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Monotonicity properties and bounds involving the two-parameter generalized Grötzsch ring function

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Abstract

In the article, we present several new monotonicity properties and bounds involving the generalized Grötzsch ring functions $\mu_{a,b}$ in the theory of Ramanujan's generalized modular equation for $0 < a, b < 1$. Our results are the variants and extensions of some previously known results.

MSC: 33C05; 33E05

Keywords: Gaussian hypergeometric function; Generalized Grötzsch ring function; Generalized elliptic integral; Monotonicity property

1 Introduction

Given $x \in (-1, 1)$ and real numbers a, b, and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function $F(a, b; c; x)$ [1–18] is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \tag{1.1}$$

where $(a, 0) = 1$ for $a \neq 0$ and $(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$ for $n = 1, 2, \dots$. $F(a, b; c; x)$ is said to be zero-balanced if $c = a + b$. If $x \to 1$, then the following asymptotic formulas

$$\begin{cases} F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, & a+b<c, \\ B(a,b)F(a,b;c;x) + \log(1-x) = R(a,b) + O((1-x)\log(1-x)), & a+b=c, \\ F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), & a+b>c, \end{cases} \tag{1.2}$$

can be found in the literature [19, Theorems 1.19 and 1.48], where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ [20–26] and $B(p, q) = [\Gamma(p)\Gamma(q)]/\Gamma(p + q)$ [27–30] are respectively the classical Euler gamma and beta functions, and

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \qquad R\left(\frac{1}{2}, \frac{1}{2}\right) = \log 16, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

and

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.5772156649\cdots$$

is the Euler–Mascheroni constant [31–33].

Ramanujan's generalized modular equation with order (or degree) $p > 0$ is given by

$$\frac{F(a,b;c;1-s^2)}{F(a,b;c;s^2)} = p\frac{F(a,b;c;1-r^2)}{F(a,b;c;r^2)}, \quad 0 < r < 1. \tag{1.3}$$

It is well known that equation (1.3) has a unique solution for s if $a,b,c > 0$ with $a+b \geq c$ [34, Lemma 4.5].

The two-parameter generalized Grötzsch ring function is defined by

$$\mu_{a,b}(r) = \frac{B(a,b)}{2}\frac{F(a,b;(a+b+1)/2;1-r^2)}{F(a,b;(a+b+1)/2;r^2)}, \quad r \in (0,1) \tag{1.4}$$

if $a+b \geq 1$.

Our interest is to focus on $c = (a+b+1)/2$, which makes the derivative formula of the two-parameter generalized Grötzsch ring function defined by (1.4) simpler.

Let $0 < a,b < 1$ with $a+b \geq 1$ and $r \in (0,1)$. Then the two-parameter generalized elliptic integrals of first and second kinds [34, (1.6)–(1.8)] are defined by

$$\mathcal{K} = \mathcal{K}_{a,b} = \mathcal{K}_{a,b}(r) = \frac{B(a,b)}{2}F\left(a,b;\frac{a+b+1}{2};r^2\right), \tag{1.5}$$

$$\mathcal{E} = \mathcal{E}_{a,b} = \mathcal{E}_{a,b}(r) = \frac{B(a,b)}{2}F\left(a-1,b;\frac{a+b+1}{2};r^2\right), \tag{1.6}$$

$$\mathcal{K}' = \mathcal{K}'_{a,b} = \mathcal{K}_{a,b}(r'), \qquad \mathcal{E}' = \mathcal{E}'_{a,b} = \mathcal{E}_{a,b}(r'), \tag{1.7}$$

where and in what follows $r' = \sqrt{1-r^2}$. Moreover, it follows from (1.2) that

$$\mathcal{K}_{a,b}(0^+) = \mathcal{E}_{a,b}(0^+) = \frac{B(a,b)}{2},$$

$$\mathcal{K}_{a,b}(1^-) = \infty, \qquad \mathcal{E}_{a,b}(1^-) = \frac{B(a,b)B((a+b+1)/2,(3-a-b)/2)}{2B((b-a+3)/2,(a-b+1)/2)}.$$

In this paper, we study the two-parameter generalized Grötzsch ring function $\mu_{a,b}(r)$ for $a,b \in (0,1)$, as well as the related functions $\mathcal{K}_{a,b}$, $\mathcal{E}_{a,b}$, and

$$m_{a,b}(r) = \frac{2}{B(a,b)}r'^2\mathcal{K}_{a,b}\mathcal{K}'_{a,b}, \quad r \in (0,1). \tag{1.8}$$

The so-called Legendre $\mathcal{M}$-function introduced in [35] can be used to study the derivative of $m_{a,b}(r)$ and satisfies the formula

$$\left[\frac{B(a,b)}{2}\right]^2\mathcal{M}(r^2) = \frac{a+b-1}{2}\mathcal{K}\mathcal{K}' + \frac{b-a+1}{2}\left(\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}'\right) \tag{1.9}$$

for $r \in (0,1)$. Furthermore, $\mathcal{M}(r)$ can be rewritten as

$$\mathcal{M}(r) = \frac{\Gamma((a+b+1)/2)^2[r(1-r)]^{(1-a-b)/2}}{\Gamma(a)\Gamma(b)}, \tag{1.10}$$

and $\mathcal{M}(r)$ becomes a constant if and only if $a+b=1$, in which case $\mathcal{M}(r^2)$ degenerates to be the generalized Legendre relation.

In the case of $a+b=1$, these functions coincide with the special functions $\mu_a(r)$, $\mathcal{K}_a(r)$, $\mathcal{E}_a(r)$, and $m_a(r)$, respectively, which were studied in [36–49]. In particular, if $a=b=1/2$, then these functions reduce to the classical cases denoted by $\mu(r)$, $\mathcal{K}(r)$, $\mathcal{E}(r)$, and $m(r)$, which appeared frequently in the geometric function theory and number theory [50–69].

The main purpose of the article is to find the sub-regions of $\{(a,b) \in \mathbb{R}^2 | 0 < a, b < 1, a+b > 1\}$ such that certain quotient functions involving $\mu_{a,b}(r)$, $\mathcal{K}_{a,b}(r)$, $\mathcal{E}_{a,b}(r)$, and $m_{a,b}(r)$ are monotonic on their corresponding sub-regions. As a consequence, several new bounds for $\mu_{a,b}(r)$ and $m_{a,b}(r)$ are discovered, which are the variants and extensions of the results given in [42, Theorems 1.1 and 1.2] for the case of zero-balanced.

2 Notations, formulas, and lemmas

In order to prove our main results, we need several derivative formulas and lemmas, which we present in this section.

