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The existence of ground state solution to elliptic equation with exponential growth on complete noncompact Riemannian manifold

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Abstract

In this paper, we consider the following elliptic problem:

$$-\operatorname{div}_g(|\nabla_g u|^{N-2} \nabla_g u) + V(x)|u|^{N-2} u = \frac{f(x, u)}{a(x)} \quad \text{in } M, \quad (P_a)$$

where (M, g) be a complete noncompact N -dimensional Riemannian manifold with negative curvature, $N \geq 2$, V is a continuous function satisfying $V(x) \geq V_0 > 0$, a is a nonnegative function and $f(x, t)$ has exponential growth with t in view of the Trudinger–Moser inequality. By proving some estimates together with the variational techniques, we get a ground state solution of (P_a) . Moreover, we also get a nontrivial weak solution to the perturbation problem.

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1 Introduction and main results

Suppose $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and we consider the following problem:

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(x, u), \quad x \in \Omega, \quad (1.1)$$

when $p = 2$, $|f(x, u)| \leq c(|u| + |u|^{q-1})$, $1 < q \leq 2^* = \frac{2N}{N-2}$, $N > 2$, for corresponding results one may refer to Brézis [1], Brézis and Nirenberg [2], Bartsch and Willem [3], Capozzi, Fortunato and Palmieri [4]. Garcia and Alonso [5] generalized Brézis and Nirenberg's existence and nonexistence results to the p -Laplace equation. When $p = N$ and $f(x, u)$ behaves like $e^{\alpha|u|^{\frac{N}{N-1}}}$ as $|u| \rightarrow \infty$, problem (1.1) was studied by Adimurthi [6], Adimurthi and Yadava [7], Ruf *et al.* [8, 9], do Ó [10], Panda [11] and the references therein. Specially, Lam and Lu [12] study the case which does not satisfy the Ambrosetti–Rabinowitz condition. All these results are based on the Trudinger–Moser inequality [13, 14] and critical point theory.

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For the following semilinear Schrödinger equation:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f(x, u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

when $p = 2$, $|f(x, u)| \leq c(|u| + |u|^{q-1})$, $1 < q \leq 2^* = \frac{2N}{N-2}$, $N > 2$, for the relevant literature one is referred to Kryszewski and Szulkin [15], Alama and Li [16], Ding and Ni [17] and Jeanjean [18]. When $p = N$, $f(x, s)$ is continuous in $\mathbb{R}^N \times \mathbb{R}$ and behaves like $e^{\alpha|s|^{\frac{N}{N-1}}}$ as $|s| \rightarrow \infty$, it was studied by Cao [19] in the case $N = 2$, by Panda [20], do Ó [21], Alves, Figueiredo [22] and do Ó *et al.* [23] in general dimensional cases. We note that the Sobolev embedding theorem can be applied to the $p < N$ case, while the Trudinger–Moser type embedding theorem can be applied to the $p = N$ case.

The Trudinger–Moser inequalities have been generalized to some Riemannian manifolds. In the case of compact Riemannian manifolds, the study of Trudinger–Moser inequalities can be traced back to Aubin [24], Cherrier [25, 26], and Fontana [27]. On a compact Riemannian surface, Ding *et al.* [28] established a Trudinger–Moser inequality and used it to deal with the prescribed Gaussian curvature problem by the method of blow-up analysis. Motivated by the ideas of Ding *et al.*, the existence of extremal functions was proved by Li in [29, 30], Yang also got extremal functions for several Trudinger–Moser type inequalities in [31, 32].

When (M, g) is any complete noncompact N -dimensional Riemannian manifold with no boundary, $N \geq 2$, if its Ricci curvature is bounded from below and injectivity radius is bounded from below by a positive number, Yang in [33] established some Trudinger–Moser inequalities and applied them to some quasilinear equations. Recently, Yang *et al.* [34] derived a sharp Trudinger–Moser inequality on a complete, simply connected N -dimensional Riemannian manifold with negative curvature. We state it below for further use.

Proposition A ([34]) *Let (M, g) be a complete, simply connected N -dimensional Riemannian manifold with negative curvature, $N \geq 2$. Then for any $\tau > 0$ there exists a positive constant $C = C(\tau, N, M)$ such that*

$$\sup_{u \in W^{1,N}(M), \|u\|_{1,\tau} \leq 1} \int_M (\Phi(\alpha_N |u_n|^{\frac{N}{N-1}})) dv_g \leq C.$$

Furthermore, the constant $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ is sharp, where ω_{N-1} is the surface measure of the unit sphere in \mathbb{R}^N . $\Phi(x) = e^x - \sum_{k=0}^{N-2} \frac{x^k}{k!}$ and

$$\|u\|_{1,\tau} = \left(\int_M |\nabla_g u|^N dv_g + \tau \int_M |u|^N dv_g \right)^{\frac{1}{N}}.$$

More recently, Kristály [35] studied some geometric features of Trudinger–Moser inequalities on complete noncompact Riemannian manifolds and proved the existence of a non-zero isometry-invariant solution for a class of N -Laplacian equation by combining variational techniques and the symmetrization-compactness principle.

In this paper, we always assume that (M, g) is a complete, simply connected N -dimensional Riemannian manifold with negative curvature, $N \geq 2$. And on (M, g) , we

consider the following quasilinear equations:

$$-\operatorname{div}_g(|\nabla_g u|^{N-2} \nabla_g u) + V(x)|u|^{N-2} u = \frac{f(x, u)}{a(x)} \quad \text{in } M, \tag{P_a}$$

$$-\operatorname{div}_g(|\nabla_g u|^{N-2} \nabla_g u) + V(x)|u|^{N-2} u = \frac{f(x, u)}{a(x)} + \epsilon h(x) \quad \text{in } M, \tag{P}$$

where $\epsilon > 0$, $h \in E^*$, $h \geq 0$ and $h \not\equiv 0$, (M, g) is a complete noncompact N -dimensional Riemannian manifold with negative curvature, $N \geq 2$, ∇_g denotes its covariant derivative (the Levi-Civita connection), and div_g denotes its divergence operator, V is a continuous function satisfying $V(x) \geq V_0 > 0$, a is a nonnegative function and $f(x, t)$ has exponential growth with t in view of the Trudinger–Moser inequality. Since we are concerned with nonnegative weak solutions, we require that $f(x, t) = 0$ for all $(x, t) \in M \times (-\infty, 0]$. Furthermore, let O be a fixed point of M and $d_g(\cdot, \cdot)$ be the geodesic distance between two points of (M, g) . We assume the function a satisfies:

(a₁) $a^{-1} \in L^s_{\text{loc}}(M)$ for some $s > 1$, i.e., $a^{-1} \in L^s(B_\rho(O))$ for any $\rho > 0$.

(a₂) a^{-1} is bounded in $M \setminus B_\rho(O)$ for any $\rho > 0$.

We assume the following conditions on V .

(V₁) There exists a constant $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in M$.

(V₂) $V(x) \rightarrow +\infty$ as $d_g(O, x) \rightarrow +\infty$.

The function f satisfies:

(f₁) There exist constants $\alpha_0, c_1, c_2 > 0$ such that, for all $(x, t) \in M \times \mathbb{R}^+$,

$$f(x, t) \leq c_1 |t|^{N-1} + c_2 \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}}),$$

where

$$\Phi(x) = e^x - \sum_{k=0}^{N-2} \frac{x^k}{k!}.$$

(f₂) $\lim_{|t| \rightarrow +\infty} \frac{F(x, t)}{|t|^N} = \infty$ uniformly on $x \in M$, where $F(x, t) = \int_0^t f(x, s) ds$.

(f₃) There exists $C > 0$ such that, for all $(x, t) \in M \times \mathbb{R}^+$, $F(x, t) \leq C|t|^N + Cf(x, t)$.

