ge \eta\bigl(t_0,t_0^{k-1}h,t_0^{-k+1}h\bigr)T\bigl(t_0^{k-1}h,t_0^{-k+1}h\bigr) \\
&= \eta\bigl(t_0,t_0^{k-1}h,t_0^{-k+1}h\bigr)T\bigl(t_0\cdot t_0^{k-2}h,t_0^{-1}\cdot t_0^{-k+2}h\bigr) \\
&\ge \eta\bigl(t_0,t_0^{k-1}h,t_0^{-k+1}h\bigr)\cdot\eta\bigl(t_0,t_0^{k-2}h,t_0^{-k+2}h\bigr)T\bigl(t_0^{k-2}h,t_0^{-k+2}h\bigr)\ge\cdots \\
&\ge \left(\left(\frac{1}{t_0}\right)^{\frac{1}{k}}\cdot t_0\right)^k T(h,h)\ge\frac{1}{t_0}\cdot t_0^k t_0 h=t_0^k h=u_0.
\end{aligned}$$

By (7), we get $T(t^{-1}x, ty) \le \eta(t, t^{-1}x, ty)^{-1}T(x,y)$, $\forall t \in (0,1)$, $x, y \in P_h$. Hence,

$$\begin{aligned} T(v_0, u_0) &= T\bigl(t_0^{-k}h, t_0^k h\bigr) \\ &= T\bigl(t_0^{-1} \cdot t_0^{-k+1}h, t_0 \cdot t_0^{k-1}h\bigr) \\ &\le \eta\bigl(t_0, t_0^{-k}h, t_0^k h\bigr)^{-1} T\bigl(t_0^{-k+1}h, t_0^{k-1}h\bigr) \\ &= \eta\bigl(t_0, t_0^{-k}h, t_0^k h\bigr)^{-1} T\bigl(t_0^{-1} \cdot t_0^{-k+2}h, t_0 \cdot t_0^{k-2}h\bigr) \\ &\le \eta\bigl(t_0, t_0^{-k}h, t_0^k h\bigr)^{-1} \cdot \eta\bigl(t_0, t_0^{-k+1}h, t_0^{k-1}h\bigr)^{-1} T\bigl(t_0^{-k+2}h, t_0^{k-2}h\bigr) \le \cdots \\ &\le \left(\frac{1}{t_0} \cdot t_0^{\frac{1}{k}}\right)^k T(h,h) \le \frac{1}{t_0^k} \cdot t_0 t_0^{-1} h = \frac{1}{t_0^k} h = v_0. \end{aligned}$$

Therefore, there is $u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0$. That is $(C2)$.

Fourthly, we show that the operator equation $T(x,x) = x$ has a unique solution $x^* \in P_h$. For u_0, v_0, construct successively the sequences as follows:

$$u_n = T(u_{n-1}, v_{n-1}), \qquad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots, n.$$

Thus, by $(C2)$, we have $u_1 \le v_1$. And then due to the mixed monotone properties of operator T, there are $u_n \le v_n$, $n = 1, 2, 3, \ldots$, and

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{8}$$

Considering $x_0 \ge r y_0$ and (8), there is $x_n \ge x_0 \ge r y_0 \ge r y_n$ $(n = 1, 2, 3, \ldots)$. Put

$$t_n = \sup\{t > 0 | x_n \ge t y_n\}, \quad n = 1, 2, 3, \ldots.$$

It is clear that $x_n \ge t_n y_n$. Then from (8), there is $x_{n+1} \ge x_n \ge t_n y_n \ge t_n y_{n+1}$, $n = 1, 2, 3, \ldots$. Hence, $t_{n+1} \ge t_n$, i.e., $\{t_n\}$ is increasing about n, and $\{t_n\} \subset (0, 1]$. Assume $\lim_{n\to\infty} t_n \to t^*$, so $t^* = 1$. Otherwise, $t \in (0,1)$. From (7) and $t_n \le t^*$,

$$\begin{aligned} x_{n+1} &= T(x_n, y_n) \ge T\bigl(t_n y_n, t_n^{-1} x_n\bigr) = T\left(\frac{t_n}{t^*} t^* y_n, \frac{t^*}{t_n} \frac{1}{t^*} x_n\right) \\ &\ge \frac{t_n}{t^*} T\left(t^* y_n, \frac{1}{t^*} x_n\right) \ge \frac{t_n}{t^*} \eta\bigl(t^*, y_n, x_n\bigr) T(y_n, x_n) = \frac{t_n}{t^*} \eta\bigl(t^*, y_n, x_n\bigr) y_{n+1}, \end{aligned}$$

combining with the definition of t_n, there is $t_{n+1} \ge \frac{t_n}{t^*} \eta(t^*, y_n, x_n)$. Then $\eta(t^*, y_n, x_n) \le \frac{t^*}{t_n} t_{n+1}$. Owing to $t^* < \frac{\delta t^* + \varphi(t^*)}{\delta + 1} < \eta(t^*, y_n, x_n) \le \frac{t^*}{t_n} t_{n+1}$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \frac{t^*}{t_n} t_{n+1} = t^*$, we know that

$$t^* < \frac{\delta t^* + \varphi(t^*)}{\delta + 1} \le t^*,$$

which is a contradiction. Thus, $\lim_{n\to\infty} t_n = 1$. For any natural number p, there is

$$\theta \le x_{n+p} - x_n \le y_n - x_n \le y_n - t_n y_n = (1 - t_n) v_n \le (1 - t_n) v_0,$$

$$\theta \le y_n - y_{n+p} \le y_n - x_n \le (1 - t_n) v_0.$$

By the normality of cone P, there is

$$\|x_{n+p} - x_n\| \leq N(1 - t_n)\|v_0\| \to 0,$$
$$\|y_n - y_{n+p}\| \leq N(1 - t_n)\|v_0\| \to 0 \quad (n, p = 1, 2, 3, \ldots),$$

where N is the normality constant. This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Because of the complete continuity of A, there are $x_n \to x^*$, $y_n \to y^*$, when $n \to \infty$. And by (8), for $\forall x^*, y^* \in P_h$, we obtain that $x_n \leq x^* \leq y^* \leq y_n$, $\theta \leq y^* - x^* \leq y_n - x_n \leq (1 - t_n)v_0$. Since cone P is normal, we have

$$\left\|y^* - x^*\right\| \leq N(1 - t_n)\|v_0\| \to 0 \quad (n \to \infty),$$

hence, $y^* = x^*$. Set $z^* := y^* = x^*$, there is

$$x_{n+1} = T(x_n, y_n) \leq T\left(z^*, z^*\right) \leq T(y_n, x_n) = y_{n+1},$$

when $n \to \infty$, $z^* = T(z^*, z^*)$, i.e., z^* is a fixed point of T in P_h.