2.1 Notations

Throughout the article, we denote $B(a,b)$ by B if no risk for confusion. Let

$$\begin{aligned}
D &= \frac{B(\frac{a+b+1}{2}, \frac{a+b-1}{2})}{2}, \\
E &= \frac{B(a,b)B(\frac{a+b+1}{2}, \frac{1-a-b}{2})}{2B(\frac{a-b+1}{2}, \frac{b-a+1}{2})}, \\
\kappa_1(a,b) &= a+b+1-2ab(4-a-b), \\
\kappa_2(a,b) &= 3+7(a+b)+2(a-b)^2-6(a+b)^3-5(a+b)^4-(a+b)^5 \\
&\quad + 8ab\big[(a+b)^2+(a+b)^3-(a+b)+8ab\big], \\
\kappa_3(a,b) &= 5+7(a+b)-3(a+b)^2-7(a+b)^3-2(a+b)^4 \\
&\quad + 4ab\big[3(a+b)^2+4(a+b)-3\big], \\
\kappa_4(a,b) &= 9+5(a+b)-9(a+b)^2-5(a+b)^3+16ab(a+b), \\
\kappa_5(a,b) &= (a+b+1)^2-12ab.
\end{aligned}$$

For the convenience of readers, we also introduce three sub-regions Ω_1, Ω_2, and Ω_3 of $\{(a,b) \in \mathbb{R}^2 | 0 < a, b < 1\}$, which are illustrated in Fig. 1.

$$\begin{aligned}
\Omega_1 &= \big\{(a,b) | 0 < a, b < 1, a+b > 1, \kappa_1(a,b) \geq 0\big\}, \\
\Omega_2 &= \big\{(a,b) | 0 < a, b < 1, a+b > 1, \kappa_2(a,b) \leq 0, \kappa_3(a,b) \leq 0\big\}, \\
\Omega_3 &= \big\{(a,b) | 0 < a, b < 1, a+b > 1, \kappa_5(a,b) \geq 0\big\}.
\end{aligned}$$

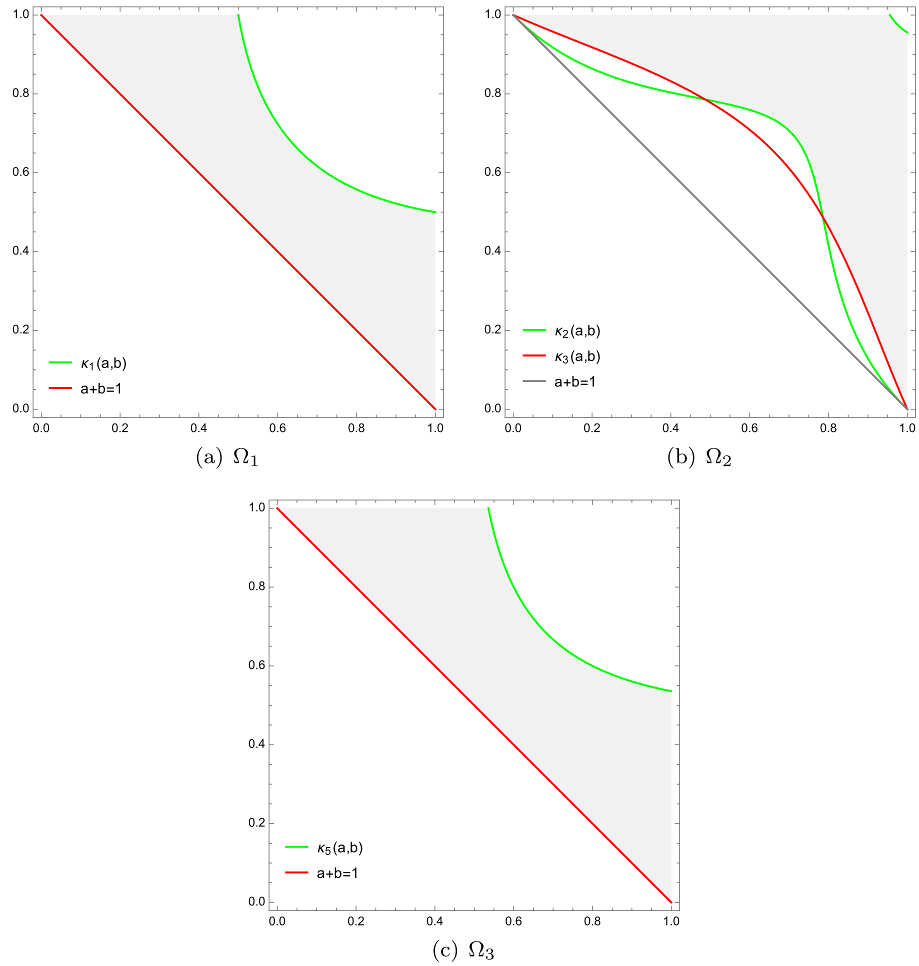

Figure 1 Visualized sub-regions Ω_1, Ω_2, and Ω_3 of $\{(a,b) \in \mathbb{R}^2 | 0 < a, b < 1\}$

2.2 Formulas

Let $r \in (0,1)$ and $0 < a, b < 1$ with $a + b > 1$. Then the following derivative formulas

$$\frac{d\mathcal{K}}{dr} = \frac{1}{rr'^2}\left[2b\left(\mathcal{E} - r'^2\mathcal{K}\right) + (a+b-1)(\mathcal{K} - \mathcal{E})\right], \tag{2.1}$$

$$\frac{d\mathcal{E}}{dr} = \frac{2(a-1)}{r}(\mathcal{K} - \mathcal{E}), \tag{2.2}$$

$$\frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{1}{rr'^2}\left[2br^2\mathcal{K} - \left(a+b-1+2(1-a)r^2\right)(\mathcal{K} - \mathcal{E})\right], \tag{2.3}$$

$$\frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = \frac{1}{r}\left[2(1-b)r^2\mathcal{K} + (a+b-1)(\mathcal{K} - \mathcal{E})\right] \tag{2.4}$$

can be found in [34, Theorem 4.15].

Note that Theorem 1.19(9) of [19] gives the derivative formula

$$\frac{d\mu_{a,b}(r)}{dr} = -\frac{(a+b-1)B^2 D}{4r^{a+b}r'^{a+b+1}\mathcal{K}^2} \tag{2.5}$$

for $\mu_{a,b}(r)$ if $d = c = (a+b+1)/2$.