Define a function space E as

$$E = \left\{ u \in W^{1,N}(M) : \int_M (|\nabla_g u|^N + V(x)|u|^N) dv_g < \infty \right\}$$

equipped with the norm

$$\|u\|_E = \left(\int_M (|\nabla_g u|^N + V(x)|u|^N) dv_g \right)^{\frac{1}{N}},$$

then the assumption $V(x) \geq V_0 > 0$ implies that E is a reflexive Banach space. For any $p \geq N$, we define

$$S_p = \inf_{u \in E \setminus \{0\}} \frac{\|u\|_E}{\left(\int_M \frac{|u|^p}{a(x)} dv_g \right)^{\frac{1}{p}}} \tag{1.3}$$

and

$$\lambda_a = \inf_{u \in E \setminus \{0\}} \frac{\|u\|_E^N}{\int_{\mathbb{R}^N} \frac{|u|^N}{a(x)} dx}.$$

The continuous embedding of $E \hookrightarrow W^{1,N}(M) \hookrightarrow L^p(M)$ ($p \geq N$) and the Hölder inequality imply

$$\begin{aligned} \int_M \frac{|u|^p}{a(x)} dv_g &= \int_{M \setminus B_\rho(O)} \frac{|u|^p}{a(x)} dv_g + \int_{B_\rho(O)} \frac{|u|^p}{a(x)} dv_g \\ &\leq C \int_{M \setminus B_\rho(O)} |u|^p dv_g + \left(\int_{B_\rho(O)} |u|^{ps'} dv_g \right)^{\frac{1}{s'}} \left(\int_{B_\rho(O)} \left(\frac{1}{a(x)} \right)^s dv_g \right)^{\frac{1}{s}} \\ &\leq C \|u\|_E^p, \end{aligned}$$

here s is stated in the condition (a_1) . Thus we have $S_p > 0$. We now introduce the following three conditions.

- (f_4) $\limsup_{t \rightarrow 0^+} \frac{NF(x,t)}{|t|^N} < \lambda_a$ uniformly in M .
- (f_5) There exist constants $p > N$ and C_p such that

$$f(x,t) \geq C_p t^{p-1},$$

where

$$C_p > \left(\frac{p-N}{p} \right)^{\frac{p-N}{N}} \left(\frac{s\alpha_0}{(s-1)\alpha_N} \right)^{\frac{(p-N)(N-1)}{N}} S_p^p.$$

- (f_6) $H(x,t) = tf(x,t) - NF(x,t)$ is strictly increasing in $t > 0$.

Our main results can be stated as follows.

Theorem 1.1 *Suppose V satisfies (V_1) and (V_2) , a satisfies (a_1) and (a_2) , f satisfies (f_1) – (f_6) . Then the problem (P_a) has a positive ground state solution.*

Remark 1.2

- (1) When (M, g) is the standard Euclidean space \mathbb{R}^N and $a(x) = |x|^\eta$ ($0 \leq \eta < N$), the equation (P_a) and the perturbation problem (P) have been studied by many authors such as Adimurthi and Yang [36], Yang [37], Lam and Lu [38].
- (2) Comparing with the existing literature, we firstly study the ground state solution of problem (P_a) on complete noncompact Riemannian manifold without the Ambrosetti–Rabinowitz condition.

Theorem 1.3 *Suppose V satisfies (V_1) and (V_2) , a satisfies (a_1) and (a_2) , f satisfies (f_1) – (f_4) and (f_6) . Then the perturbation problem (P) has a nontrivial weak solution of mountain-pass type.*

This paper is organized as follows: In Sect. 2, some preliminary results are introduced. In Sect. 3, we study the functionals and related compact analysis. In Sect. 4, we give a proof of Theorem 1.1. Finally, in Sect. 5, we prove Theorem 1.3.

2 Preliminaries

In this section, we give some preliminaries for later use.

Lemma 2.1 *Suppose a satisfies (a_1) and (a_2) , V satisfies condition (V_1) . Let $\alpha > 0$ and $\{u_n\}$ be a sequence satisfying*

$$\limsup_{n \rightarrow \infty} \|u_n\|_E^{\frac{N}{N-1}} < \frac{\alpha_N}{\alpha},$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ is the sharp constant in Proposition A. Then there exist constants $t > 1, C > 0$, independent of n , such that

$$\int_M \frac{[\Phi(\alpha|u_n|^{\frac{N}{N-1}})]^t}{a(x)} dv_g \leq C.$$

Proof Since

$$\limsup_{n \rightarrow \infty} \|u_n\|_E < \left(\frac{\alpha_N}{\alpha}\right)^{\frac{N-1}{N}},$$

passing to a subsequence, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n\|_E^{\frac{N}{N-1}} \leq m < \frac{\alpha_N}{\alpha}, \quad \forall n \geq n_0.$$

Choose $k > ts' > 1$ such that $k\alpha m < \alpha_N$ and $t < s$, here s is stated in the condition (a_1) and $\frac{1}{s} + \frac{1}{s'} = 1$. Now for each $n \geq n_0$, combining with $\frac{m}{\|u_n\|_E^{\frac{N}{N-1}}} \geq 1$,

$$\begin{aligned} & \int_M \frac{[\Phi(\alpha|u_n|^{\frac{N}{N-1}})]^t}{a(x)} dv_g \\ &= \int_{B_\rho(O)} \frac{[\Phi(\alpha|u_n|^{\frac{N}{N-1}})]^t}{a(x)} dv_g + \int_{M \setminus B_\rho(O)} \frac{[\Phi(\alpha|u_n|^{\frac{N}{N-1}})]^t}{a(x)} dv_g \\ &\leq \|a^{-1}\|_{L^s(B_\rho(O))} \left(\int_{B_\rho(O)} (\Phi(\alpha|u_n|^{\frac{N}{N-1}}))^{ts'} dv_g \right)^{\frac{1}{s'}} + C \int_{M \setminus B_\rho(O)} (\Phi(\alpha|u_n|^{\frac{N}{N-1}}))^t dv_g \\ &\leq \|a^{-1}\|_{L^s(B_\rho(O))} \left(\int_M \Phi(ts'\alpha|u_n|^{\frac{N}{N-1}}) dv_g \right)^{\frac{1}{s'}} + C \int_M \Phi(t\alpha|u_n|^{\frac{N}{N-1}}) dv_g \\ &\leq C \int_M \Phi(k\alpha|u_n|^{\frac{N}{N-1}}) dv_g \leq C \int_M \Phi\left(k\alpha m \left(\frac{|u_n|}{\|u_n\|_E}\right)^{\frac{N}{N-1}}\right) dv_g. \end{aligned}$$

The property $\Phi(q)^p \leq \Phi(pq)$ (see [37]) has been used above. Finally, the result is derived from Proposition A by taking $\tau = V_0$ in condition (V_1) . □

Lemma 2.2 *Suppose V satisfies (V_1) and (V_2) , a satisfies (a_1) and (a_2) , we have $E \hookrightarrow L^q(M), q \geq N$. Moreover, this continuous embedding is compact.*

Proof From the condition (V_1) , we have $E \hookrightarrow W^{1,N}(M)$, together with the standard Sobolev embedding theorem $W^{1,N}(M) \hookrightarrow L^q(M)$ for any $q \geq N$, we immediately derive that E can be continuously embedded into $L^q(M)$.