Next, we show that z^* is the unique fixed point of T. Assume that $\overline{z}$ is another fixed point of T. Thanks to $z^*, \overline{z} \in P_h$, there exist positive numbers a_1, a_2, b_1, b_2 such that

$$a_1 h \leq z^* \leq a_2 h, \qquad b_1 h \leq \overline{z} \leq b_2 h,$$

then $\overline{z} \leq b_2 h = \frac{b_2}{a_1} a_1 h \leq \frac{b_2}{a_1} z^*$, $\overline{z} \geq b_1 h = \frac{b_1}{a_2} a_2 h \geq \frac{b_1}{a_2} z^*$. We put

$$e_1 = \sup\left\{t > 0, tz^* \leq \overline{z} \leq t^{-1} z^*\right\}.$$

Consequently, $0 < e_1 \leq 1$, $e_1 z^* \leq \overline{z} \leq e_1^{-1} z^*$. Then $e_1 = 1$. Otherwise, $0 < e_1 < 1$. There is

$$\overline{z} = T(\overline{z}, \overline{z}) \geq T\left(e_1 z^*, e_1^{-1} z^*\right) \geq \eta\left(e_1, z^*, z^*\right) T\left(z^*, z^*\right) = \eta\left(e_1, z^*, z^*\right) z^*.$$

Hence, $\eta(e_1, z^*, z^*) \geq e_1$, which contradicts the definition of e_1. So, $e_1 = 1$. Therefore, $z^* = \overline{z}$, i.e., T has a unique fixed point x^* in P_h.

Eventually, we show that conclusion $(C4)$ holds. For any initial values $x_0, y_0 \in P_h$, we construct the iterative sequences:

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, 3, \ldots.$$

Due to $x_0, y_0 \in P_h$, we have $e_2 h \leq x_0 \leq e_2^{-1} h$, $e_3 h \leq y_0 \leq e_3^{-1} h$, where $e_2, e_3 \in (0, 1)$ are two small numbers. Put $e_* = \min\{e_2, e_3\}$, we deduce that $e_* \in (0, 1)$ and $e_* h \leq x_0$, $y_0 \leq e_*^{-1} h$. Then, by $e_* < \eta(e_*, x, y) \leq 1$, there is $1 < \frac{\eta(e_*, x, y)}{e_*} \leq \frac{1}{e_*}$. By the Archimedes principle, there exists a sufficiently large positive integer m such that

$$\frac{\eta(e_*, x, y)}{e_*} \geq \left(\frac{1}{e_*}\right)^{\frac{1}{m}}.$$

Set $\overline{u}_0 = e_*^m h$, $\overline{v}_0 = \frac{1}{e_*^m} h$. Obviously, $\overline{u}_0, \overline{v}_0 \in P_h$, and $\overline{u}_0 < x_0$, $y_0 < \overline{v}_0$. Put

$$\overline{u}_n = T(\overline{u}_{n-1}, \overline{v}_{n-1}), \qquad \overline{v}_n = T(\overline{v}_{n-1}, \overline{u}_{n-1}), \quad n = 1, 2, 3, \ldots.$$

Similarly, there exists $y^* \in P_h$ such that $T(y^*, y^*) = y^*$, $\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \overline{v}_n = y^*$. Because T has the unique fixed points in P_h, we get $x^* = y^*$. And by induction, $\overline{u}_n \leq x_n$, $y_n \leq \overline{v}_n$, $n = 1, 2, 3, \ldots$. Thanks to cone P being normal, there is $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$, i.e., conclusion $(C4)$. The proof is complete. □

Corollary 3.1 *There are four operators A, B, C, D, where $A, B : P \times P \to P$ are two mixed monotone operators, $C : P \to P$ is an increasing homogeneous operator, and $D : P \to P$ is a decreasing operator. Then, if (A_2)–(A_5) of Theorem 3.1 are satisfied and meet the following*:

(A_6) *There exists $\gamma \in (0, 1)$ such that*

$$A(\lambda x, \lambda^{-1} y) \geq \lambda^\gamma A(x, y), \qquad D(\lambda^{-1} y) \geq \lambda D y, \quad \forall \lambda \in (0, 1), x, y \in P;$$

then we can get conclusions $(C1)$–$(C4)$ of Theorem 3.1.

Corollary 3.2 *There are four operators A, B, C, D, where $A, B : P_h \times P_h \to P_h$ are two mixed monotone operators, $C : P_h \to P_h$ is an increasing sub-homogeneous operator, and $D : P_h \to P_h$ is a decreasing operator. Then, if (A_3), (A_5) of Theorem 3.1 are satisfied and have the following*:

(A_7) *There exists $\varphi(\lambda) \in (\lambda, 1]$ such that*

$$A(\lambda x, \lambda^{-1} y) \geq \varphi(\lambda) A(x, y), \qquad D(\lambda^{-1} y) \geq \lambda D y, \quad \forall x, y \in P_h;$$

(A_8) *For any fixed $y \in P_h$, $B(\cdot, y) : P_h \to P_h$ is concave; for any $x \in P_h$, $B(x, \cdot) : P_h \to P_h$ is convex; then we can get conclusions $(C2)$–$(C4)$ of Theorem 3.1.*

Corollary 3.3 *There are four operators A, B, C, D, where $A, B : P_h \times P_h \to P_h$ are two mixed monotone operators, $C : P_h \to P_h$ is an increasing homogeneous operator, and $D : P_h \to P_h$ is a decreasing operator. Then, if (A_3), (A_5), (A_7), (A_8) are satisfied, then we can get conclusions $(C2)$–$(C4)$ of Theorem 3.1.*

Theorem 3.2 *There are four operators A, B, C, D, where $A, B : P \times P \to P$ are two mixed monotone operators, $C : P \to P$ is an increasing operator, and $D : P \to P$ is a decreasing operator. Then, if conditions (A_2)–(A_4) of Theorem 3.1 and the following conditions are satisfied*:

(A_9) *There exists $\varphi(\lambda) \in (\lambda, 1]$ such that*

$$\begin{aligned} &A(\lambda x, \lambda^{-1} y) \geq \lambda A(x, y), \qquad C(\lambda x) \geq \varphi(\lambda) C x, \\ &D(\lambda^{-1} y) \geq \lambda D y, \quad \forall \lambda \in (0, 1), x, y \in P; \end{aligned} \tag{9}$$

(A_{10}) *There exists $\delta > 0$ such that $\lambda_1 A(x, y) + \lambda_2 B(x, y) + \lambda_4 D y \leq \delta \lambda_3 C x$, $\forall x, y \in P_h$; then we can get conclusions $(C1)$–$(C4)$ of Theorem 3.1.*

Proof Define the operator $T = \lambda_1 A + \lambda_2 B + \lambda_3 C + \lambda_4 D$ by

$$T(x, y) = \lambda_1 A(x, y) + \lambda_2 B(x, y) + \lambda_3 C x + \lambda_4 D y, \quad \forall x, y \in P.$$

Firstly, we show that $T : P_h \times P_h \to P_h$. According to Theorem 3.1, there is $B : P_h \times P_h \to P_h$, $D : P_h \to P_h$. Thus, we only prove that $A : P_h \times P_h \to P_h$, $C : P_h \to P_h$. Since (9), we obtain

$$A(\lambda^{-1}x, \lambda y) \le \lambda^{-1}A(x, y), \qquad C(\lambda^{-1}x) \le \varphi(\lambda)^{-1}Cx, \quad \forall \lambda \in (0, 1), x, y \in P. \tag{10}$$

By $A(h, h), Ch \in P_h$, (3)–(4), (6) hold. By deduction, for any $x, y \in P_h$, we can get

$$\begin{aligned}
&A(x,y) \le A(c_1^{-1}h, c_2h) \le A(c^{-1}h, ch) \le c^{-1}A(h,h) \le c^{-1}\overline{v_1}h,\\
&A(x,y) \ge A(c_1h, c_2^{-1}h) \ge A(ch, c^{-1}h) \ge cA(h,h) \ge c\overline{u_1}h,\\
&Cx \le C(c_1^{-1}h) \le C(c^{-1}h) \le \varphi(c)^{-1}Ch \le \varphi(\lambda)^{-1}\overline{v_3}h,\\
&Cx \ge C(c_1h) \ge C(ch) \ge \varphi(\lambda)Ch \ge \varphi(\lambda)\overline{u_3}h,
\end{aligned}$$

where $c = \min\{c_1, c_2\}$. It shows that $A : P_h \times P_h \to P_h$, $C : P_h \to P_h$. With $B : P_h \times P_h \to P_h$, $D : P_h \to P_h$, $T : P_h \times P_h \to P_h$ holds. Clearly, $T(h, h) \in P_h$.