From (1.7), (1.9), (1.10), and (2.1) we clearly see that

$$\frac{dm_{a,b}(r)}{dr} = \frac{4}{Br}\left[\big((2b-1)r^2\mathcal{K} - (b-a+1)(\mathcal{K}-\mathcal{E})\big)\mathcal{K}' - \frac{(a+b-1)BD}{4(rr')^{a+b-1}}\right]. \tag{2.6}$$

2.3 Lemmas

Lemma 2.1 ([70, Theorem 2.1]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \{0, 1, 2, \ldots\}$. Let $h(x) = f(x)/g(x)$ and $H_{f,g} = (f'/g')g - f$. Then the following statements hold true:*

(1) *If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for all $n \geq 0$, then $h(x)$ is strictly increasing (decreasing) on $(0, r)$;*
(2) *If there exists $n_0 > 0$ such that the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for $0 \leq n \leq n_0$ and decreasing (increasing) for $n \geq n_0$, then $h(x)$ is strictly increasing (decreasing) on $(0, r)$ if and only if $H_{f,g}(r^-) \geq (\leq) 0$. Moreover, if $H_{f,g}(r^-) < (>) 0$, then there exists $x_0 \in (0, r)$ such that $h(x)$ is strictly increasing (decreasing) on $(0, x_0)$ and strictly decreasing (increasing) on (x_0, r).*

Lemma 2.2 ([19, Theorem 1.25]) *Suppose that $-\infty < a < b < \infty$, $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.3 *Let $0 < a, b < 1$ with $a + b > 1$. Then the following assertions are valid:*

(i) *The function $(\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})$ is strictly increasing from $(0, 1)$ onto $(2b/(a + b + 1), 1)$;*
(ii) *The function $r'^{a+b-1}\mathcal{K}$ has positive Maclaurin coefficients and maps $(0, 1)$ onto $(B/2, D)$;*
(iii) *The function $r'^p\mathcal{K}$ is strictly decreasing from $(0, 1)$ onto $(0, B/2)$ if $p \geq 4ab/(a + b + 1)$.*

Proof Items (i) and (ii) follow directly from [34, Lemma 4.22]. We only need to prove item (iii).

It follows from (2.1) that

$$\begin{aligned}\frac{d(r'^p\mathcal{K})}{dr} &= \frac{r'^{p-2}}{r}\left[(2b-p)r^2\mathcal{K} - (b-a+1)(\mathcal{K}-\mathcal{E})\right] \\ &= (b-a+1)rr'^{p-2}\mathcal{K}\left[\frac{2b-p}{b-a+1} - \frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}}\right].\end{aligned} \tag{2.7}$$

Lemma 2.3(i) and (2.7) enable us to know that $r'^p\mathcal{K}$ is strictly decreasing on $(0, 1)$ if $(2b - p)/(b - a + 1) \leq 2b/(a + b + 1)$, that is, $p \geq 4ab/(a + b + 1)$.

Note that

$$r'^p\mathcal{K} = \frac{B(a,b)}{2} r'^{p+1-a-b} F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; r^2\right). \tag{2.8}$$

If $p \geq 4ab/(a + b + 1)$, then $p + 1 - a - b \geq (a - b + 1)(b - a + 1)/(a + b + 1) > 0$. This in conjunction with (1.2) and (2.8) gives $\lim_{r\to 1^-} r'^p\mathcal{K} = 0$. □

In the following Lemma 2.4 we provide an asymptotic formula for $\mathcal{K}$ as $r \to 1$ in the case of $a+b>1$, which is the analog for the zero-balanced hypergeometric function (1.2).

Lemma 2.4 *Let $0<a,b<1$ with $a+b>1$. Then one has*

$$\mathcal{K}(\sqrt{r}) = D(1-r)^{(1-a-b)/2} + E + o\big((1-r)^{\frac{a+b-1}{2}}\log(1-r)\big)$$

as $r \to 1$.

Proof It follows from $F(a,b;(a+b+1)/2;r)$ is asymptotic to $2D(1-r)^{(1-a-b)/2}/B$ [19, Theorem 1.19(5)] as $r \to 1$ for $a+b>1$ and the derivative formula

$$\frac{dF(a,b;c;r)}{dr} = \frac{ab}{c}F(a+1,b+1;c+1;r) \tag{2.9}$$

given in [19, (1.16)] for the hypergeometric function together with (1.2), and L'Hôpital's rule that

$$\begin{aligned}
&\lim_{r\to 1^-}\frac{\mathcal{K}(\sqrt{r}) - D(1-r)^{(1-a-b)/2} - E}{(1-r)^{(a+b-1)/2}\log(1-r)}\\
&\quad= \lim_{r\to 1^-}\frac{BF(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};r) - 2D - 2E(1-r)^{(a+b-1)/2}}{2(1-r)^{a+b-1}\log(1-r)}\\
&\quad= \lim_{r\to 1^-}\frac{[(b-a)^2-1]BF(\frac{b-a+3}{2},\frac{a-b+3}{2};\frac{a+b+3}{2};r) - 2[(a+b)^2-1]E(1-r)^{(a+b-3)/2}}{4(a+b+1)(1-r)^{a+b-2}[(a+b-1)\log(1-r)+1]}\\
&\quad= \lim_{r\to 1^-}\frac{[(b-a)^2-1]BF(a,b;\frac{a+b+3}{2};r) - 2[(a+b)^2-1]E}{4(a+b+1)(1-r)^{(a+b-1)/2}[(a+b-1)\log(1-r)+1]}\\
&\quad= \lim_{r\to 1^-}\frac{ab[1-(b-a)^2]B(1-r)^{2-(a+b)}F(\frac{b-a+3}{2},\frac{a-b+3}{2};\frac{a+b+5}{2};r)}{(a+b+3)[(a+b)^2-1][(a+b-1)\log(1-r)+3]}\\
&\quad= 0.
\end{aligned}$$

This completes the proof. □

Lemma 2.4 leads to Corollary 2.5 immediately.

Corollary 2.5 *Let $0<a,b<1$ and $a+b>1$. Then*

$$Dr^{1-a-b} + E - \mu_{a,b}(r) \to 0 \quad \textit{and} \quad Dr^{1-a-b} + E - m_{a,b}(r) \to 0$$

as $r \to 0$.

Proof By replacing r with $1-r^2$ in Lemma 2.4, we clearly see that

$$\mathcal{K}' = Dr^{1-a-b} + E + o\big(r^{a+b-1}\log r^2\big). \tag{2.10}$$

By definition, it is easy to know that $(\mathcal{K}-B/2)/r \to 0$ as $r \to 0$. This in conjunction with (2.10) and $a+b<2$ yields

$$\begin{aligned} &Dr^{1-a-b}+E-\mu_{a,b}(r) \\ &\quad = \frac{B}{2\mathcal{K}}\big(Dr^{1-a-b}+E-\mathcal{K}'\big)+\frac{1}{\mathcal{K}}(\mathcal{K}-B/2)\big(Dr^{1-a-b}+E\big) \to 0 \end{aligned}$$

as $r \to 0$. The second asymptotic formula can be proved by similar arguments. □

Lemma 2.6 *Let* $0<a,b<1$ *with* $a+b>1$. *Then the following assertions are valid*:

(i) *If* $\kappa_1(a,b) \geq 0$, *then* $\kappa_5(a,b)>0$ *and* $a+b<3/2$;

(ii) $\kappa_4(a,b)<\kappa_3(a,b)$;

(iii) *If* $\kappa_5(a,b) \geq 0$ *and* $a \leq b$, *then* $3-3a-b>0$.