Next, for any $q \geq N$, one needs to prove that the above continuous embedding $E \hookrightarrow L^q(M)$ is compact. Let $\{u_n\}$ be a sequence of functions with $\|u_n\|_E^N \leq C$ for some constant C , it suffices to prove that, up to a subsequence, $\{u_n\}$ strongly converges in $L^q(M)$. Obviously $\{u_n\}$ is also bounded in $W^{1,N}(M)$, thus we can assume that, for any $q \geq N$, there exists $u_0 \in E$ such that up to a subsequence

$$\begin{cases} u_n \rightharpoonup u_0 & \text{in } E, \\ u_n \rightarrow u_0 & \text{in } L^q(B_\rho(O)), \\ u_n(x) \rightarrow u_0(x) & \text{a.e. in } M. \end{cases}$$

Hence for any $\epsilon > 0$, there exists some positive integral n_0 , such that when $n > n_0$,

$$\int_{B_\rho(O)} |u_n - u_0|^N dv_g < \frac{\epsilon}{2}. \tag{2.1}$$

On the other hand, for any $\epsilon > 0$, from the condition (V_2) , we can assume $V(x) > 4C/\epsilon$ when $x \in M \setminus B_\rho(O)$, so

$$\frac{4C}{\epsilon} \int_{M \setminus B_\rho(O)} |u_n - u_0|^N dv_g < \int_{M \setminus B_\rho(O)} V|u_n - u_0|^N dv_g \leq 2C,$$

thus

$$\int_{M \setminus B_\rho(O)} |u_n - u_0|^N dv_g < \frac{\epsilon}{2}. \tag{2.2}$$

From (2.1) and (2.2), when $n > n_0$, we obtain

$$\int_M |u_n - u_0|^N dv_g < \epsilon. \tag{2.3}$$

By Hölder’s inequality and the continuous embedding $E \hookrightarrow L^q(M)$ ($q \geq N$), we have

$$\begin{aligned} \int_M |u_n - u_0|^q dv_g &= \int_M |u_n - u_0| \cdot |u_n - u_0|^{q-1} dv_g \\ &\leq \left(\int_M |u_n - u_0|^N dv_g \right)^{\frac{1}{N}} \left(\int_M |u_n - u_0|^{\frac{N(q-1)}{N-1}} dv_g \right)^{\frac{N-1}{N}} \\ &\leq C \left(\int_M |u_n - u_0|^N dv_g \right)^{\frac{1}{N}}. \end{aligned}$$

Combining with (2.3), we see that $\{u_n\}$ strongly converges in $L^q(M)$. □

Lemma 2.3 *Suppose V satisfies (V_1) , a satisfies (a_1) and (a_2) . Let $\beta > 0$ and $\|u\|_E \leq M$ such that $\beta M^{\frac{N}{N-1}} < \alpha_N$ and $q > N$, then*

$$\int_M |u|^q \frac{\Phi(\beta|u|^{\frac{N}{N-1}})}{a(x)} dv_g \leq C(\beta, N) \|u\|_E^q.$$

Proof On the one hand, using the Hölder inequality, we have

$$\begin{aligned} & \int_{B_\rho(O)} |u|^q \frac{\Phi(\beta|u|^{\frac{N}{N-1}})}{a(x)} dv_g \\ & \leq \|a^{-1}\|_{L^s(B_\rho(O))} \left(\int_{B_\rho(O)} |u|^{qk} dv_g \right)^{\frac{1}{k}} \left(\int_M (\Phi(\beta|u|^{\frac{N}{N-1}}))^t dv_g \right)^{\frac{1}{t}}, \end{aligned} \tag{2.4}$$

where $t > 1$ is sufficiently close 1 such that $\beta t M^{\frac{N}{N-1}} < \alpha_N$ and $1/s + 1/k + 1/t = 1$.

On the other hand, we get

$$\begin{aligned} & \int_{M \setminus B_\rho(O)} |u|^q \frac{\Phi(\beta|u|^{\frac{N}{N-1}})}{a(x)} dv_g \\ & \leq C \left(\int_M |u|^{q\gamma'} dv_g \right)^{\frac{1}{\gamma'}} \left(\int_M (\Phi(\beta|u|^{\frac{N}{N-1}}))^\gamma dv_g \right)^{\frac{1}{\gamma}}, \end{aligned} \tag{2.5}$$

where $\gamma > 1$ is sufficiently close 1 such that $\beta\gamma M^{\frac{N}{N-1}} < \alpha_N$, $1/\gamma' + 1/\gamma = 1$.

From (2.4) and (2.5), by Proposition A and the continuous embedding of $E \hookrightarrow L^q(\mathbb{R}^N)$ ($q \geq N$), we have

$$\int_M |u|^q \frac{\Phi(\beta|u|^{\frac{N}{N-1}})}{a(x)} dv_g \leq C(\beta, N) \|u\|_E^q.$$

This completes the proof. □

3 Functionals and compactness analysis

We say that $u \in E$ is a weak solution of problem (P_a) if, for all $\phi \in E$,

$$\int_M (|\nabla_g u|^{N-2} \nabla_g u \nabla_g \phi + V(x)|u|^{N-2} u \phi) dv_g - \int_M \frac{f(x, u)}{a(x)} \phi dv_g = 0.$$

Define the functional $I : E \rightarrow \mathbb{R}$ by

$$I(u) = \frac{1}{N} \|u\|_E^N - \int_M \frac{F(x, u)}{a(x)} dv_g, \tag{3.1}$$

where $F(x, t) = \int_0^t f(x, s) ds$. I is well defined and $I \in C^1(E, \mathbb{R})$ thanks to the Trudinger–Moser inequality. A straightforward calculation shows that

$$\langle I'(u), \phi \rangle = \int_M |\nabla_g u|^{N-2} \nabla_g u \nabla_g \phi dv_g + \int_M V(x)|u|^{N-2} u \phi dv_g - \int_M \frac{f(x, u)}{a(x)} \phi dv_g, \tag{3.2}$$

for all $u, \phi \in E$. Hence, a critical point of I defined in (3.1) is a weak solution of (P_a) .

In this paper, we will use the mountain-pass theorem for the existence of the Cerami sequence which was introduced in [39, 40].

Definition A Let $(E, \|\cdot\|_E)$ be a real Banach space with its dual space $(E^*, \|\cdot\|_{E^*})$. Suppose $I \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$, we call $\{u_n\} \subset E$ a $(C)_c$ sequence of the functional I , if

$$I(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|_E) \|I'(u_n)\|_{E^*} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proposition B Let $(E, \| \cdot \|_E)$ be a real Banach space, $I \in C^1(E, \mathbb{R})$, $I(0) = 0$ and satisfies:

(i) There exist positive constants δ and r such that

$$I(u) \geq \delta \quad \text{for } \|u\|_E = r$$

and

(ii) there exists $e \in E$ with $\|e\|_E > r$ such that

$$I(e) \leq 0.$$

Define c by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma =: \{ \gamma \in C([0, 1] : E) : \gamma(0) = 0, \gamma(1) = e \}.$$

Then I possesses a $(C)_c$ sequence.

Next we will check the geometry of the functional I .

Lemma 3.1 Assume that (V_1) , (f_1) and (f_4) hold, a satisfies (a_1) and (a_2) . Then there exist positive constants δ and r such that

$$I(u) \geq \delta \quad \text{for } \|u\|_E = r.$$

Proof From (f_4) , there exist $\sigma, \epsilon > 0$, such that if $|u| \leq \epsilon$,

$$F(x, u) \leq \frac{\lambda_a - \sigma}{N} |u|^N$$

for all $x \in M$. On the other hand, using (f_1) for each $q > N$, we have

$$\begin{aligned} F(x, u) &\leq \frac{c_1}{N} |u|^N + c_2 |u| \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}}) \\ &\leq C |u|^q \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}}) \end{aligned}$$

for $|u| \geq \epsilon$ and $x \in M$. Combining the above estimates, we obtain

$$F(x, u) \leq \frac{\lambda_a - \sigma}{N} |u|^N + C |u|^q \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}}),$$

for all $(x, u) \in M \times \mathbb{R}$. Fix $r > 0$ small enough such that $\alpha_0 r^{\frac{N}{N-1}} < \alpha_N$, then Lemma 2.3 implies

$$\begin{aligned} I(u) &= \frac{1}{N} \|u\|_E^N - \int_M \frac{F(x, u)}{a(x)} dv_g \\ &\geq \frac{1}{N} \|u\|_E^N - \frac{\lambda_a - \sigma}{N} \int_M \frac{|u|^N}{a(x)} dv_g - C \int_M \frac{|u|^q \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}})}{a(x)} dv_g \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{N} \|u\|_E^N - \frac{\lambda_a - \sigma}{N} \int_M \frac{|u|^N}{a(x)} dv_g - C \|u\|_E^q \\ &\geq \frac{1}{N} \left(1 - \frac{\lambda_a - \sigma}{\lambda_a}\right) \|u\|_E^N - C \|u\|_E^q = \frac{\sigma}{N\lambda_a} \|u\|_E^N - C \|u\|_E^q, \end{aligned}$$

for $\|u\|_E \leq r$. Hence, I is bounded from below for $\|u\|_E \leq r$. Since $\sigma > 0$ and $q > N$, we may choose a small r that satisfies the inequality

$$\frac{\sigma}{N\lambda_a} r^N - Cr^q \geq \frac{\sigma}{2N\lambda_a} r^N,$$

and we derive that

$$I(u) \geq \frac{\sigma}{2N\lambda_a} r^N := \delta > 0 \quad \text{for } \|u\|_E = r.$$