Secondly, we prove that there exists $\eta(t, x, y) \in (t, 1]$ such that

$$T(tx, t^{-1}y) \ge \eta(t, x, y)T(x, y), \quad \forall t \in (0, 1), x, y \in P_h. \tag{11}$$

By $A, B : P_h \times P_h \to P_h$, $C, D : P_h \to P_h$, we get $\lambda_1A(x, y) + \lambda_2B(x, y) + \lambda_4Dy \sim \lambda_3Cx$. Set

$$J^*(x,y) = K\left(\frac{\lambda_1A(x,y) + \lambda_2B(x,y) + \lambda_4Dy}{\lambda_3Cx}\right).$$

Since (A_{10}), $J^*(x, y) \le \delta$. Considering a similar function $h^*(s) = \frac{\varphi(t)+J(x,y)t}{(J^*(x,y)+1)s}$ of Theorem 3.1, $J^*(x, y) = \frac{\varphi(t)-\eta(t,x,y)}{\eta(t,x,y)-t}$ holds by Theorem 3.1. Thus, there is

$$\lambda_1A(x,y) + \lambda_2B(x,y) + \lambda_4Dy \le \frac{\varphi(t) - \eta(t,x,y)}{\eta(t,x,y) - t}\lambda_3Cx, \quad \forall t \in (0,1), x, y \in P_h.$$

Combining with (5) and (9), for $\forall t \in (0, 1)$, $x, y \in P_h$, we obtain

$$\begin{aligned}
T(tx, t^{-1}y) &= \lambda_1A(tx, t^{-1}y) + \lambda_2B(tx, t^{-1}y) + \lambda_3C(tx) + \lambda_4D(t^{-1}y)\\
&\ge t\lambda_1A(x,y) + t\lambda_2B(x,y) + \varphi(t)\lambda_3Cx + t\lambda_4Dy\\
&= \eta(t,x,y)\lambda_3Cx + (\varphi(t) - \eta(t,x,y))\lambda_3Cx\\
&\quad + t[\lambda_1A(x,y) + \lambda_2B(x,y) + \lambda_4Dy]\\
&= \eta(t,x,y)[\lambda_1A(x,y) + \lambda_2B(x,y) + \lambda_3Cx + \lambda_4Dy]\\
&= \eta(t,x,y)T(x,y).
\end{aligned}$$

The next steps are the same as those of Theorem 3.1, then we can get conclusions $(C1)$–$(C4)$ of Theorem 3.1. □

Corollary 3.4 *There are four operators A, B, C, D, where $A, B : P \times P \to P$ are two mixed monotone operators, $C : P \to P$ is an increasing operator, and $D : P \to P$ is a decreasing operator. Then, if conditions (A_2)–(A_4) and (A_{10}) of Theorem 3.1 and the following conditions are satisfied*:

(A_{11}) *There exists $\gamma \in (0,1)$ such that*

$$A\bigl(\lambda x, \lambda^{-1}y\bigr) \geq \lambda A(x,y), \qquad C(\lambda x) \geq \lambda^{\gamma} Cx,$$
$$D\bigl(\lambda^{-1}y\bigr) \geq \lambda Dy, \quad \forall \lambda \in (0,1), x, y \in P;$$

then we can get conclusions $(C1)$–$(C4)$ of Theorem 3.1.

Corollary 3.5 *There are four operators A, B, C, D, where $A, B : P_h \times P_h \to P_h$ are two mixed monotone operators, $C : P_h \to P_h$ is an increasing sub-homogeneous operator, and $D : P_h \to P_h$ is a decreasing operator. Then, if conditions (A_3), (A_8), (A_{10}) and the following conditions are satisfied*:

(A_{12}) *There exists $\gamma \in (0,1)$ such that*

$$A\bigl(\lambda x, \lambda^{-1}y\bigr) \geq \lambda A(x,y), \qquad C(\lambda x) \geq \varphi(\lambda) Cx,$$
$$D\bigl(\lambda^{-1}y\bigr) \geq \lambda Dy, \quad \forall x, y \in P_h;$$

then we can get conclusions $(C2)$–$(C4)$ of Theorem 3.1.

4 Application

Nonlinear boundary value problems have attracted much attention for their applications in a variety of different areas such as mathematics, physics, biology, and so on. Thereafter there are a lot of interesting and important results, which include the uniqueness, existence, and multiplicity of positive solutions for the differential equation with the two-point, three-point, infinite-point boundary value problems or some integral boundary problems, etc. In the following, we study a class of integro-differential boundary problems for fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} x(t) + \lambda_1 f(t, x(t), x(t)) + \lambda_2 g(t, x(t), x(t)) + \lambda_3 \phi(t, x(t)) + \lambda_4 \psi(t, x(t)) = 0, \\ x^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-2, \\ D_{0^+}^{\beta} x(1) = \int_0^{\eta} x(s)\,\mathrm{d}s, \end{cases} \tag{12}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ is Riemann–Liouville fractional derivative, $0 < \beta < n-1 < \alpha \leq n$ $(n > 1)$, $\alpha - \beta - 1 > 0$, $t \in [0,1]$, $\eta \in (0,1]$.

Definition 4.1 ([8]) Suppose that $h \in C[0,1]$, $\alpha > 0$. Then the Riemann–Liouville fractional derivative of α order is defined to be

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} h(s)\,\mathrm{d}s.$$

The Riemann–Liouville fractional integral of α order is defined to be

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number $[\alpha]$, provided that the right-hand side is pointwise defined on $(0,1)$.

Lemma 4.1 *Let $h \in C[0,1]$, the function x is the solution of the following fractional differential equation*:

$$\begin{cases} D_{0^+}^{\alpha} x(t) + h(t) = 0, \quad n-1 < \alpha \leq n, \\ x^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ D_{0^+}^{\beta} x(1) = \int_0^{\eta} x(s)\,\mathrm{d}s, \end{cases} \tag{13}$$

if and only if x satisfies

$$x(t) = \int_0^1 G(t,s) h(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - A(t-s)^{\alpha-1} - \Gamma(\alpha-\beta)t^{\alpha-1}(\eta-s)^{\alpha}}{A\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - A(t-s)^{\alpha-1}}{A\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \Gamma(\alpha-\beta)t^{\alpha-1}(\eta-s)^{\alpha}}{A\Gamma(\alpha)}, & 0 \leq t \leq s \leq \eta \leq 1, \\ \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \eta \leq s, \end{cases} \tag{14}$$

where $A = \Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}$.