Proof (i) We only need to prove that it is not possible for $\kappa_1(a,b) \geq 0$ and $\kappa_5(a,b) \leq 0$. By calculations, the inequality $\kappa_1(a,b) \geq 0$ is equivalent to $0<a \leq 1/2$ and $1-a<b<1$ or $1/2<a<1$ and $1-a<b \leq b_1(a)$, where $b_1(a)=\frac{1}{4a}[-1+8a-2a^2-\sqrt{1-24a+60a^2-32a^3+4a^4}]$ and $\kappa_5(a,b) \leq 0$ is equivalent to $1/2<2(2-\sqrt{3})<a<1$ and $b_2(a,b) \leq b<1$, where $b_2(a)=-1+5a-2\sqrt{3(2a^2-a)}$.

It remains to show that $b_2(a)>b_1(a)$ for $2(2-\sqrt{3})<a<1$. A simple calculation leads to

$$\begin{aligned} b_2(a)-b_1(a) = {}&\frac{1}{4a}\big[1-12a+22a^2+\sqrt{1-24a+60a^2-32a^3+4a^4}\big] \\ &-2\sqrt{3a(2a-1)}>0 \end{aligned}$$

if and only if

$$\begin{aligned} &\big(1-12a+22a^2+\sqrt{1-24a+60a^2-32a^3+4a^4}\big)^2-\big(8a\sqrt{3a(2a-1)}\big)^2 \\ &\quad = 2\big[\big(22a^2-12a+1\big)\sqrt{1-24a+60a^2-32a^3+4a^4} \\ &\qquad +1-24a+124a^2-184a^3+52a^4\big]>0, \end{aligned}$$

which is also equivalent to

$$\begin{aligned} &\big(22a^2-12a+1\big)^2\big(1-24a+60a^2-32a^3+4a^4\big) \\ &\qquad -\big(1-24a+124a^2-184a^3+52a^4\big)^2 \\ &\quad = 64a^3(a+1)(3-2a)(2a-1)\big(3a^2-3a+1\big)>0 \end{aligned}$$

for $1/2<a<1$. On the other hand, as we know, $\kappa_1(a,b)$ can be thought of as a quadratic function of b and the parabola opens up. It is easy to verify that $\kappa_1(a,1-a)=2(1-3a+3a^2)>0$ and $\kappa_1(a,1)=-(2-a)(2a-1)<0$ for $1/2<a<1$. Combining this with $\kappa_1(a,3/2-a)=-5(1-a)(a-1/2)<0$ for $1/2<a<1$, we conclude that $3/2-a<b<1$ makes $\kappa_1(a,b)$ negative. This completes the first assertion.

(ii) Observe that $\kappa_4(a,b)-\kappa_3(a,b)=2(a+b+1)(a+b-1)\mathcal{Q}(b)$, where

$$\mathcal{Q}(b)=a^2+a-2-(4a-1)b+b^2 \tag{2.11}$$

is a quadratic function in terms of b. Since the parabola of $\mathcal{Q}(b)$ opens up, it follows from $\mathcal{Q}(1-a) = -6a(1-a) < 0$ and $\mathcal{Q}(1) = -a(3-a) < 0$ that $\mathcal{Q}(b) < 0$ for $0 < a < 1$ and $1-a < b < 1$. This in conjunction with (2.11) yields $\kappa_4(a,b) < \kappa_3(a,b)$.

(iii) If the conclusion is not true, that is, $3-3a-b \le 0$, it follows that $b \ge \max\{a, 3(1-a)\}$. As we know, $\kappa_5(a,b) = b^2 - 2(5a-1)b + (a+1)^2$ is a quadratic function of b. We divide the proof into two cases.

Case 1: $a \ge 3(1-a)$. Then we clearly see that $a \le b < 1$ and $3/4 \le a < 1$. Since the symmetric axis $5a-1 > 1$, $\kappa_5(a,b)$ is strictly decreasing for $a < b < 1$. This gives $\kappa_5(a,b) \le \kappa_5(a,a) = -[8(a-3/4)^2 + 8(a-3/4) + 1/2] < 0$, which is a contradiction.

Case 2: $a < 3(1-a)$. In other words, $3(1-a) < b < 1$ and $2/3 < a < 3/4$. Similarly, the monotonicity of $\kappa_5(a,b)$ gives rise to $\kappa_5(a,b) \le \kappa_5(a, 3(1-a)) = 4(2a-1)(5a-4) < 0$, which is also a contradiction. □

Lemma 2.7 *Let $0 < a, b < 1$ with $a+b > 1$ and $a+b+1 \ge 4ab$, and $\varphi(r)$ be defined by*

$$\varphi(r) = \frac{1/r'^{a+b-1} - 1}{B^2/(4r'^{a+b+1}\mathcal{K}^2) - 1}.$$

Then $\varphi(r)$ is strictly decreasing from $(0,1)$ onto $(0, \frac{(a+b-1)(a+b+1)}{1+2a+2b+a^2+b^2-6ab})$.

Proof Let $\varphi_1(r) = 1/r'^{a+b-1} - 1$ and $\varphi_2(r) = B^2/(4r'^{a+b+1}\mathcal{K}^2) - 1$. Then $\varphi(r) = \varphi_1(r)/\varphi_2(r)$ and $\varphi_1(0) = \varphi_2(0) = 0$. Combining this with Lemma 2.2, we clearly see that the monotonicity of $\varphi(r)$ depends on $\varphi_1'(r)/\varphi_2'(r)$, that is,

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{4(a+b-1)}{B^2} \cdot \left(r'\mathcal{K}\right)^2 \cdot \frac{r^2\mathcal{K}}{(a+1-3b)r^2\mathcal{K} + 2(b-a+1)(\mathcal{K}-\mathcal{E})}. \tag{2.12}$$

It follows from Lemma 2.3(i) that $(a+1-3b) + 2(b-a+1)(\mathcal{K}-\mathcal{E})/(r^2\mathcal{K})$ is strictly increasing from $(0,1)$ onto $(\frac{1+2a+2b+a^2+b^2-6ab}{a+b+1}, 3-a-b)$. Since $a+b+1 \ge 4ab$, Lemma 2.3(iii) leads to the conclusion that $r'\mathcal{K}$ is strictly decreasing from $(0,1)$ onto $(0, B/2)$. This in conjunction with (2.12) implies that $\varphi_1'(r)/\varphi_2'(r)$ is strictly decreasing on $(0,1)$.