This completes the proof. □

Lemma 3.2 *If the condition (f_2) is satisfied, then there exists $e \in B_r^c(0)$ such that*

$$I(e) < \inf_{\|u\|_E=r} I(u),$$

where r is given in Lemma 3.1.

Proof Let $u \in E \setminus \{0\}$, $u \geq 0$ with compact support $\Omega = \text{supp}(u)$, by (f_2) , there exists d such that, for all $(x, s) \in \Omega \times \mathbb{R}^+$,

$$F(x, s) \geq Ms^N - d.$$

Then

$$\begin{aligned} I(tu) &\leq \frac{t^N}{N} \|u\|_E^N - Mt^N \int_{\Omega} \frac{|u|^N}{a(x)} dv_g + O(1) \\ &\leq t^N \left(\frac{\|u\|_E^N}{N} - M \int_{\Omega} \frac{|u|^N}{a(x)} dv_g \right) + O(1). \end{aligned}$$

Now, choose $M > \frac{\|u\|_E^N}{\int_{\Omega} \frac{|u|^N}{a(x)} dv_g}$, which implies that $I(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. Setting $e = tu$ with t sufficiently large, the proof of the lemma is completed. □

From Lemmas 3.1 and 3.2, we get a $(C)_c$ sequence $\{u_n\} \subset E$ by using Proposition B, i.e.,

$$I(u_n) \rightarrow c > 0 \quad \text{and} \quad (1 + \|u_n\|_E) \|I'(u_n)\|_{E^*} \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{3.3}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

and

$$\Gamma =: \{\gamma \in C([0, 1] : E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.3 *Suppose V satisfies (V_1) and (V_2) , a satisfies (a_1) and (a_2) , f satisfies (f_5) , then the level $c \in (0, \frac{1}{N}((1 - \frac{1}{s})^{\frac{\alpha N}{\alpha_0}})^{N-1})$.*

Proof Firstly, we claim the best constant S_p ($p > N$) in (1.3) can be obtained. In fact, since

$$S_p = \inf_{u \in E \setminus \{0\}} \frac{\|u\|_E}{\left(\int_M \frac{|u|^p}{a(x)} dv_g\right)^{\frac{1}{p}}},$$

we can choose u_n such that

$$\int_M \frac{|u_n|^p}{a(x)} dv_g = 1 \quad \text{and} \quad \|u_n\|_E \rightarrow S_p \quad \text{as } n \rightarrow \infty,$$

so u_n is bounded in E . From Lemma 2.2, there exists $u \in E$ such that up to a subsequence $u_n \rightharpoonup u$ in E , $u_n \rightarrow u$ in $L^p(M)$ and $u_n(x) \rightarrow u(x)$ almost everywhere in M , so $u_n \rightarrow u$ in $L^q(B_\rho(O))$ for all $q > 1$ and $a \in L^s(B_\rho(O))$, by the Hölder inequality

$$\int_{B_\rho(O)} \frac{|u_n|^p}{a(x)} dv_g \rightarrow \int_{B_\rho(O)} \frac{|u|^p}{a(x)} dv_g.$$

From (a_2) , we have

$$\begin{aligned} \int_{M \setminus B_\rho(O)} \left| \frac{|u_n|^p - |u|^p}{a(x)} \right| dv_g &\leq pC \int_M (|u_n|^{p-1} + |u|^{p-1}) |u_n - u| dv_g \\ &\leq C \left(\int_M |u_n - u|^p dv_g \right)^{\frac{1}{p}} \rightarrow 0. \end{aligned}$$

This implies

$$\int_M \frac{|u|^p}{a(x)} dx = \lim_{n \rightarrow \infty} \int_M \frac{|u_n|^p}{a(x)} dx = 1.$$

We also have $\|u\|_E \leq \lim_{n \rightarrow \infty} \|u_n\|_E = S_p$, thus $\|u\|_E = S_p$. We finished the proof of the claim.

We claim that there exists a number $t_0 > r$ such that for the above u we have

$$I(t_0u) < 0,$$

namely, we can take $e = t_0u$ as in Lemma 3.2. In fact, for $t > 0$, by condition (f_5) , we have

$$I(tu) \leq \frac{t^N}{N} S_p^N - \frac{t^p}{p} C_p.$$

Since $p > N$, by choosing t_0 large enough we get the claim clearly.

From the definition of c , take $\gamma : [0, 1] \rightarrow E$, $\gamma(t) = tt_0u$ with $e = t_0u$ as claimed above.

We have $\gamma \in \Gamma$ and therefore

$$\begin{aligned} c &\leq \max_{t \in [0,1]} I(\gamma(t)) \leq \max_{t \geq 0} I(tu) \\ &\leq \max_{t \geq 0} \left(\frac{t^N}{N} S_p^N - \frac{t^p}{p} C_p \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{p - N}{Np} \frac{S_p^{\frac{Np}{p-N}}}{C_p^{\frac{N}{p-N}}} \\
 &< \frac{1}{N} \left(\left(1 - \frac{1}{s} \right) \frac{\alpha_N}{\alpha_0} \right)^{N-1}.
 \end{aligned}$$

In the last estimation we have used the inequality in condition (f_5) . The proof of Lemma 3.3 is completed. \square

It is well known that the absence of the Cerami compactness condition brings about difficulties in studying this class of elliptic problems involving critical growth and unbounded domains. In the next lemma, we will analyze the compactness of Cerami sequences for I .

Lemma 3.4 *Suppose that the conditions (V_1) – (V_2) , (f_1) – (f_4) and (f_6) are satisfied, a satisfies (a_1) and (a_2) , let $\{u_n\} \subset E$ be an arbitrary Cerami sequence of I , then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u \in E$ such that*

$$\begin{cases}
 \frac{f(x, u_n)}{a(x)} \rightarrow \frac{f(x, u)}{a(x)} & \text{strongly in } L^1_{\text{loc}}(M), \\
 \frac{F(x, u_n)}{a(x)} \rightarrow \frac{F(x, u)}{a(x)} & \text{strongly in } L^1(M), \\
 \nabla_g u_n \rightarrow \nabla_g u & \text{almost everywhere in } M, \\
 |\nabla_g u_n|^{N-2} \nabla_g u_n \rightharpoonup |\nabla_g u|^{N-2} \nabla_g u & \text{weakly in } (L^{N/(N-1)}(M))^N.
 \end{cases}$$

Furthermore, u is a weak solution of (P_a) .