Proof Integrating on both sides of the first formula of (13), we can obtain

$$\begin{aligned} x(t) &= -I_{0^+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\mathrm{d}s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}. \end{aligned}$$

By the boundary condition $x^{(i)}(0) = 0$ $(2 \leq i \leq n-2)$, we can easily get $c_n = c_{n-1} = \cdots = c_2 = 0$. Then we can get the solution of equation (13) of the following form:

$$x(t) = -I_{0^+}^{\alpha} h(t) + c_1 t^{\alpha-1} = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\mathrm{d}s + c_1 t^{\alpha-1}. \tag{15}$$

By the equality $D_{0^+}^{\beta} t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}$, we get

$$D_{0^+}^{\beta} x(t) = -I_{0^+}^{\alpha-\beta} h(t) + D_{0^+}^{\beta} c_1 t^{\alpha-\beta-1} = -\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s)\,\mathrm{d}s + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.$$

Then integrating formula (15) from 0 to η, there is

$$\begin{aligned} \int_0^{\eta} x(s)\,\mathrm{d}s &= -\int_0^{\eta} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^{\eta} c_1 s^{\alpha-1}\,\mathrm{d}s \\ &= -\int_0^{\eta} \frac{(\eta-s)^{\alpha}}{\Gamma(\alpha+1)} h(s)\,\mathrm{d}s + c_1 \frac{\eta^{\alpha}}{\alpha}. \end{aligned}$$

By the condition $D_{0^+}^{\beta}x(1)=\int_0^{\eta}x(s)\,\mathrm{d}s$, we have

$$c_1=\frac{1}{A}\left\{\int_0^1\alpha(1-s)^{\alpha-\beta-1}h(s)\,\mathrm{d}s-\int_0^{\eta}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}(\eta-s)^{\alpha}h(s)\,\mathrm{d}s\right\},\tag{16}$$

where A is $\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^{\alpha}$. Substituting (16) in (15), we conclude

$$\begin{aligned}x(t)&=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,\mathrm{d}s+c_1t^{\alpha-1}\\&=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,\mathrm{d}s+\frac{\alpha t^{\alpha-1}}{A}\int_0^1(1-s)^{\alpha-\beta-1}h(s)\,\mathrm{d}s\\&\quad-\frac{t^{\alpha-1}\Gamma(\alpha-\beta)}{A\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha}h(s)\,\mathrm{d}s\\&=\int_0^1G(t,s)h(s)\,\mathrm{d}s,\end{aligned}$$

where $G(t,s)$ is defined as in (14). □

Lemma 4.2 *Let $G(t,s)$ be as given in (14). If $\Gamma(\alpha+1)\geq\Gamma(\alpha-\beta)\eta^{\alpha}$, then for any $t,s\in[0,1]$, we have*

$$\begin{aligned}0&\leq\frac{\Gamma(\alpha-\beta)\eta^{\alpha}[1-(1-s)^{\beta+1}](1-s)^{\alpha-\beta-1}t^{\alpha-1}}{A\Gamma(\alpha)}\\&\leq G(t,s)\\&\leq\frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)}.\end{aligned}\tag{17}$$

Proof When $0\leq s\leq t\leq 1$, $s\leq\eta$, by $s\leq\eta$, $\alpha\geq 0$, we observe that

$$(1-s)^{\alpha}>\left(1-\frac{s}{\eta}\right)^{\alpha}.$$

By $s\leq t$, $\alpha-\beta-1\geq 0$, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1}>(t-s)^{\alpha-1}.\tag{18}$$

Thus,

$$\begin{aligned}G(t,s)&=\frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}-A(t-s)^{\alpha-1}-\Gamma(\alpha-\beta)t^{\alpha-1}(\eta-s)^{\alpha}}{A\Gamma(\alpha)}\\&=\frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\Gamma(\alpha-\beta)t^{\alpha-1}\eta^{\alpha}(1-\frac{s}{\eta})^{\alpha}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\&\geq\frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\Gamma(\alpha-\beta)t^{\alpha-1}\eta^{\alpha}(1-s)^{\alpha}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\&=\frac{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^{\alpha}(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\&=\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-\beta)\eta^{\alpha}[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)}\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad + \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} \\
&\geq \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)} \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
G(t,s) &= \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - A(t-s)^{\alpha-1} - \Gamma(\alpha-\beta)t^{\alpha-1}(\eta-s)^\alpha}{A\Gamma(\alpha)} \\
&\leq \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)}.
\end{aligned}
$$

When $0 \leq \eta \leq s \leq t \leq 1$, from (18), there is

$$
\begin{aligned}
G(t,s) &= \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - A(t-s)^{\alpha-1}}{A\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\geq \frac{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} \\
&\geq \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1)-\Gamma(\alpha-\beta)\eta^\alpha]\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha-\beta)\eta^\alpha[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)} \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
G(t,s) &= \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - A(t-s)^{\alpha-1}}{A\Gamma(\alpha)} \\
&\leq \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)}.
\end{aligned}
$$

When $0 \leq t \leq s \leq \eta \leq 1$,

$$
G(t,s) = \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \Gamma(\alpha-\beta)t^{\alpha-1}(\eta-s)^\alpha}{A\Gamma(\alpha)}
$$

$$\begin{aligned}
&\geq \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \Gamma(\alpha-\beta)t^{\alpha-1}\eta^{\alpha}(1-s)^{\alpha}}{[\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)} \\
&= \frac{[\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\beta)\eta^{\alpha}[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)} \\
&\geq \frac{\Gamma(\alpha-\beta)\eta^{\alpha}[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)} \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
G(t,s) &= \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \Gamma(\alpha-\beta)t^{\alpha-1}(\eta-s)^{\alpha}}{A\Gamma(\alpha)} \\
&\leq \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)}.
\end{aligned}$$

When $0 \leq t \leq s \leq \eta \leq 1$,

$$\begin{aligned}
G(t,s) &= \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)} \\
&\geq \frac{[\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{[\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha}]\Gamma(\alpha)} \\
&\geq \frac{\Gamma(\alpha-\beta)\eta^{\alpha}[1-(1-s)^{\beta+1}]t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)} \geq 0.
\end{aligned}$$

It is obvious that

$$G(t,s) \leq \frac{\Gamma(\alpha+1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{A\Gamma(\alpha)}.$$

Then, for $\forall t, s \in [0,1]$, we obtain that $G(t,s)$ meets (17). □

Theorem 4.1 *Let $f, g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $\phi, \psi : [0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous with $f(t,0,1), g(t,0,1), \phi(t,0), \psi(t,1) \not\equiv 0$ and satisfy the following conditions*:

(N_1) *For any fixed $t \in [0,1]$, $y \in [0,+\infty)$, $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x)$ are increasing in $x \in [0,+\infty)$; for any fixed $t \in [0,1]$, $x \in [0,+\infty)$, $f(t,x,y)$, $g(t,x,y)$, $\psi(t,y)$ are decreasing in $y \in [0,+\infty)$;*