On the other hand, it follows from L'Hôpital's rule and (2.12) that

$$\varphi\left(0^+\right) = \lim_{r\to 0^+} \frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{(a+b-1)(a+b+1)}{1+2a+2b+a^2+b^2-6ab}, \qquad \varphi\left(1^-\right) = 0.$$ □

Lemma 2.8 *Let $(a,b) \in \Omega_1$ and $f(r)$ be defined by*

$$f(r) = \frac{(1-2b)r^2\mathcal{K} + (b-a+1)(\mathcal{K}-\mathcal{E})}{\frac{B^2}{4r'^{a+b+1}\mathcal{K}^2} - 1}.$$

Then $f(r)$ is strictly decreasing from $(0,1)$ onto $(0, \frac{(a+b+1-4ab)B}{1+2a+2b+a^2+b^2-6ab})$.

Proof Let $f_1(r) = (1-2b)r^2\mathcal{K} + (b-a+1)(\mathcal{K}-\mathcal{E})$ and $f_2(r) = B^2/(4r'^{a+b+1}\mathcal{K}^2) - 1$. Then we clearly see that $f(r) = f_1(r)/f_2(r)$ and $f_1(0) = f_2(0) = 0$.

By calculations, one has

$$\frac{f_1'(r)}{f_2'(r)} = \frac{4}{B^2} \cdot \left[r'^{(a+b+1)/3}\mathcal{K}\right]^3 \cdot \widehat{f}(r), \tag{2.13}$$

where

$$\widehat{f}(r)=\frac{[\sigma_1(a,b)+\sigma_2(a,b)r^2]r^2\mathcal{K}-[\sigma_3(a,b)+\sigma_4(a,b)r^2](\mathcal{K}-\mathcal{E})}{(a+1-3b)r^2\mathcal{K}+2(b-a+1)(\mathcal{K}-\mathcal{E})}\triangleq\frac{\widehat{f_1}(r)}{\widehat{f_2}(r)}\tag{2.14}$$

and

$$\sigma_1(a,b)=2\bigl(1-b-ab+b^2\bigr),\qquad \sigma_2(a,b)=2(1-b)(2b-1),$$
$$\sigma_3(a,b)=(a+b-1)(b-a+1),\qquad \sigma_4(a,b)=(b-a+1)(3-2a-2b).$$

Let

$$\widehat{f_{11}}(r)=\sigma_1(a,b)-\sigma_3(a,b)\frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}},\qquad \widehat{f_{12}}(r)=-r^2\left[\sigma_4(a,b)\frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}}-\sigma_2(a,b)\right].$$

Then $\widehat{f_1}(r)/(r^2\mathcal{K})=\widehat{f_{11}}(r)+\widehat{f_{12}}(r)$.

It follows from Lemma 2.3(i) and $\sigma_3(a,b)>0$ that $\widehat{f_{11}}(r)$ is strictly decreasing on $(0,1)$. For $(a,b)\in\Omega_1$, namely $0<a,b<1$, $a+b>1$ and $\kappa_1(a,b)\geq 0$, we clearly see from Lemma 2.6(i) that $\kappa_5(a,b)=(a+b+1)^2-12ab>0$, $a+b<3/2$, and then $\sigma_4(a,b)>0$. This in conjunction with Lemma 2.3(i), (iii) implies that $r'^{(a+b+1)/3}\mathcal{K}$ is strictly decreasing on $(0,1)$ and

$$\sigma_4(a,b)\frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}}-\sigma_2(a,b)>\sigma_4(a,b)\cdot\frac{2b}{a+b+1}-\sigma_2(a,b)=\frac{2\kappa_1(a,b)}{a+b+1}\geq 0.\tag{2.15}$$

Lemma 2.3(i) and (2.15) enable us to know that $\widehat{f_{12}}(r)$ is strictly decreasing on $(0,1)$. This gives the monotonicity of $\widehat{f_1}(r)/(r^2\mathcal{K})$. So $\widehat{f_1}(r)/(r^2\mathcal{K})>\sigma_1(a,b)+\sigma_2(a,b)-\sigma_3(a,b)-\sigma_4(a,b)=(2-a-b)(a+b-1)>0$. Moreover, it is easy to verify from Lemma 2.3(i) that $\widehat{f_2}(r)/(r^2\mathcal{K})$ is strictly increasing from $(0,1)$ onto $(\frac{1+2a+2b+a^2+b^2-6ab}{a+b+1},3-a-b)$. Combining with (2.14), the monotonicity of $\widehat{f_1}(r)/(r^2\mathcal{K})$ and $\widehat{f_2}(r)/(r^2\mathcal{K})$ leads to the conclusion that $\widehat{f}(r)$ is strictly decreasing on $(0,1)$.

Therefore, the monotonicity of $f(r)$ follows from Lemma 2.2 and (2.13) together with the monotonicity of $r'^{(a+b+1)/3}\mathcal{K}$ and $\widehat{f}(r)$.

To this end, by L'Hôpital's rule and (2.13), (2.14),

$$f\bigl(0^+\bigr)=\lim_{r\to 0^+}\frac{f_1'(r)}{f_2'(r)}=\frac{(a+b+1-4ab)B}{1+2a+2b+a^2+b^2-6ab},\qquad f\bigl(1^-\bigr)=0.\qquad\square$$

Lemma 2.9 *Let $(a,b)\in\Omega_2$ and $g(r)$ be defined by*

$$g(r)=\frac{B^2/(4r'^{a+b+1}\mathcal{K}^2)-1}{[2br^2\mathcal{K}+(a-b-1)(\mathcal{K}-\mathcal{E})]/r'^2}.$$

Then $g(r)$ is strictly decreasing from $(0,1)$ onto $(0,\frac{1+2a+2b+a^2+b^2-6ab}{4abB})$.

Proof Let $g_1(r)=B^2/(4r'^{a+b+1}\mathcal{K}^2)-1$ and $g_2(r)=[2br^2\mathcal{K}+(a-b-1)(\mathcal{K}-\mathcal{E})]/r'^2$. Then $g(r)=g_1(r)/g_2(r)$ and $g_1(0)=g_2(0)=0$.