Proof We shall prove that $\{u_n\}$ is bounded in E . Indeed, suppose by contradiction that

$$\|u_n\|_E \rightarrow +\infty$$

and set

$$v_n = \frac{u_n}{\|u_n\|_E},$$

then $\|v_n\| = 1$. From Lemma 2.2, we can assume that, for any $q \geq N$, there exists $v \in E$ such that up to a subsequence

$$\begin{cases}
 v_n^+ \rightharpoonup v^+ & \text{in } E, \\
 v_n^+ \rightarrow v^+ & \text{in } L^q(M), \\
 v_n^+ \rightarrow v^+ & \text{a.e. in } M.
 \end{cases}$$

We will show that $v^+ = 0$ a.e. in M . In fact, if $\Lambda^+ = \{x \in M : v^+(x) > 0\}$ has a positive measure, then in Λ^+ , we have

$$\lim_{n \rightarrow \infty} u_n^+ = \lim_{n \rightarrow \infty} v_n^+ \|u_n\| = +\infty.$$

From (f_2) we have

$$\lim_{n \rightarrow \infty} \frac{F(x, u_n^+(x))}{a(x)|u_n^+(x)|^N} = +\infty \quad \text{a.e. in } \Lambda^+$$

and

$$\lim_{n \rightarrow \infty} \frac{F(x, u_n^+(x))}{a(x)|u_n^+(x)|^N} |v_n^+(x)|^N = +\infty \quad \text{a.e. in } \Lambda^+.$$

Thus

$$\int_M \liminf_{n \rightarrow \infty} \frac{F(x, u_n^+(x))}{a(x)|u_n^+(x)|^N} |v_n^+(x)|^N dv_g = +\infty.$$

Since $\{u_n\} \subset E$ is an arbitrary Cerami sequence of I , we have

$$\|u_n\|_E^N = Nc + N \int_M \frac{F(x, u_n^+(x))}{a(x)} dv_g + o_n(1)$$

then

$$\int_M \frac{F(x, u_n^+(x))}{a(x)} dv_g \rightarrow +\infty,$$

so

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_M \frac{F(x, u_n^+(x))}{a(x)|u_n^+(x)|^N} |v_n^+(x)|^N dv_g \\ &= \liminf_{n \rightarrow \infty} \int_M \frac{F(x, u_n^+(x))}{a(x)\|u_n\|^N} dv_g \\ &= \liminf_{n \rightarrow \infty} \frac{\int_M \frac{F(x, u_n^+(x))}{a(x)} dv_g}{Nc + N \int_M \frac{F(x, u_n^+(x))}{a(x)} dv_g + o_n(1)} \\ &= \frac{1}{N}. \end{aligned}$$

This is a contradiction. Hence $v \leq 0$ a.e. and $v_n^+ \rightarrow 0$ in E .

Let $t_n \in [0, 1]$ be such that

$$I(t_n u_n) = \max_{t \in [0, 1]} I(t u_n).$$

For any given $A \in (0, ((1 - \frac{1}{s}) \frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}})$, for the sake of simplicity, let

$$\epsilon = \frac{(1 - \frac{1}{s})\alpha_N}{A^{\frac{N}{N-1}}} - \alpha_0 > 0.$$

In the following argument we will take $A \rightarrow ((1 - \frac{1}{s}) \frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$ and so we have $\epsilon \rightarrow 0$. From condition (f_1) , there exists $C > 0$ such that

$$F(x, t) \leq C|t|^N + \epsilon \Phi((\alpha_0 + \epsilon)|t|^{\frac{N}{N-1}}), \quad \forall (x, t) \in M \times \mathbb{R}^+. \tag{3.4}$$

In fact, from condition (f_1) , we have

$$F(x, t) \leq \frac{C}{N}|t|^N + |t|\Phi(\alpha_0|t|^{\frac{N}{N-1}}).$$

By using the Young inequality, for $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$, we have

$$ab \leq \epsilon \frac{a^p}{p} + \epsilon^{-q/p} \frac{b^q}{q}.$$

So we have

$$F(x, t) \leq \frac{C}{N} |t|^N + \frac{\epsilon \Phi(p\alpha_0 |t|^{\frac{N}{N-1}})}{p} + \epsilon^{-q/p} \frac{|t|^q}{q}.$$

Now we take $p = \frac{\alpha_0 + \epsilon}{\alpha_0}$ and $q = \frac{\alpha_0 + \epsilon}{\epsilon} > N$. One can see that near infinity $|t|^q$ can be estimated from above by $\Phi((\alpha_0 + \epsilon)|t|^{\frac{N}{N-1}})$, and near the origin $|t|^q$ can be estimated from above by $|t|^N$.

We also have $\frac{A}{\|u_n\|} \in [0, 1]$ with sufficient large n , so by using (3.4)

$$\begin{aligned} I(t_n u_n) &\geq I\left(\frac{A}{\|u_n\|} u_n\right) = I(Av_n) \\ &= \frac{A^N}{N} - \int_M \frac{F(x, Av_n)}{a(x)} dv_g \\ &= \frac{A^N}{N} - \int_M \frac{F(x, Av_n^+)}{a(x)} dv_g \\ &\geq \frac{A^N}{N} - \frac{C}{N} A^N \int_M \frac{|v_n^+|^N}{a(x)} dv_g - \epsilon \int_M \frac{\Phi((\alpha_0 + \epsilon)|Av_n^+|^{\frac{N}{N-1}})}{a(x)} dv_g \\ &\geq \frac{A^N}{N} - \frac{C}{N} A^N \int_M \frac{|v_n^+|^N}{a(x)} dv_g - \epsilon \int_M \frac{\Phi((1 - \frac{1}{s})\alpha_N |v_n^+|^{\frac{N}{N-1}})}{a(x)} dv_g. \end{aligned}$$

Since $v_n^+ \rightarrow 0$ in E and the embedding $E \hookrightarrow L^q(M)$ ($q \geq N$) is compact, by using the Hölder inequality, we have $\int_M \frac{|v_n^+|^N}{a(x)} dv_g \rightarrow 0$. By Proposition A, $\int_M \frac{\Phi((1 - \frac{1}{s})\alpha_N |v_n^+|^{\frac{N}{N-1}})}{a(x)} dv_g$ is bounded. Note that when $A \rightarrow (\frac{(s-1)\alpha_N}{s\alpha_0})^{\frac{N-1}{N}}, \epsilon \rightarrow 0$, we can show

$$\liminf_{n \rightarrow \infty} I(t_n u_n) \geq \frac{1}{N} \left(\left(1 - \frac{1}{s}\right) \frac{\alpha_N}{\alpha_0} \right)^{N-1} > c. \tag{3.5}$$

Notice $I(0) = 0$ and $I(u_n) \rightarrow c$, we can assume $t_n \in (0, 1)$, and so $I'(t_n u_n)t_n u_n = 0$, it follows from (f₆) that

$$\begin{aligned} NI(t_n u_n) &= NI(t_n u_n) - I'(t_n u_n)t_n u_n \\ &= \|t_n u_n\|^N - N \int_M \frac{F(x, t_n u_n)}{a(x)} dv_g - \|t_n u_n\|^N + \int_M \frac{f(x, t_n u_n)t_n u_n}{a(x)} dv_g \\ &= \int_M \frac{H(x, t_n u_n)}{a(x)} dv_g \leq \int_M \frac{H(x, u_n)}{a(x)} dv_g \\ &= NI(u_n) - I'(u_n)u_n \\ &= NI(u_n) + o_n(1) = Nc + o_n(1), \end{aligned}$$

which is a contradiction to (3.5). This proves that $\{u_n\}$ is bounded in E . It then follows from (3.3) that

$$\int_M \frac{f(x, u_n)u_n}{a(x)} dv_g \leq C, \quad \int_M \frac{F(x, u_n)}{a(x)} dv_g \leq C.$$

By (f_3) , there exists $C > 0$ such that

$$F(x, u_n) \leq C|u_n|^N + Cf(x, u_n).$$

From Lemma 2.2 and the generalized Lebesgue dominated convergence theorem, discussed as Lemma 7.6 in [33], we can derive that

$$\frac{f(x, u_n)}{a(x)} \rightarrow \frac{f(x, u)}{a(x)} \text{ strongly in } L^1_{loc}(M) \tag{3.6}$$

and

$$\frac{F(x, u_n)}{a(x)} \rightarrow \frac{F(x, u)}{a(x)} \text{ strongly in } L^1(M). \tag{3.7}$$