(N_2) *For $\forall \lambda \in (0,1)$, $t \in [0,1]$, $x, y \in [0,+\infty)$, there exists $\varphi(\lambda) \in (\lambda, 1]$ such that*

$$\begin{aligned}
&f\bigl(t, \lambda x, \lambda^{-1}y\bigr) \geq \varphi(\lambda) f(t,x,y), \qquad \phi(t,\lambda x) \geq \lambda\phi(t,x), \\
&\psi\bigl(t, \lambda^{-1}y\bigr) \geq \lambda\psi(t,y);
\end{aligned}$$

and for fixed $t \in [0,1]$, $y \in [0,+\infty)$, $g(t,\cdot,y)$ is concave; for fixed $t \in [0,1]$, $x \in [0,+\infty)$, $g(t,x,\cdot)$ is convex;

(N_3) *Let $h(t) = t^{\alpha-1}$, there exists $c \geq 0$ such that $g(s,\theta,lh) \geq cg(s,lh,\theta)$, $l \geq 1$;*

(N_4) *There exists a constant $\delta > 0$ such that $[\lambda_2 g(t,x,y) + \lambda_3\phi(t,x) + \lambda_4\psi(t,y)] \leq \delta\lambda_1 f(t,x,y)$, $\forall t \in [0,1]$, $x, y \in [0,+\infty)$.*

Then

($S1$) *There exist* $u_0, v_0 \in P_h$, $r \in (0,1)$ *such that* $rv_0 \leq u_0 < v_0$,

$$\begin{aligned} u_0 &\leq \lambda_1 \int_0^1 G(t,s) f\bigl(s, u_0(s), v_0(s)\bigr)\, ds + \lambda_2 \int_0^1 G(t,s) g\bigl(s, u_0(s), v_0(s)\bigr)\, ds \\ &\quad + \lambda_3 \int_0^1 G(t,s)\phi\bigl(s, u_0(s)\bigr)\, ds + \lambda_4 \int_0^1 G(t,s)\psi\bigl(s, v_0(s)\bigr)\, ds \\ &\leq \lambda_1 \int_0^1 G(t,s) f\bigl(s, v_0(s), u_0(s)\bigr)\, ds + \lambda_2 \int_0^1 G(t,s) g\bigl(s, v_0(s), u_0(s)\bigr)\, ds \\ &\quad + \lambda_3 \int_0^1 G(t,s)\phi\bigl(s, v_0(s)\bigr)\, ds + \lambda_4 \int_0^1 G(t,s)\psi\bigl(s, u_0(s)\bigr)\, ds \leq v_0; \end{aligned}$$

($S2$) *Equation* (12) *has a unique positive solution* x^* *in* P_h, *which meets* $\mu_1 t^{\alpha-1} \leq x^* \leq \nu_1 t^{\alpha-1}$, μ_1, $\nu_1 > 0$ *are two constants;*

($S3$) *For any initial values* $x_0, y_0 \in P_h$, *we construct successively the iterative sequences*

$$\begin{aligned} x_n(t) &= \lambda_1 \int_0^1 G(t,s) f\bigl(s, x_{n-1}(s), y_{n-1}(s)\bigr)\, ds + \lambda_2 \int_0^1 G(t,s) g\bigl(s, x_{n-1}(s), y_{n-1}(s)\bigr)\, ds \\ &\quad + \lambda_3 \int_0^1 G(t,s)\phi\bigl(s, x_{n-1}(s)\bigr)\, ds \\ &\quad + \lambda_4 \int_0^1 G(t,s)\psi\bigl(s, y_{n-1}(s)\bigr)\, ds, \quad n = 1, 2, \ldots, \\ y_n(t) &= \lambda_1 \int_0^1 G(t,s) f\bigl(s, y_{n-1}(s), x_{n-1}(s)\bigr)\, ds + \lambda_2 \int_0^1 G(t,s) g\bigl(s, y_{n-1}(s), x_{n-1}(s)\bigr)\, ds \\ &\quad + \lambda_1 \int_0^1 G(t,s)\phi\bigl(s, y_{n-1}(s)\bigr)\, ds \\ &\quad + \lambda_4 \int_0^1 G(t,s)\phi\bigl(s, x_{n-1}(s)\bigr)\, ds, \quad n = 1, 2, \ldots. \end{aligned}$$

Here, $x_n \to x^*$, $y_n \to x^*$, *when* $n \to \infty$.

Proof Let $E = C[0,1]$ and $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$. It is obvious that $(E, \|\cdot\|)$ is a Banach space. Set $P = \{x \in E | x(t) \geq 0, t \in [0,1]\}$.

By Lemma 4.1, the unique solution of problem (12) has an integral formulation:

$$x(t) = \int_0^1 G(t,s)\bigl[\lambda_1 f\bigl(s, x(s), x(s)\bigr) + \lambda_2 g\bigl(s, x(s), x(s)\bigr) + \lambda_3 \phi\bigl(s, x(s)\bigr) + \lambda_4 \psi\bigl(s, x(s)\bigr)\bigr]\, ds.$$

Define four operators $A, B, C, D : P \times P \to E$ by

$$\begin{aligned} A(x,y)(t) &= \int_0^1 G(t,s) f\bigl(s, x(s), y(s)\bigr)\, ds, \\ B(x,y)(t) &= \int_0^1 G(t,s) g\bigl(s, x(s), y(s)\bigr)\, ds, \\ Cx(t) &= \int_0^1 G(t,s)\phi\bigl(s, x(s)\bigr)\, ds, \end{aligned}$$

$$Dy(t) = \int_0^1 G(t,s)\psi\bigl(s,y(s)\bigr)\,\mathrm{d}s.$$

Therefore, x is the solution of equation (12) if and only if x is the solution of operator equation $x = \lambda_1 A(x,x) + \lambda_2 B(x,x) + \lambda_3 Cx + \lambda_4 Dx$.

Firstly, for $\forall x,y \in P$, in view of the definition of f, g, ϕ, ψ and Lemma 4.2, it is easy to obtain $A(x,y) \geq 0$, $B(x,y) \geq 0$, $Cx \geq 0$, $Dy \geq 0$, which shows that $A,B : P \times P \to P$, $C,D : P \to P$. In addition, due to (N_1), A, B are two monotone operators, C is an increasing operator, and D is a decreasing operator.

Secondly, we illustrate that C is a sub-homogeneous operator and A, D satisfy condition (A_1) of Theorem 3.1. From (N_2), we see that, for any $\lambda \in (0,1)$, $x \in P$,

$$\begin{aligned} C(\lambda x) &= \int_0^1 G(t,s)\phi\bigl(s,\lambda x(s)\bigr)\,\mathrm{d}s \\ &\geq \int_0^1 G(t,s)\lambda\phi\bigl(s,x(s)\bigr)\,\mathrm{d}s \\ &= \lambda\int_0^1 G(t,s)\phi\bigl(s,x(s)\bigr)\,\mathrm{d}s \\ &= \lambda Cx, \end{aligned}$$

which shows that C is a sub-homogeneous operator. Using (N_2) again, $\forall \lambda \in (0,1)$, $x,y \in P$, there exists $\varphi(\lambda) \in (\lambda, 1]$ such that

$$\begin{aligned} A\bigl(\lambda x, \lambda^{-1}y\bigr) &= \int_0^1 G(t,s)f\bigl(s,\lambda x(s),\lambda^{-1}y(s)\bigr)\,\mathrm{d}s \\ &\geq \int_0^1 G(t,s)\varphi(\lambda)f\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s \\ &= \varphi(\lambda)\int_0^1 G(t,s)f\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s \\ &= \varphi(\lambda)A(x,y), \end{aligned}$$

and for any $\lambda \in (0,1)$, $y \in P$,

$$\begin{aligned} D\bigl(\lambda^{-1}y\bigr) &= \int_0^1 G(t,s)\psi\bigl(s,\lambda^{-1}y(s)\bigr)\,\mathrm{d}s \\ &\geq \int_0^1 G(t,s)\lambda\psi\bigl(s,y(s)\bigr)\,\mathrm{d}s \\ &= \lambda\int_0^1 G(t,s)\psi\bigl(s,y(s)\bigr)\,\mathrm{d}s \\ &= \lambda Dy. \end{aligned}$$

Thus, operator A, D satisfies (A_1) of Theorem 3.1.