By calculations, one has

$$\frac{g_1'(r)}{g_2'(r)}=\frac{B^2}{4[r'^{a+b-1}\mathcal{K}]^2}\cdot\frac{\widehat{g_1}(r)}{\widehat{g_2}(r)},\tag{2.16}$$

where

$$\widehat{g}_1(r) = \frac{r'^{a+b-1}[(a-3b+1)r^2\mathcal{K} + 2(b-a+1)(\mathcal{K}-\mathcal{E})]}{r^2\mathcal{K}}, \tag{2.17}$$

$$\widehat{g}_2(r) = \frac{[\lambda_1(a,b) + \lambda_2(a,b)r^2]r^2\mathcal{K} + [\lambda_3(a,b) + \lambda_4(a,b)r^2](\mathcal{K}-\mathcal{E})}{r^2} \tag{2.18}$$

and

$$\lambda_1(a,b) = 2b(a-b+1), \qquad \lambda_2(a,b) = 4b^2,$$
$$\lambda_3(a,b) = (b-a+1)(a+b-1), \qquad \lambda_4(a,b) = -2(a+b)(b-a+1).$$

By (2.2) and (2.9), we clearly see that

$$\begin{aligned}\frac{\mathcal{K}-\mathcal{E}}{r^2} &= \frac{B(a,b)}{4(a-1)r}\frac{dF(a-1,b;(a+b+1)/2;r^2)}{dr}\\ &= \frac{bB(a,b)}{a+b+1}F\big(a,b+1;(a+b+3)/2;r^2\big).\end{aligned} \tag{2.19}$$

It follows from (1.2), (1.5), and (2.19) that

$$r'^{a+b-1}\mathcal{K} = \frac{B(a,b)}{2}F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};r^2\right), \tag{2.20}$$

$$r'^{a+b-1}\frac{\mathcal{K}-\mathcal{E}}{r^2} = \frac{B(a,b)b}{a+b+1}F\left(\frac{b-a+3}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};r^2\right). \tag{2.21}$$

Combining with (2.17), (2.18), (2.20), and (2.21), we rewrite $\widehat{g}_1(r)$ and $\widehat{g}_2(r)$ in terms of power series:

$$\widehat{g}_1(r) = \frac{\sum_{n=0}^{\infty}\frac{(\frac{a-b+1}{2},n)(\frac{b-a+1}{2},n)}{(\frac{a+b+1}{2},n)n!}\xi_{a,b}(n)r^{2n}}{\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(\frac{a+b+1}{2},n)n!}r^{2n}}, \tag{2.22}$$

$$\widehat{g}_2(r) = \frac{B(a,b)}{2}\sum_{n=0}^{\infty}\frac{(a,n-1)(b,n-1)}{(\frac{a+b+1}{2},n+1)n!}\zeta_{a,b}(n)r^{2n}, \tag{2.23}$$

where

$$\begin{aligned}\xi_{a,b}(n) &= \frac{1+2a+2b+a^2+b^2-6ab+2(a+b+1)n}{1+a+b+2n},\\ \zeta_{a,b}(n) &= \big[(a+b)^2-1\big]n^3 + 2(a+b-1)(a+b+2ab-1)n^2\\ &\quad + \big[4ab(a+b+ab-1) - 3(a+b)^2 + 4(a+b) - 1\big]n\\ &\quad + 4ab(1-a)(1-b).\end{aligned}$$

We now claim that $\widehat{g}_1(r)$ is strictly decreasing on $(0,1)$ and $\widehat{g}_2(r)$ is strictly increasing on $(0,1)$; furthermore, $\widehat{g}_2(r)$ has positive Maclaurin coefficients.

- Lemma 2.1 and (2.22) enable us to know that the monotonicity of $\widehat{g}_1(r)$ depends on the monotonicity of the following sequence:

$$\{\alpha_n\}_{n\geq 0} = \left\{ \frac{(\frac{a-b+1}{2}, n)(\frac{b-a+1}{2}, n)}{(a,n)(b,n)} \xi_{a,b}(n) \right\}_{n\geq 0}.$$

A simple calculation yields

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{(1+b-a+2n)(1+a-b+2n)\xi_{a,b}(n+1)}{4(b+n)(a+n)\xi_{a,b}(n)} \leq 1$$

if and only if

$$\begin{aligned}\Delta_{a,b}(n) &= (1+b-a+2n)(1+a-b+2n)\xi_{a,b}(n+1) \\ &\quad - 4(a+n)(b+n)\xi_{a,b}(n) \\ &= \frac{\widehat{\Delta}_{a,b}(n)}{(1+a+b+2n)(3+a+b+2n)} \leq 0,\end{aligned} \tag{2.24}$$

where

$$\widehat{\Delta}_{a,b}(n) = \kappa_2(a,b) + 4\kappa_3(a,b)n + 4\kappa_4(a,b)n^2 - 16\big[(a+b)^2 - 1\big]n^3. \tag{2.25}$$

For $(a,b) \in \Omega_2$, namely $0 < a,b < 1$, $a+b > 1$, $\kappa_2(a,b) \leq 0$, $\kappa_3(a,b) \leq 0$, and then $\kappa_4(a,b) \leq 0$ by Lemma 2.6(ii). This in conjunction with (2.24) and (2.25) implies that the sequence $\{\alpha_n\}_{n\geq 0}$ is decreasing. So the first assertion is valid.

- We mention that the Pochhammer symbol $(a,-1)(b,-1) = \frac{1}{(a-1)(b-1)} > 0$ for $0 < a,b < 1$. It only needs to prove $\zeta_{a,b}(n) > 0$ for $n \geq 0$ with $0 < a,b < 1$ and $a+b > 1$.

  Clearly, $\zeta_{a,b}(0) = 4ab(1-a)(1-b) > 0$ and $\zeta_{a,b}(1) = 4ab(a+b+2ab-1) > 0$. Moreover, $\zeta'_{a,b}(n)$ is strictly increasing for $n \geq 0$. This gives $\zeta'_{a,b}(n) \geq \zeta'_{a,b}(1) = 4q(b)$ for $n \geq 1$, where $q(b) = (a^2+3a+1)b^2 + (3a^2-a-1)b + a(a-1)$ is regarded as a quadratic function in terms of b and its parabola opens up.

  Observe that

$$-\frac{3a^2-a-1}{2(a^2+3a+1)} - (1-a) = -\frac{a(1-a)(2a+3)+1}{2(a^2+3a+1)} < 0,$$

that is to say, the symmetric axis of $q(b)$ lies on the left side of the interval $[1-a, 1]$. This in conjunction with $q(1-a) = a^2(a-1)^2 > 0$ implies that $q(b) > 0$ for $1-a < b < 1$. So $\zeta_{a,b}(n)$ is strictly increasing for $n \geq 1$ and $\zeta_{a,b}(n) \geq \zeta_{a,b}(1) > 0$ for $n \geq 1$. This completes the second assertion.

Therefore, $\widehat{g}_1(r)/\widehat{g}_2(r)$ is strictly decreasing on $(0,1)$ follows from the above assertions together with $\widehat{g}_1(r) > 0$ and $\widehat{g}_2(r) > 0$. Combining this with (2.16), Lemma 2.2 and Lemma 2.3(ii), we conclude that $g(r)$ is strictly decreasing on $(0,1)$.