Now we prove the remaining part of the lemma. Up to a subsequence, we can define an energy concentration set for any fixed $\delta > 0$,

$$\Sigma_\delta = \left\{ x \in M : \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x)} (|\nabla_g u_n|^N + |u_n|^N) dv_g \geq \delta \right\}.$$

Since $\{u_n\}$ is bounded in E , Σ_δ must be a finite set. For any $x^* \in M \setminus \Sigma_\delta$, there exists $r : 0 < r < d_g(x^*, \Sigma_\delta)$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} (|\nabla_g u_n|^N + |u_n|^N) dv_g < \delta.$$

It follows that, for large n ,

$$\int_{\mathbb{B}_r(x^*)} (|\nabla_g u_n|^N + |u_n|^N) dv_g < \delta. \tag{3.8}$$

Thanks to the Trudinger–Moser inequality in Lemma 2.1, for sufficiently small $\delta > 0$, there exists $q > 1$ such that

$$\int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n)|^q}{a(x)} dv_g \leq C. \tag{3.9}$$

For any $L > 0$, we denote

$$A_L = \{x \in \mathbb{B}_r(x^*) : |u(x)| \geq L\}.$$

It can be estimated that

$$\begin{aligned} & \int_{A_L} \frac{|f(x, u_n) - f(x, u)||u|}{a(x)} dv_g \\ & \leq \left(\int_{A_L} \frac{|f(x, u_n) - f(x, u)|^q}{a(x)} dv_g \right)^{\frac{1}{q}} \left(\int_{A_L} \frac{|u|^{q' t'}}{a(x)} dv_g \right)^{\frac{1}{q'}} \\ & \leq \left[\left(\int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n)|^q}{a(x)} dv_g \right)^{\frac{1}{q}} + \left(\int_{\mathbb{B}_r(x^*)} \frac{|f(x, u)|^q}{a(x)} dv_g \right)^{\frac{1}{q}} \right] \\ & \quad \times \left(\int_{\mathbb{B}_r(x^*)} \frac{1}{a(x)^s} dv_g \right)^{\frac{1}{q's}} \left(\int_{A_M} |u|^{q' t'} dx \right)^{\frac{1}{q' t'}} \\ & \leq C \left(\int_{A_L} |u|^{q' t'} dv_g \right)^{\frac{1}{q' t'}}, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$. Here we have used (3.9) in the last inequality. Since $u \in L^{q' t'}(\mathbb{B}_r(x^*))$, we have, for any $\beta > 0$,

$$\int_{A_L} \frac{|f(x, u_n) - f(x, u)||u|}{a(x)} dv_g < \beta, \tag{3.10}$$

provided that L is chosen sufficiently large. It follows from (3.6) that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*) \setminus A_L} \frac{|f(x, u_n) - f(x, u)||u|}{a(x)} dv_g = 0. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n) - f(x, u)||u|}{a(x)} dv_g \leq \beta,$$

and thanks to the fact that $\beta > 0$ is arbitrary,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n) - f(x, u)||u|}{a(x)} dv_g = 0. \tag{3.12}$$

On the other hand, we have by using the Hölder inequality, (3.6), and (3.9),

$$\begin{aligned} & \int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n)||u_n - u|}{a(x)} dv_g \\ & \leq \left(\int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n)|^q}{a(x)} dv_g \right)^{\frac{1}{q}} \left(\int_{\mathbb{B}_r(x^*)} \frac{1}{a(x)^s} dv_g \right)^{\frac{1}{q's}} \|u_n - u\|_{L^{q' s'}(\mathbb{B}_r(x^*))} \\ & \leq C \|u_n - u\|_{L^{q' s'}(\mathbb{B}_r(x^*))} \rightarrow 0, \end{aligned} \tag{3.13}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$. Combining (3.12) and (3.13), we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} \frac{|f(x, u_n) - f(x, u)u|}{a(x)} dv_g = 0.$$

A covering argument implies that, for any compact set $K \subset\subset M \setminus \Sigma_\delta$,

$$\lim_{n \rightarrow \infty} \int_K \frac{|f(x, u_n) - f(x, u)u|}{a(x)} dv_g = 0. \tag{3.14}$$

Next, we will prove for any compact set $K \subset\subset M \setminus \Sigma_\delta$

$$\lim_{n \rightarrow \infty} \int_K |\nabla_g u_n - \nabla_g u|^N dv_g = 0. \tag{3.15}$$

It suffices to prove for any $x^* \in M \setminus \Sigma_\delta$ and r given by (3.9), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_{\frac{r}{2}}(x^*)} |\nabla_g u_n - \nabla_g u|^N dv_g = 0. \tag{3.16}$$

For this purpose, we take $\phi \in C_0^\infty(\mathbb{B}_r(x^*))$ with $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in $\mathbb{B}_{\frac{r}{2}}(x^*)$. By the Hölder inequality and the compact embedding of $E \hookrightarrow L^N(M)$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} |\nabla_g u_n|^{N-2} \nabla_g u_n \nabla_g \phi (u - u_n) dv_g = 0. \tag{3.17}$$

Since $\nabla u_n \rightharpoonup \nabla u$ in $(L^N(M))^N$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} \phi |\nabla_g u_n|^{N-2} \nabla_g u (\nabla_g u - \nabla_g u_n) dv_g = 0. \tag{3.18}$$

The Hölder inequality and (3.13) implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{B}_r(x^*)} \phi (u_n - u) \frac{f(x, u_n)}{a(x)} dv_g \rightarrow 0. \tag{3.19}$$

Note that

$$\langle |x|^{N-2}x - |y|^{N-2}y, x - y \rangle = \frac{|x|^{N-2} + |y|^{N-2}}{2} |x - y|^2 + \frac{(|x|^{N-2} - |y|^{N-2})(|x|^2 - |y|^2)}{2}.$$

Hence,

$$\langle |x|^{N-2}x - |y|^{N-2}y, x - y \rangle \geq \frac{|x|^{N-2} + |y|^{N-2}}{2} |x - y|^2 \geq 2^{2-N} |x - y|^N. \tag{3.20}$$

Obviously ϕu_n is a bounded sequence in E . Inserting $\varphi = \phi u_n$ and $\varphi = \phi u$ into (3.3), respectively, we have from (3.17)–(3.20) and $\tau_n \rightarrow 0$ as $n \rightarrow \infty$

$$\begin{aligned} 0 &\leq 2^{2-N} \int_{\mathbb{B}_r(x^*)} |\nabla_g u_n - \nabla_g u|^N dv_g \\ &\leq \int_{\mathbb{B}_r(x^*)} \phi \langle |\nabla_g u_n|^{N-2} \nabla_g u_n - |\nabla_g u|^{N-2} \nabla_g u, \nabla_g u_n - \nabla_g u \rangle dv_g \\ &\leq \int_{\mathbb{B}_r(x^*)} |\nabla_g u_n|^{N-2} \nabla_g u_n \nabla_g \phi (u - u_n) dv_g + \int_{B_r(x^*)} \phi |\nabla_g u|^{N-2} \nabla_g u (\nabla_g u - \nabla_g u_n) dv_g \\ &\quad + \int_{\mathbb{B}_r(x^*)} \phi (u_n - u) \frac{f(x, u_n)}{a(x)} dv_g + \tau_n \|\phi u_n\| + \tau_n \|\phi u\| \rightarrow 0. \end{aligned} \tag{3.21}$$

We derive (3.16) from (3.21). Hence, (3.15) holds thanks to a covering argument. Since Σ_δ is a finite set, it follows that $\nabla_g u_n$ converges to $\nabla_g u$ almost everywhere in M . This immediately implies, up to a subsequence,

$$|\nabla_g u_n|^{N-2} \nabla_g u_n \rightharpoonup |\nabla_g u|^{N-2} \nabla_g u \quad \text{in } (L^{\frac{N}{N-1}}(\mathbb{B}_R(0)))^N$$

for all $R > 0$. For any fixed $\varphi \in C_0^\infty(M)$, there exists some $R_0 > 0$ such that the support of φ is contained in the ball $\mathbb{B}_{R_0}(O)$. Hence,

$$\lim_{n \rightarrow \infty} \int_M (|\nabla_g u_n|^{N-2} \nabla_g u_n - |\nabla_g u|^{N-2} \nabla_g u) \varphi \, dv_g \rightarrow 0.$$