Thirdly, we verified condition (A_2) of Theorem 3.1. From (N_2), for fixed $t \in (0,1)$, $y \in P$,

$$\begin{aligned} &B\bigl(\tau x_1 + (1-\tau)x_2, y\bigr)(t) \\ &\quad= \int_0^1 G(t,s)g\bigl(s,\tau x_1(s) + (1-\tau)x_2(s),y(s)\bigr)\,\mathrm{d}s \end{aligned}$$

$$\begin{aligned}&\geq \int_0^1 G(t,s)\big[\tau g\big(s,x_1(s),y(s)\big)+(1-\tau)g\big(s,x_2(s),y(s)\big)\big]\,\mathrm{d}s\\&=\tau\int_0^1 G(t,s)g\big(s,x_1(s),y(s)\big)\,\mathrm{d}s+(1-\tau)\int_0^1 G(t,s)g\big(s,x_2(s),y(s)\big)\,\mathrm{d}s\\&=\tau B(x_1,y)+(1-\tau)B(x_2,y),\quad \forall\tau\in(0,1),x_1,x_2\in P;\end{aligned}$$

for fixed $t\in(0,1)$, $x\in P$,

$$\begin{aligned}&B\big(x,\tau y_1+(1-\tau)y_2\big)(t)\\&\quad=\int_0^1 G(t,s)g\big(s,x(s),\tau y_1(s)+(1-\tau)y_2(s)\big)\,\mathrm{d}s\\&\quad\leq\int_0^1 G(t,s)\big[\tau g\big(s,x(s),y_1(s)\big)+(1-\tau)g\big(s,x(s),y_2(s)\big)\big]\,\mathrm{d}s\\&\quad=\tau\int_0^1 G(t,s)g\big(s,x(s),y_1(s)\big)\,\mathrm{d}s+(1-\tau)\int_0^1 G(t,s)g\big(s,x(s),y_2(s)\big)\,\mathrm{d}s\\&\quad=\tau B(x,y_1)+(1-\tau)B(x,y_2),\quad\forall\tau\in(0,1),y_1,y_2\in P,\end{aligned}$$

which indicate that, for fixed $y\in P$, $B(\cdot,y)$ is concave; for fixed $x\in P$, $B(x,\cdot)$ is convex. That is condition (A_2) of Theorem 3.1.

Fourthly, we check condition (A_3) of Theorem 3.1. By (N_3), there exists $\widetilde{c}\geq 0$ such that

$$\begin{aligned}B(\theta,lh)(t)&=\int_0^1 G(t,s)g\big(s,\theta,lh(s)\big)\,\mathrm{d}s\\&\geq\widetilde{c}\int_0^1 G(t,s)g\big(s,lh(s),\theta\big)\,\mathrm{d}s\\&=\widetilde{c}B(lh,\theta)(t).\end{aligned}$$

In the next step, we prove that $A(h,h),B(h,h),Ch,Dh\in P_h$. By Lemma 4.2, we have

$$\begin{aligned}A(h,h)(t)&=\int_0^1 G(t,s)f\big(s,s^{\alpha-1},s^{\alpha-1}\big)\,\mathrm{d}s\\&\geq\frac{\Gamma(\alpha-\beta)\eta^\alpha}{A\Gamma(\alpha)}\int_0^1\big[1-(1-s)^{\beta+1}\big](1-s)^{\alpha-\beta-1}f\big(s,s^{\alpha-1},s^{\alpha-1}\big)\,\mathrm{d}s\\&\geq\left\{\frac{\Gamma(\alpha-\beta)\eta^\alpha}{A\Gamma(\alpha)}\int_0^1\big[1-(1-s)^{\beta+1}\big](1-s)^{\alpha-\beta-1}f(s,0,1)\,\mathrm{d}s\right\}h(t),\end{aligned}$$

and

$$\begin{aligned}A(h,h)(t)&=\int_0^1 G(t,s)f\big(s,s^{\alpha-1},s^{\alpha-1}\big)\,\mathrm{d}s\\&\leq\frac{\Gamma(\alpha+1)t^{\alpha-1}}{A\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}f\big(s,s^{\alpha-1},s^{\alpha-1}\big)\,\mathrm{d}s\\&\leq\left\{\frac{\Gamma(\alpha+1)}{A\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}f(s,1,0)\,\mathrm{d}s\right\}h(t),\end{aligned}$$

where

$$c_1 = \frac{\Gamma(\alpha-\beta)\eta^\alpha}{A\Gamma(\alpha)}\int_0^1 \bigl[1-(1-s)^{\beta+1}\bigr](1-s)^{\alpha-\beta-1}f(s,0,1)\,\mathrm{d}s,$$
$$c_2 = \frac{\Gamma(\alpha+1)}{A\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}f(s,1,0)\,\mathrm{d}s.$$

Thus, we can get $c_1h(t) \le A(h,h)(t) \le c_2h(t)$. Besides, due to $f(s,1,0) \ge f(s,0,1) \ge 0$ and $f(s,0,1) \not\equiv 0$, we derive that

$$\int_0^1 f(s,1,0)\,\mathrm{d}s \ge \int_0^1 f(s,0,1)\,\mathrm{d}s > 0,$$

thus, $A(h,h) \in P_h$. Similarly, there are

$$\begin{aligned}&\left\{\frac{\Gamma(\alpha-\beta)\eta^\alpha}{A\Gamma(\alpha)}\int_0^1 \bigl[1-(1-s)^{\beta+1}\bigr](1-s)^{\alpha-\beta-1}g(s,0,1)\,\mathrm{d}s\right\}h(t)\\&\quad\le B(h,h)(t) \le \left\{\frac{\Gamma(\alpha+1)}{A\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}g(s,1,0)\,\mathrm{d}s\right\}h(t),\\&\left\{\frac{\Gamma(\alpha-\beta)\eta^\alpha}{A\Gamma(\alpha)}\int_0^1 \bigl[1-(1-s)^{\beta+1}\bigr](1-s)^{\alpha-\beta-1}\phi(s,0)\,\mathrm{d}s\right\}h(t)\\&\quad\le Ch(t) \le \left\{\frac{\Gamma(\alpha+1)}{A\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\phi(s,1)\,\mathrm{d}s\right\}h(t),\end{aligned}$$

and

$$\begin{aligned}&\left\{\frac{\Gamma(\alpha-\beta)\eta^\alpha}{A\Gamma(\alpha)}\int_0^1 \bigl[1-(1-s)^{\beta+1}\bigr](1-s)^{\alpha-\beta-1}\psi(s,1)\,\mathrm{d}s\right\}h(t)\\&\quad\le Dh(t) \le \left\{\frac{\Gamma(\alpha+1)}{A\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\psi(s,0)\,\mathrm{d}s\right\}h(t).\end{aligned}$$

Then, by $g(s,0,1), \phi(s,0), \psi(s,1) \not\equiv 0$, we can obtain $B(h,h), Ch, Dh \in P_h$.

