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# On Hardy-type integral inequalities in the whole plane related to the extended Hurwitz-zeta function

Michael Th. Rassias<sup>1,2\*</sup>, Bicheng Yang<sup>3</sup> and Andrei Raigorodskii<sup>4,5,6,7</sup>

\*Correspondence:  
[michail.rassias@math.uzh.ch](mailto:michail.rassias@math.uzh.ch);  
[mthrassias@yahoo.com](mailto:mthrassias@yahoo.com)

<sup>1</sup>Institute of Mathematics, University  
of Zurich, Zurich, Switzerland

<sup>2</sup>Program in Interdisciplinary  
Studies, Institute for Advanced  
Study, Princeton, USA

Full list of author information is  
available at the end of the article

## Abstract

Using weight functions, we establish a few equivalent statements of two kinds of Hardy-type integral inequalities with nonhomogeneous kernel in the whole plane. The constant factors related to the extended Hurwitz-zeta function are proved to be the best possible. In the form of applications, we deduce some special cases involving homogeneous kernel. We additionally consider some particular inequalities and operator expressions.

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## 1 Introduction

If  $f(x), g(y) \geq 0$ ,

$$0 < \int_0^\infty f^2(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(y) dy < \infty,$$

we have the following well-known Hilbert integral inequality (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left( \int_0^\infty f^2(x) dx \int_0^\infty g^2(y) dy \right)^{\frac{1}{2}}, \quad (1)$$

with the best possible constant factor  $\pi$ .

Recently, by the use of weight functions, several extensions of (1) have been established in [2] and [3]. Some Hilbert-type inequalities were also presented in [4–9]. Furthermore, Hong [10] considered as well an equivalent condition between a Hilbert-type inequality with homogenous kernel and a few parameters. Some additional kinds of Hilbert-type inequalities were also obtained in [11–19]. Most of these results are constructed in the quarter plane of the first quadrant.

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In 2007, Yang [20] proved the following Hilbert-type integral inequality in the whole plane:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(1 + e^{x+y})^{\lambda}} dx dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left( \int_{-\infty}^{\infty} e^{-\lambda x} f^2(x) dx \int_{-\infty}^{\infty} e^{-\lambda y} g^2(y) dy \right)^{\frac{1}{2}}, \quad (2)$$

with the best possible constant factor  $B(\frac{\lambda}{2}, \frac{\lambda}{2})$  ( $\lambda > 0$ , where  $B(u, v)$  stands for the beta function) (see [21]). He et al. [22–35] also established some Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In the present paper, using weight functions, we establish a few equivalent statements of two kinds of Hardy-type integral inequalities with nonhomogeneous kernel and multi-parameters in the whole plane. The constant factors related to the extended Hurwitz-zeta function are proved to be the best possible. In the form of applications, we deduce a few equivalent statements of two kinds of Hardy-type integral inequalities with homogeneous kernel in the whole plane. As corollaries, we also consider some particular cases and operator expressions.

## 2 An example and two lemmas

*Example 1* We set

$$H(xy) := \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^{\beta}}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|},$$

wherefrom

$$H(-xy) = \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^{\beta}}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy + 1|} \quad (x, y \in \mathbf{R} = (-\infty, \infty)),$$

$$H(u) = \frac{(\min\{|u|, 1\})^{1+\alpha} |\ln |u||^{\beta}}{(\max\{|u|, 1\})^{\lambda+\alpha} |u - 1|},$$

and

$$H(-u) = \frac{(\min\{|u|, 1\})^{1+\alpha} |\ln |u||^{\beta}}{(\max\{|u|, 1\})^{\lambda+\alpha} |u + 1|} \quad (u \in \mathbf{R}).$$

For  $\beta > 0$ ,  $\sigma > -\alpha - 1$ , it follows that

$$\begin{aligned} K^{(1)}(\sigma) &:= \int_{-1}^1 H(u) |u|^{\sigma-1} du = \int_0^1 (H(-u) + H(u) u^{\sigma-1}) du \\ &= \int_0^1 \frac{(\min\{u, 1\})^{1+\alpha} (-\ln u)^{\beta} u^{\sigma-1}}{(\max\{u, 1\})^{\lambda+\alpha}} \left( \frac{1}{u+1} + \frac{1}{|u-1|} \right) du \\ &= \int_0^1 (-\ln u)^{\beta} \left( \frac{1}{u+1} + \frac{1}{1-u} \right) u^{\sigma+\alpha} du \\ &= 2 \int_0^1 (-\ln u)^{\beta} \frac{u^{\sigma+\alpha}}{1-u^2} du = 2 \int_0^1 (-\ln u)^{\beta} \sum_{k=0}^{\infty} u^{2k+\sigma+\alpha} du. \end{aligned}$$