This equality holds for all $\varphi \in L^{\frac{N}{N-1}}(M)$, thanks to the density of $C_0^\infty(M)$ in $L^{\frac{N}{N-1}}(M)$. Hence, we obtain

$$|\nabla_g u_n|^{N-2} \nabla_g u_n \rightharpoonup |\nabla_g u|^{N-2} \nabla_g u \quad \text{in } (L^{\frac{N}{N-1}}(M))^N, \tag{3.22}$$

we obtain by combining (3.6) and (3.22),

$$\langle I'(u), \varphi \rangle = 0, \quad \forall \varphi \in C_0^\infty(M).$$

Since $C_0^\infty(M)$ is dense in E , the above equation implies that u is a weak solution of (P_a) . This completes the proof of the lemma. \square

4 The ground state solution

Proof of Theorem 1.1 By the process of proof in Lemma 3.4, we see that the Cerami sequence $\{u_n\}$ is bounded in E and its weak limit u is a critical point of the functional I .

We will show that u is nonzero. If $u \equiv 0$, since $F(x, 0) = 0$ for all $x \in \mathbb{R}^N$, from Lemma 3.4, we have

$$\lim_{n \rightarrow \infty} \|u_n\|_E^N = Nc > 0. \tag{4.1}$$

Lemma 3.3 implies $0 < c < \frac{1}{N}((1 - \frac{1}{s})\frac{\alpha_N}{\alpha_0})^{N-1}$, so there exist some $\epsilon_0 > 0$ and $n_* > 0$ such that $\|u_n\|_E^N \leq ((1 - \frac{1}{s})\frac{\alpha_N}{\alpha_0} - \epsilon_0)^{N-1}$ for all $n > n_*$. Choose $q > 1$ sufficiently close to 1 such that $q\alpha_0 \|u_n\|_E^{\frac{N}{N-1}} \leq (1 - \frac{1}{s})\alpha_N - \epsilon_0\alpha_0/2$ for all $n > n_*$. By (f_1) ,

$$f(x, u_n) \leq c_1 |u_n|^N + c_2 |u_n| \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}}),$$

thus

$$\begin{aligned} & \int_M \frac{|f(x, u_n)u_n|}{a(x)} \, dv_g \\ & \leq c_1 \int_M \frac{|u_n|^N}{a(x)} \, dv_g + c_2 \int_M \frac{|u_n| \Phi(\alpha_0 |u_n|^{\frac{N}{N-1}})}{a(x)} \, dv_g \end{aligned}$$

$$\begin{aligned} &\leq c_1 \int_M \frac{|u_n|^N}{a(x)} dv_g + c_2 \left(\int_M \frac{\Phi(q\alpha_0|u_n|^{\frac{N}{N-1}})}{a(x)} dv_g \right)^{\frac{1}{q}} \left(\int_M \frac{|u_n|^{q'}}{a(x)} dv_g \right)^{\frac{1}{q'}} \\ &\leq c_1 \int_M \frac{|u_n|^N}{a(x)} dv_g + C \left(\int_M \frac{|u_n|^{q'}}{a(x)} dv_g \right)^{\frac{1}{q'}} \rightarrow 0, \end{aligned}$$

here we used Lemma 2.2 in the last step. From $I'(u_n)u_n \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \|u_n\|_E^N = 0, \tag{4.2}$$

which is a contradiction with (4.1), so u is nonzero.

Now, we will show the existence of positive ground state solution for (P_a) . Setting

$$m = \inf_{u \in \Lambda} I(u), \quad \Lambda := \{u \in E \setminus \{0\} : I'(u)u = 0\}.$$

Let c be the mountain-pass level, obviously $m \leq c$.

On the other hand, let $u \in \Lambda$, then $u > 0$. In fact, denote $u^- := \min\{u, 0\}$, from $I'(u)u^- = 0$, we have $\|u^-\| = 0$, thus $u \geq 0$ on M , from $I(u) = c > 0$, we know u is positive on M . Define $h : (0, +\infty) \rightarrow \mathbb{R}$ by $h(t) = I(tu)$. We see that h is differentiable and

$$h'(t) = I'(tu)u = t^{N-1} \|u\|^N - \int_M \frac{f(x, tu)u}{a(x)} dv_g, \quad \forall t > 0.$$

From $I'(u)u = 0$, we get

$$h'(t) = I'(tu)u - t^{N-1} I'(u)u,$$

so

$$h'(t) = t^{N-1} \int_M \left(\frac{f(x, u)}{u^{N-1}} - \frac{f(x, tu)}{(tu)^{N-1}} \right) \frac{u^N}{a(x)} dv_g, \quad \forall t > 0.$$

By (f_6) and $u > 0$, we conclude that $h'(t) > 0$ for $0 < t < 1$ and $h'(t) < 0$ for $t > 1$, since $h'(1) = 0$, thus,

$$I(u) = \max_{t \geq 0} I(tu).$$

Now, define $\gamma : [0, 1] \rightarrow E$, $\gamma(t) = tt_0u$, where t_0 is a real number which satisfies $I(t_0u) < 0$, we have $\gamma \in \Gamma$, and therefore

$$c \leq \max_{t \in [0,1]} I(\gamma(t)) \leq \max_{t \geq 0} I(tu) = I(u).$$

Since $u \in \Lambda$ is arbitrary, we have $c \leq m$, thus $c = m$. This completes the proof of Theorem 1.1. □

5 The nontrivial solution of perturbation problem

In this section, we consider the following perturbation problem:

$$-\operatorname{div}_g(|\nabla_g u|^{N-2} \nabla_g u) + V(x)|u|^{N-2}u = \frac{f(x, u)}{a(x)} + \epsilon h(x) \quad \text{in } M \tag{P}$$

where $\epsilon > 0$, $h \in E^*$, $h \geq 0$ and $h \not\equiv 0$.

We say that $u \in E$ is a weak solution of problem (P) if for all $\phi \in E$,

$$\int_M (|\nabla_g u|^{N-2} \nabla_g u \nabla_g \phi + V(x)|u|^{N-2} u \phi) dv_g - \int_M \frac{f(x, u)}{a(x)} \phi dv_g - \epsilon \int_{\mathbb{R}^N} h(x) \phi dv_g = 0.$$

Define the functional $I_\epsilon : E \rightarrow \mathbb{R}$ by

$$I_\epsilon(u) = \frac{1}{N} \|u\|_E^N - \int_{\mathbb{R}^N} \frac{F(x, u)}{a(x)} dv_g - \epsilon \int_M h(x) u dv_g, \tag{5.1}$$

where $F(x, t) = \int_0^t f(x, s) ds$. I is well defined and $I \in C^1(E, \mathbb{R})$ thanks to the singular Trudinger–Moser inequality. A straightforward calculation shows that

$$\begin{aligned} \langle I'(u), \phi \rangle &= \int_M (|\nabla_g u|^{N-2} \nabla_g u \nabla_g \phi + V(x)|u|^{N-2} u \phi) dv_g \\ &\quad - \int_M \frac{f(x, u)}{a(x)} \phi dv_g - \epsilon \int_M h(x) \phi dv_g, \end{aligned} \tag{5.2}$$

for all $u, \phi \in E$, hence, a critical point of (5.2) is a weak solution of (P).

Next, we will prove Theorem 1.3, one solution is obtained by using the mountain-pass theorem without Cerami condition.