Eventually, we test condition (A_5) of Theorem 3.1. From (N_5), for any $t \in [0,1]$, $x,y \in P$,

$$\begin{aligned}&\bigl[\lambda_2B(x,y)(t) + \lambda_3Cx(t) + \lambda_4Dy(t)\bigr]\\&\quad= \int_0^1 G(t,s)\bigl[\lambda_2g\bigl(s,x(s),y(s)\bigr) + \lambda_3\phi\bigl(s,x(s)\bigr) + \lambda_4\psi\bigl(s,y(s)\bigr)\bigr]\,\mathrm{d}s\\&\quad\le \delta\int_0^1 G(t,s)\lambda_1f\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s\\&\quad= \delta\lambda_1A(x,y)(t), \quad n = 1,2,\ldots.\end{aligned}$$

From the above six steps, we verified all the conditions of Theorem 3.1, thus the conclusions of Theorem 4.1 hold with Theorem 3.1. □

Theorem 4.2 *Let f, $g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $\phi : [0,1] \times [0,+\infty) \to [0,+\infty)$, and $\psi : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous with $f(t,0,1), g(t,0,1), \phi(t,0), \psi(t,1) \not\equiv 0$. Then, if conditions (N_1), (N_4) and the following conditions are satisfied:*

(N_5) *For* $\forall\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in [0,+\infty)$, *there exists* $\varphi(\lambda) \in (\lambda, 1]$ *such that*

$$\begin{aligned} &f\left(t, \lambda x, \lambda^{-1} y\right) \geq \lambda f(t,x,y), \qquad \phi(t,\lambda x) \geq \varphi(\lambda)\phi(t,x), \\ &\psi\left(t, \lambda^{-1} y\right) \geq \lambda^{-1}\psi(t,y); \end{aligned}$$

and for fixed $t \in [0,1]$, $y \in [0,+\infty)$, $g(t,\cdot,y)$ *is concave; for fixed* $t \in [0,1]$, $x \in [0,+\infty)$, $g(t,x,\cdot)$ *is convex;*

(N_6) *There exists a constant* $\delta > 0$ *such that* $\lambda_1 f(t,x,y) + \lambda_2 g(t,x,y) + \lambda_4\psi(t,y) \leq \delta\lambda_3\phi(t,x)$, $\forall t \in [0,1]$, $x, y \in [0,+\infty)$;

then we can get conclusions $(S1)$–$(S3)$ *of Theorem* 4.1.

Proof The proof process is similar to that of Theorem 4.1. □

Example 4.1 Consider the following problem:

$$\begin{cases} D_{0^+}^{\frac{13}{2}} x(t) + 2(x+3)^{\frac{1}{4}} + 3(x+2)^{\frac{1}{4}} + 3(x+4)^{-\frac{1}{6}} + 4(x+5)^{-\frac{1}{6}} \\ \quad - \frac{22}{5} e^{-x} + 4e^{-y} + 4t + 2t^2 + 3t^3 + 4t^4 + 6 = 0, \quad 0 < t < 1, \\ x^i(0) = 0, \quad 0 \leq i \leq 4, \qquad D_{0^+}^{\frac{5}{3}} x(1) = \int_0^{\frac{3}{4}} x(s)\,ds. \end{cases}$$

Here, $\alpha = \frac{13}{2} \in (5,6)$, $\beta = \frac{5}{3} \in (1,2)$, $\alpha - \beta - 1 = \frac{23}{6} \geq 0$, $\eta = \frac{3}{4}$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$, and

$$\begin{aligned} &f(t,x,y) = 2(x+3)^{\frac{1}{4}} + 3(y+4)^{-\frac{1}{6}} + 6 + 4t, \\ &g(t,x,y) = -\frac{11}{5} e^{-x} + 2e^{-y} + 3 + t^2, \\ &\phi(t,x) = (x+2)^{\frac{1}{4}} + t^3, \\ &\psi(t,y) = (y+5)^{-\frac{1}{6}} + t^4. \end{aligned}$$

Thus, $\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^\alpha = \Gamma(\frac{15}{2}) - \Gamma(\frac{29}{6})\frac{3}{4}^{\frac{13}{2}} = 1.868 \times 10^{-3}$, $f(t,0,1) = 2 \cdot 3^{\frac{1}{4}} + 3 \cdot 5^{-\frac{1}{5}} + 6 + 4t \not\equiv 0$, $g(t,0,1) = -\frac{11}{5} + 2e^{-1} + t^2 \not\equiv 0$, $\phi(t,0) = 2^{\frac{1}{4}} + t^3 \not\equiv 0$, $\psi(t,1) = 3 \times 6^{-\frac{1}{6}} + t^4 \not\equiv 0$.

For $\forall\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in [0,+\infty)$, there exists $\gamma = \frac{1}{4} \in (0,1)$, we can get that

$$\begin{aligned} f\left(t, \lambda x, \lambda^{-1} y\right) &= 2(\lambda x+3)^{\frac{1}{4}} + 3\left(\lambda^{-1} y + 4\right)^{-\frac{1}{6}} + 6 + 4t \\ &\geq \lambda^{\frac{1}{4}} 2(x+3)^{\frac{1}{4}} + \lambda^{\frac{1}{6}} 3(y+4)^{-\frac{1}{5}} + 6 + 4t \\ &\geq \lambda^{\frac{1}{4}} \left(2(x+3)^{\frac{1}{4}} + 3(y+4)^{-\frac{1}{6}} + 6 + 4t\right) \\ &= \lambda^{\frac{1}{4}} f(t,x,y). \end{aligned}$$

For $\forall t \in [0,1]$, $x, y \in [0,+\infty)$, there is

$$g''_{xx}(t,x,y) = -\frac{11}{5} e^{-x} \leq 0, \qquad g''_{yy}(t,x,y) = 2e^{-y} \geq 0,$$

we obtain that, for fixed $t \in [0,1]$, $y \in [0,+\infty)$, $g(t,\cdot,y)$ is concave; for fixed $t \in [0,1]$, $x \in [0,+\infty)$, $g(t,x,\cdot)$ is convex. For $\forall\lambda, t \in [0,1]$, $x,y \in [0,+\infty)$, we have

$$\begin{aligned}\phi(t,\lambda x) &= (\lambda x+2)^{\frac{1}{4}} + t^3 \geq \lambda^{\frac{1}{4}}(x+2)^{\frac{1}{4}} + t^3 \\ &\geq \lambda\big((\lambda x+2)^{\frac{1}{4}} + t^3\big) = \lambda\phi(t,x)\end{aligned}$$

and

$$\psi\big(t,\lambda^{-1}y\big) = \big(\lambda^{-1}y+5\big)^{-\frac{1}{6}} + t^4 \geq \lambda\big((y+5)^{-\frac{1}{6}} + t^4\big) = \lambda\psi(t,y).$$