It remains to compute two end values of $g(r)$. By L'Hôpital's rule and (2.16) together with Lemma 2.3(i), (ii),

$$g\big(0^+\big) = \lim_{r\to 0^+} \frac{g_1'(r)}{g_2'(r)} = \frac{1+2a+2b+a^2+b^2-6ab}{4abB}, \qquad g\big(1^-\big) = 0. \qquad \square$$

Lemma 2.10 *Let* $0 < a \leq b < 1$ *with* $a + b > 1$ *and* $\kappa_5(a, b) \geq 0$, *and* $h(r)$ *be defined by*

$$h(r) = \frac{2br^2\mathcal{K} - [2(a + b - 1) + (3 - 3a - b)r^2](\mathcal{K} - \mathcal{E})}{(a - 3b + 1)r^2\mathcal{K} + 2(b - a + 1)(\mathcal{K} - \mathcal{E})}.$$

Then $h(r)$ *is strictly decreasing from* $(0, 1)$ *onto* $(\frac{a+b-1}{3-a-b}, \frac{2b(3-a-b)}{1+2a+2b+a^2+b^2-6ab})$.

Proof We denote by $h_1(r) = 2b - [2(a + b - 1) + (3 - 3a - b)r^2](\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})$ and $h_2(r) = (a - 3b + 1) + 2(b - a + 1)(\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})$.

If $0 < a \leq b < 1$, $a + b > 1$, and $\kappa_5(a, b) \geq 0$, then $3 - 3a - b > 0$ follows from Lemma 2.6(iii). Combining this with Lemma 2.3(i), we conclude that $h_1(r)$ is strictly decreasing from $(0, 1)$ onto $(a + b - 1, \frac{2b(3-a-b)}{a+b+1})$ and $h_2(r)$ is strictly increasing from $(0, 1)$ onto $(\frac{1+2a+2b+a^2+b^2-6ab}{a+b+1}, 3 - a - b)$. This gives the monotonicity of $h(r) = h_1(r)/h_2(r)$ together with two limiting values $h(0^+)$ and $h(1^-)$. □

3 Main results

Theorem 3.1 *Let* $(a, b) \in \Omega_1$ *and* $F(r)$ *be defined on* $(0, 1)$ *by*

$$F(r) = \frac{Dr^{1-a-b} + E - m_{a,b}(r)}{Dr^{1-a-b} + E - \mu_{a,b}(r)}.$$

Then $F(r)$ *is strictly decreasing from* $(0, 1)$ *onto* $(1, L_0)$, *where*

$$L_0 = \frac{(a + b + 1)[(a + b - 1)^2 + 4] - 16ab}{(a + b - 1)(1 + 2a + 2b + a^2 + b^2 - 6ab)}.$$

In particular, the double inequality

$$m_{a,b}(r) < \mu_{a,b}(r) < \frac{1}{L_0}m_{a,b}(r) + \left(1 - \frac{1}{L_0}\right)\left(Dr^{1-a-b} + E\right)$$

holds for $r \in (0, 1)$.

Proof Let $F_1(r) = Dr^{1-a-b} + E - m_{a,b}(r)$ and $F_2(r) = Dr^{1-a-b} + E - \mu_{a,b}(r)$. Clearly, $F(r) = F_1(r)/F_2(r)$ and $F_1(0^+) = F_2(0^+) = 0$ follow from Corollary 2.5.

By calculations, one has

$$\begin{aligned}\frac{F_1'(r)}{F_2'(r)} &= \frac{(\frac{1}{r'^{a+b-1}} - 1) + \frac{4r^{a+b-1}\mathcal{K}'}{(a+b-1)BD}[(1 - 2b)r^2\mathcal{K} + (b - a + 1)(\mathcal{K} - \mathcal{E})]}{\frac{B^2}{4r'^{a+b+1}\mathcal{K}^2} - 1} \\ &= \varphi(r) + \frac{4}{(a + b - 1)BD} \cdot r^{a+b-1}\mathcal{K}' \cdot f(r), \qquad (3.1)\end{aligned}$$

where $\varphi(r)$ and $f(r)$ are defined as in Lemma 2.7 and Lemma 2.8, respectively.

Since $r^{a+b-1}\mathcal{K}'$ can be regarded as the composition of $x'^{a+b-1}\mathcal{K}(x)$ and $x = r' = \sqrt{1 - r^2}$, Lemma 2.3(ii) enables us to know that $r^{a+b-1}\mathcal{K}'$ is strictly decreasing from $(0, 1)$ onto $(B/2, D)$. This in conjunction with (3.1) together with Lemma 2.2, Lemma 2.7, and Lemma 2.8 gives rise to the monotonicity of $F(r)$ and also, by L'Hôpital's rule and (3.1),

$$F(0^+) = \lim_{r\to 0^+} \frac{F_1'(r)}{F_2'(r)} = \varphi(0^+) + \frac{4}{(a+b-1)BD} \cdot D \cdot f(0^+) = L_0,$$

and $F(1^-) = 1$ follows directly from $m_{a,b}(1^-) = \mu_{a,b}(1^-) = 0$. □

Corollary 3.2 *Let* $(a,b) \in \Omega_1$ *and* $\widehat{F}(r)$ *be defined on* $(0,1)$ *by*

$$\widehat{F}(r) = \frac{m_{a,b}(r) - D(r^{1-a-b} - 1)}{\mu_{a,b}(r) - D(r^{1-a-b} - 1)}.$$

Then $\widehat{F}(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$.

Proof Let $\widehat{F}_1(r) = m_{a,b}(r) - D(r^{1-a-b} - 1)$ and $\widehat{F}_2(r) = \mu_{a,b}(r) - D(r^{1-a-b} - 1)$. Then $\widehat{F}(r) = \widehat{F}_1(r)/\widehat{F}_2(r)$ and $\widehat{F}_1(1^-) = \widehat{F}_2(1^-) = 0$.

Since $\widehat{F}_1'(r)/\widehat{F}_2'(r) = F_1'(r)/F_2'(r)$, Lemma 2.2 enables us to know the monotonicity of $\widehat{F}(r)$ depends on that of $F_1'(r)/F_2'(r)$, which follows from Theorem 3.1. It only remains to compute two limiting values $\widehat{F}(0^+)$ and $\widehat{F}(1^-)$.

By Corollary 2.5, it is easy to see that $\widehat{F}(0^+) = (D+E)/(D+E) = 1$. By L'Hôpital's rule and (3.1) together with Lemma 2.7, Lemma 2.8,

$$\widehat{F}(1^-) = \lim_{r\to 1^-} \frac{\widehat{F}_1'(r)}{\widehat{F}_2'(r)} = \varphi(1^-) + \frac{4}{(a+b-1)BD} \cdot \frac{B}{2} \cdot f(1^-) = 0.$$ □

Theorem 3.3 *Let* $(a,b) \in \Omega_2$ *and* $G(r)$ *be defined on* $(0,1)$ *by*

$$G(r) = \frac{Dr^{1-a-b} + E - \mu_{a,b}(r)}{\mathcal{K} - B/2}.$$

Then $G(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\infty)$.

Proof We denote $G_1(r) = Dr^{1-a-b} + E - \mu_{a,b}(r)$ and $G_2(r) = \mathcal{K} - B/2$. Then we clearly see that $G(r) = G_1(r)/G_2(r)$ and $G_1(0^+) = G_2(0^+) = 0$.