Proof of Theorem 1.3 As in previous sections, it is similar to the first part of Theorem 1.1, I_ϵ satisfies the geometry condition of the mountain-pass theorem; i.e., $I_\epsilon \in C^1(E, \mathbb{R})$, there exists $\epsilon_1 > 0$ such that when $0 < \epsilon < \epsilon_1$, $I_\epsilon(u) \geq \delta_\epsilon > 0$ for $\|u\|_E = r_\epsilon$; $I_\epsilon(e) < 0$ for some $e \in E$ with $\|e\|_E > r_\epsilon$, where r_ϵ can be further chosen such that $r_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. Then, by the mountain-pass theorem without the Cerami condition, we get a $(C)_c$ sequence $\{u_n\} \subset E$; i.e.,

$$I_\epsilon(u_n) \rightarrow c_\epsilon \quad \text{and} \quad (1 + \|u_n\|_E) \|I'_\epsilon(u_n)\|_{E^*} \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{5.3}$$

where

$$c_\epsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\epsilon(\gamma(t)) \geq \delta_\epsilon$$

is the min–max value of I_ϵ . By a similar argument to Lemma 3.4 to I_ϵ , there exists a subsequence of $\{u_n\}$ that is bounded and its weak limit u is a critical point of the functional I_ϵ . This completes the proof of Theorem 1.3. □

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Authors' contributions

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References

1. Brézis, H.: Elliptic equations with limiting Sobolev exponents. *Commun. Pure Appl. Math.* **39**, 517–539 (1986)
2. Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.* **36**, 437–477 (1983)
3. Bartsch, T., Willem, M.: On an elliptic equation with concave and convex nonlinearities. *Proc. Am. Math. Soc.* **123**, 3555–3561 (1995)
4. Capozzi, A., Fortunato, D., Palmieri, G.: An existence result for nonlinear elliptic problems involving critical Sobolev exponent. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **2**, 463–470 (1985)
5. Garcia, A., Alonso, P.: Existence and non-uniqueness for the p -Laplacian. *Commun. Partial Differ. Equ.* **12**, 1389–1430 (1987)
6. Adimurthi, A.: Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the N -Laplacian. *Ann. Sc. Norm. Super. Pisa* **XVII**, 393–413 (1990)
7. Adimurthi, A., Yadava, S.: Multiplicity results for semilinear elliptic equations in a bounded domain of \mathbb{R}^2 involving critical exponent. *Ann. Sc. Norm. Super. Pisa* **XVII**, 481–504 (1990)
8. de Figueiredo, D.G., do Ó, J.M., Ruf, B.: On an inequality by N. Trudinger and J. Moser and related elliptic equations. *Commun. Pure Appl. Math.* **IV**, 135–152 (2002)
9. de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in \mathbb{R}^2 with nonlinearities in the critical growth range. *Calc. Var. Partial Differ. Equ.* **3**, 139–153 (1995)
10. do Ó, J.M.: Semilinear Dirichlet problems for the N -Laplacian in \mathbb{R}^N with nonlinearities in the critical growth range. *J. Differ. Equ.* **9**, 967–979 (1996)
11. Panda, R.: On semilinear Neumann problems with critical growth for the N -Laplacian. *Nonlinear Anal.* **26**, 1347–1366 (1996)
12. Lam, N., Lu, G.: Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti–Rabinowitz conditions. *J. Geom. Anal.* **24**, 118–143 (2014)
13. Trudinger, N.S.: On embeddings into Orlicz spaces and some applications. *J. Math. Mech.* **17**, 473–484 (1967)
14. Moser, J.: A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.* **20**, 1077–1091 (1971)
15. Kryszewski, W., Szulkin, A.: Generalized linking theorem with an application to semilinear Schrödinger equation. *Adv. Differ. Equ.* **3**, 441–472 (1998)
16. Alama, S., Li, Y.: Existence of solutions for semilinear elliptic equations with indefinite linear part. *J. Differ. Equ.* **96**, 89–115 (1992)
17. Ding, W., Ni, W.: On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Ration. Mech. Anal.* **31**, 283–308 (1986)
18. Jeanjean, L.: Solutions in spectral gaps for a nonlinear equation of Schrödinger type. *J. Differ. Equ.* **112**, 53–80 (1994)
19. Cao, D.: Nontrivial solution of semilinear elliptic equations with critical exponent in \mathbb{R}^2 . *Commun. Partial Differ. Equ.* **17**, 407–435 (1992)
20. Panda, R.: Nontrivial solution of a quasilinear elliptic equation with critical growth in \mathbb{R}^N . *Proc. Indian Acad. Sci. Math. Sci.* **105**, 425–444 (1995)
21. do Ó, J.M.: N -Laplacian equations in \mathbb{R}^N with critical growth. *Abstr. Appl. Anal.* **2**, 301–315 (1997)
22. Alves, C.O., Figueiredo, G.M.: On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in \mathbb{R}^N . *J. Differ. Equ.* **246**, 1288–1311 (2009)
23. do Ó, J.M., Medeiros, E., Severo, U.: On a quasilinear nonhomogeneous elliptic equation with critical growth in \mathbb{R}^N . *J. Differ. Equ.* **246**, 1363–1386 (2009)
24. Aubin, T.: Sur la fonction exponentielle. *C. R. Acad. Sci. Paris, Ser. A* **270**, 1514–1516 (1970)
25. Cherrier, P.: Une inégalité de Sobolev sur les variétés Riemanniennes. *Bull. Sci. Math.* **103**, 353–374 (1979)
26. Cherrier, P.: Cas d'exception du théorème d'inclusion de Sobolev sur les variétés Riemanniennes et applications. *Bull. Sci. Math.* **105**, 235–288 (1981)
27. Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.* **68**, 415–454 (1993)
28. Ding, W., Jost, J., Li, J., Wang, G.: The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. *Asian J. Math.* **1**, 230–248 (1997)
29. Li, Y.: Moser–Trudinger inequality on compact Riemannian manifolds of dimension two. *J. Partial Differ. Equ.* **14**, 163–192 (2001)
30. Li, Y.: The extremal functions for Moser–Trudinger inequality on compact Riemannian manifolds. *Sci. China Ser. A* **48**, 618–648 (2005)
31. Yang, Y.: A sharp form of Moser–Trudinger inequality on compact Riemannian surface. *Trans. Am. Math. Soc.* **359**, 5761–5776 (2007)
32. Yang, Y.: A sharp form of trace Moser–Trudinger inequality on compact Riemannian surface with boundary. *Math. Z.* **255**, 373–392 (2007)

33. Yang, Y.: Trudinger–Moser inequalities on complete noncompact Riemannian manifolds. *J. Funct. Anal.* **263**, 1894–1938 (2012)
34. Yang, Q., Su, D., Kong, Y.: Sharp Moser–Trudinger inequalities on Riemannian manifolds with negative curvature. *Ann. Mat. Pura Appl. (4)* **195**, 459–471 (2016)
35. Kristály, A.: New geometric aspects of Moser–Trudinger inequalities on Riemannian manifolds: the non-compact case. *J. Funct. Anal.* **276**, 2359–2396 (2019)
36. Adimurthi, Yang, Y.: An interpolation of Hardy inequality and Trudinger–Moser inequality in \mathbb{R}^N and its applications. *Int. Math. Res. Not.* **13**, 2394–2426 (2010)
37. Yang, Y.: Existence of positive solutions to quasilinear elliptic equations with exponential growth in the whole Euclidean space. *J. Funct. Anal.* **262**, 1679–1704 (2012)
38. Lam, N., Lu, G.: Existence and multiplicity of solutions to equations of n -Laplacian type with critical exponential growth in \mathbb{R}^N . *J. Funct. Anal.* **262**, 1132–1165 (2012)
39. Cerami, G.: An existence criterion for the critical points on unbounded manifolds. *Ist. Lomb., Accad. Sci. Lett., Rend. A* **112**, 332–336 (1978) (Italian)
40. Cerami, G.: On the existence of eigenvalues for a nonlinear boundary value problem. *Ann. Mat. Pura Appl.* **124**(4), 161–179 (1980) (Italian)

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