Let l' be a sufficiently large constant and $x,y \leq l'$, $\widetilde{c} = \frac{1}{25}$, there is

$$g\big(t,\theta,l'\big) = -\frac{11}{5} + 2e^{-l'} + 3 + t^2 \geq \frac{1}{25}\left(-\frac{11}{5}e^{-l'} + 2 + 3 + t^2\right) = \widetilde{c}g\big(t,l',\theta\big).$$

Let $\delta = 2$, by calculation, we have

$$\begin{aligned}&\lambda_2 g(t,x,y) + \lambda_3\phi(t,x) + \lambda_4\psi(t,y) \\ &\quad= 2\left[-\frac{11}{5}e^{-x} + 2e^{-y} + 3 + t^2\right] + 3\big[(x+2)^{\frac{1}{4}} + t^3\big] + 4\big[(y+5)^{-\frac{1}{6}} + t^4\big] \\ &\quad\leq 3(x+2)^{\frac{1}{4}} + 4(y+5)^{-\frac{1}{6}} + 6 - \frac{22}{5}e^{-x} + 4e^{-y} + 2t^2 + 3t^3 + 4t^4 \\ &\quad\leq 4(x+3)^{\frac{1}{4}} + 6(y+4)^{-\frac{1}{6}} + 12 + 8t \\ &\quad\leq 2\big[2(x+2)^{\frac{1}{4}} + 3(y+4)^{-\frac{1}{6}} + 6 + 4t\big] \\ &\quad= \delta\lambda_1 f(t,x,y).\end{aligned}$$

Therefore, all the assumptions of Theorem 4.1 are satisfied, then Example 4.1 has a unique positive solution $x^* \in P_h$, where $h(t) = t^{\frac{11}{2}}$, $t \in [0,1]$. Besides, the other condition of Theorem 4.1 holds.

Example 4.2 Consider the following problem:

$$\begin{cases} D_{0^+}^{\frac{7}{3}}x(t) + (x+3)^{\frac{1}{4}} + 2(x+2)^{\frac{1}{4}} + (y+4)^{-\frac{1}{5}} + 3(y+5)^{-\frac{1}{6}} \\ \quad - \frac{11}{5}e^{-x} + 2e^{-y} + 4t + t^2 + t^3 + t^4 + 6 = 0, \quad 0 < t < 1, \\ x^i(0) = 0, \quad 0 \leq i \leq 4, \qquad D_{0^+}^{\frac{1}{4}}x(1) = \int_0^{\frac{3}{4}} x(s)\,ds. \end{cases}$$

Here, $\alpha = \frac{7}{3} \in (2,3)$, $\beta = \frac{1}{4} \in (0,1)$, $\alpha - \beta - 1 = \frac{13}{12} \geq 0$, $\eta = \frac{1}{2}$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 4$, $\lambda_4 = 3$, and

$$\begin{aligned}f(t,x,y) &= 2(x+1)^{\frac{1}{2}} + \frac{1}{1+y} + \sin t, \\ g(t,x,y) &= -\frac{3}{4}e^{-x} + \frac{12}{5}e^{-y} + 3 + \cos t, \\ \phi(t,x) &= (x+4)^{\frac{1}{2}} + 2,\end{aligned}$$

$$\psi(t,y) = \frac{1}{1+y} + t^2.$$

Therefore, $\Gamma(\alpha+1) - \Gamma(\alpha-\beta)\eta^{\alpha} = \Gamma(\frac{10}{3}) - \Gamma(\frac{25}{12})\frac{1}{2}^{\frac{7}{3}} = 2.5722$, $f(t,0,1) = 2^{\frac{1}{2}} + \frac{1}{2} + \sin t$, $g(t,0,1) = -\frac{3}{4} + \frac{12}{5}e^{-1} + \cos t$, $\phi(t,0) = 4^{\frac{1}{2}} + 2$, $\psi(t,1) = \frac{1}{2} + t^2 \not\equiv 0$.

For $\forall\lambda \in (0,1)$, $t \in [0,1]$, $x,y \in [0,+\infty)$, we can derive that

$$\begin{aligned} f\bigl(t,\lambda x,\lambda^{-1}y\bigr) &= 2(\lambda x+1)^{\frac{1}{2}} + \frac{1}{1+y} + \sin t \\ &\geq \lambda^{\frac{1}{2}}2(x+1)^{\frac{1}{2}} + \lambda\frac{1}{1+y} + \sin t \\ &\geq \lambda\left(2(x+1)^{\frac{1}{2}} + \frac{1}{1+y} + \sin t\right) \\ &= \lambda f(t,x,y). \end{aligned}$$

For expression of g, there is

$$g''_{xx}(t,x,y) = -\frac{3}{4}e^{-x} \leq 0, \qquad g''_{yy}(t,x,y) = \frac{12}{5}e^{-y} \geq 0,$$

i.e., $g(t,\cdot,y)$ is concave, $g(t,x,\cdot)$ is convex. Further, there exist $\gamma = \frac{1}{2} \in (0,1)$,

$$\phi(t,\lambda x) = (\lambda x+4)^{\frac{1}{2}} + 2 \geq \lambda^{\frac{1}{2}}\bigl((x+2)^{\frac{1}{4}} + 2\bigr) = \lambda^{\frac{1}{2}}\phi(t,x),$$

and

$$\psi\bigl(t,\lambda^{-1}y\bigr) = \frac{1}{1+\lambda^{-1}y} + t^2 \geq \lambda\left(\frac{1}{1+y} + t^2\right) = \lambda\psi(t,y).$$

Let l' be a sufficiently large constant with $x,y \leq l'$. $c = \frac{1}{9}$, we can obtain

$$g\bigl(t,\theta,l'\bigr) = -\frac{3}{4} + \frac{12}{5}e^{-l'} + 3 + \cos t \geq \frac{1}{9}\left(-\frac{3}{4}e^{-l'} + \frac{12}{5} + 3 + \cos t\right) = cg\bigl(t,l',\theta\bigr).$$

Let $\delta = 2.5$, we get

$$\begin{aligned} &\lambda_1 f(t,x,y) + \lambda_2 g(t,x,y) + \lambda_4\psi(t,y) \\ &\quad= 2\left[2(x+1)^{\frac{1}{2}} + \frac{1}{1+y} + \sin t\right] + \left[-\frac{3}{4}e^{-x} + \frac{12}{5}e^{-y} + 3 + \cos t\right] + 3\left[\frac{1}{1+y} + t^2\right] \\ &\quad= 4(x+1)^{\frac{1}{2}} + \frac{5}{1+y} + 2\sin t - \frac{3}{4}e^{-x} + \frac{12}{5}e^{-y} + 3 + \cos t + 3t^2 \\ &\quad\leq 4(x+4)^{\frac{1}{2}} + 17 \\ &\quad\leq 2.5 \times 4\bigl[(x+4)^{\frac{1}{2}} + 2\bigr] = \delta\lambda_3\phi(t,x). \end{aligned}$$

Then Example 4.2 has a unique positive solution $x^* \in P_h$, where $h(t) = t^{\frac{4}{3}}$, $t \in [0,1]$.

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Authors' contributions
NZ and LZ carried out the concepts, design, definition of content, literature search, and manuscript preparation. NZ drafted the manuscript. BZ and HT performed manuscript review. All authors have read and approved the content of the manuscript.

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