By taking the derivative of $G_1(r)$ and $G_2(r)$, one has

$$\begin{aligned}\frac{G_1'(r)}{G_2'(r)} &= \frac{(a+b-1)D}{r^{a+b-1}} \cdot \frac{B^2/(4r'^{a+b+1}\mathcal{K}^2) - 1}{[2br^2\mathcal{K} + (a-b-1)(\mathcal{K}-\mathcal{E})]/r'^2} \\ &= \frac{(a+b-1)D}{r^{a+b-1}} \cdot g(r),\end{aligned} \tag{3.2}$$

where $g(r)$ is defined as in Lemma 2.9.

Therefore, the monotonicity of $G(r)$ follows from Lemma 2.9 and that of $1/r^{a+b-1}$.

To this end, by L'Hôpital's rule and (3.2),

$$G(0^+) = \lim_{r\to 0^+} \frac{G_1'(r)}{G_2'(r)} = \lim_{r\to 0^+} \frac{(a+b-1)D}{r^{a+b-1}} \cdot g(0^+) = \infty, \qquad G(1^-) = 0.$$ □

Theorem 3.4 *Let* $(a,b) \in \Omega_3$ *and* $H(r)$ *be defined on* $(0,1)$ *by*

$$H(r) = \frac{r^{1-a-b}(B/2 - \mathcal{E})}{Dr^{1-a-b} + E - \mu_{a,b}(r)}.$$

Then $H(r)$ is strictly decreasing from $(0,1)$ onto (L_1, L_2), where

$$L_1 = \frac{2b(1-a)(3-a-b)B}{(a+b-1)(1+2a+2b+a^2+b^2-6ab)D},$$
$$L_2 = \frac{(b-a+1)B+2(a+b-1)E}{2(b-a+1)(D+E)}.$$

As a consequence, the double inequality

$$r^{1-a-b}\left(D-\frac{B}{2L_1}+\mathcal{E}\right)+E<\mu_{a,b}(r)<r^{1-a-b}\left(D-\frac{B}{2L_2}+\mathcal{E}\right)+E$$

holds for $r\in(0,1)$.

Proof Since $H(r)$ is symmetric with respect to a, b, we may assume that $0<a\le b<1$. Let $H_1(r)=r^{1-a-b}(B/2-\mathcal{E})$ and $H_2(r)=Dr^{1-a-b}+E-\mu_{a,b}(r)$. Then we clearly see from Corollary 2.5 and $a+b<2$ that $H_1(r)=H_1(r)/H_2(r)$ and $H_1(0^+)=H_2(0^+)=0$.

Moreover,

$$\begin{aligned}\frac{H_1'(r)}{H_2'(r)} &= \frac{r^{-(a+b)}[(1-a-b)B/2+(3a+b-3)\mathcal{E}+2(1-a)\mathcal{K}]}{(a+b-1)Dr^{-(a+b)}(\frac{B^2}{4r'^{a+b+1}\mathcal{K}^2}-1)}\\ &= \frac{1}{(a+b-1)D}\cdot\frac{(1-a-b)B/2+(3a+b-3)\mathcal{E}+2(1-a)\mathcal{K}}{(\frac{B^2}{4r'^{a+b+1}\mathcal{K}^2}-1)}\\ &\triangleq \frac{H_{11}(r)}{H_{22}(r)},\end{aligned} \tag{3.3}$$
$$H_{11}(0^+)=H_{22}(0^+)=0,$$

and

$$\frac{H_{11}'(r)}{H_{22}'(r)}=\frac{8(1-a)}{(a+b-1)B^2D}\cdot\left(r'^{\frac{a+b+1}{3}}\mathcal{K}\right)^3\cdot h(r), \tag{3.4}$$

where $h(r)$ is defined as in Lemma 2.10.

If $(a,b)\in\Omega_3$, in other words, $0<a\le b<1$, $a+b>1$, and $\kappa_5(a,b)=(a+b+1)^2-12ab\ge 0$, then it follows from Lemma 2.3(iii) and Lemma 2.10 that $r'^{\frac{a+b+1}{3}}\mathcal{K}$ is strictly decreasing on $(0,1)$ and $h(r)$ is strictly decreasing on $(0,1)$. This in conjunction with (3.3), (3.4), and Lemma 2.2 implies that $H(r)$ is strictly decreasing on $(0,1)$. By L'Hôpital's rule together with Lemma 2.10 and (3.3), (3.4),

$$\begin{aligned}H(0^+) &= \lim_{r\to 0^+}H(r)=\lim_{r\to 0^+}\frac{H_1'(r)}{H_2'(r)}=\lim_{r\to 0^+}\frac{H_{11}'(r)}{H_{22}'(r)}\\ &= \frac{8(1-a)}{(a+b-1)B^2D}\cdot\left(\frac{B}{2}\right)^3\cdot h(0^+)=L_1,\end{aligned}$$

and $H(1^-)=L_2$ follows easily from $\mu_{a,b}(1^-)=0$. □

Open Problem What is the sub-region of $\{(a,b)\in\mathbb{R}^2 | 0<a,b<1\}$ such that the function

$$\widehat{G}(r)=\frac{Dr^{1-a-b}+E-\mu_{a,b}(r)}{r^{1-a-b}(\mathcal{K}-B/2)}$$

is strictly decreasing from $(0,1)$ onto $(0,L_3)$, where

$$L_3 = \frac{(a+b-1)(1+2a+2b+a^2+b^2-6ab)D}{2ab(3-a-b)B}.$$

4 Consequences and discussion

In the article, we study the monotonicity of the functions $F(r)$, $G(r)$, and $H(r)$ related to generalized Grötzsch ring function and generalized elliptic integrals, where $F(r)$, $G(r)$, and $H(r)$ are explicitly given by

$$F(r) = \frac{Dr^{1-a-b}+E-m_{a,b}(r)}{Dr^{1-a-b}+E-\mu_{a,b}(r)}, \qquad G(r) = \frac{Dr^{1-a-b}+E-\mu_{a,b}(r)}{\mathcal{K}-B/2},$$

and

$$H(r) = \frac{r^{1-a-b}(B/2-\mathcal{E})}{Dr^{1-a-b}+E-\mu_{a,b}(r)}.$$

5 Conclusion

In the article, we have found the sub-regions of $\{(a,b)\in\mathbb{R}^2|0<a,b<1,a+b>1\}$ such that several quotient functions involving $\mu_{a,b}(r)$, $\mathcal{K}_{a,b}(r)$, $\mathcal{E}_{a,b}(r)$, and $m_{a,b}(r)$ are monotonic on their corresponding sub-regions, and established several inequalities for $\mu_{a,b}(r)$ and $m_{a,b}(r)$. Our results are the variants and extensions of the previous results of [42, Theorems 1.1 and 1.2] in the case of zero-balanced.

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Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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