By the Lebesgue term-by-term integration theorem (cf. [36]), for  $v = -(2k + \sigma + \alpha + 1) \ln u$ , we obtain

$$\begin{aligned} K^{(1)}(\sigma) &= 2 \sum_{k=0}^{\infty} \int_0^1 (-\ln u)^{\beta} u^{2k+\sigma+\alpha} du \\ &= 2 \sum_{k=0}^{\infty} \frac{1}{(2k + \sigma + \alpha + 1)^{\beta+1}} \int_0^{\infty} v^{\beta} e^{-v} dv \\ &= \frac{1}{2^{\beta}} \sum_{k=0}^{\infty} \frac{1}{[k + (\sigma + \alpha + 1)/2]^{\beta+1}} \int_0^{\infty} v^{(\beta+1)-1} e^{-v} dv \\ &= \frac{\Gamma(\beta + 1)}{2^{\beta}} \zeta\left(\beta + 1, \frac{\sigma + \alpha + 1}{2}\right) \in \mathbf{R}_+, \end{aligned} \quad (3)$$

where

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k + a)^s} \quad (\operatorname{Re} s > 1; 0 < a \leq 1)$$

stands for the Hurwitz-zeta function. Note that

$$\zeta(s, 1) = \zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is the Riemann-zeta function. Moreover,

$$\zeta\left(\beta + 1, \frac{\sigma + \alpha + 1}{2}\right)$$

stands for the extended Hurwitz-zeta function (cf. [21]).

In particular, for  $\sigma = -\alpha + 1$  ( $> -\alpha - 1$ ), it follows that

$$K^{(1)}(-\alpha + 1) = \int_{-1}^1 H(u) |u|^{-\alpha} du = \frac{\Gamma(\beta + 1)}{2^{\beta}} \zeta(\beta + 1).$$

Similarly, for  $\beta > 0$ ,  $\mu > -\alpha - 1$  ( $\sigma + \mu = \lambda$ ), we obtain that

$$\begin{aligned} K^{(2)}(\sigma) &:= \int_{\{u' | u| \geq 1\}} H(u) |u|^{\sigma-1} du \\ &= \int_1^{\infty} (H(-u) + H(u)) u^{\sigma-1} du \\ &= \int_{-1}^1 \frac{(\min\{|v|, 1\})^{1+\alpha} |\ln |v||^{\beta}}{(\max\{|v|, 1\})^{\lambda+\alpha} |v-1|} |v|^{\mu-1} dv \\ &= \frac{\Gamma(\beta + 1)}{2^{\beta}} \zeta\left(\beta + 1, \frac{\mu + \alpha + 1}{2}\right) = K^{(1)}(\mu) \in \mathbf{R}_+, \end{aligned} \quad (4)$$

$$K^{(2)}(\lambda + \alpha - 1) = \int_{\{u' | u| \geq 1\}} H(u) |u|^{\lambda+\alpha-2} du = \frac{\Gamma(\beta + 1)}{2^{\beta}} \zeta(\beta + 1).$$

**Remark 1** For  $\sigma + \mu = \lambda$ , it is clear that

$$K^{(1)}(\sigma) < \infty \quad (\text{resp. } K^{(2)}(\sigma) = K^{(1)}(\mu) < \infty)$$

if and only if  $\sigma > -\alpha - 1$  and  $\beta > 0$  (resp.  $\mu > -\alpha - 1$  and  $\beta > 0$ ).

In the sequel, we assume that  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\sigma + \mu = \lambda$ .

**Lemma 1** *If  $\sigma_1 \in \mathbf{R}$ , there exists a constant  $M_1$  such that, for any nonnegative measurable functions  $f(x)$  and  $g(y)$  in  $\mathbf{R}$ , the following inequality*

$$\begin{aligned} & \int_{-\infty}^{\infty} g(y) \left[ \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} f(x) dx \right] dy \\ & \leq M_1 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \quad (5)$$

holds true, then we have  $\sigma_1 = \sigma > -\alpha - 1$  and  $\beta > 0$ .

*Proof* If  $\sigma_1 > \sigma$ , then for  $n \geq \frac{1}{\sigma_1 - \sigma}$  ( $n \in \mathbf{N}$ ) we consider the following functions:

$$f_n(x) := \begin{cases} |x|^{\sigma + \frac{1}{pn}-1}, & 0 < |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \quad g_n(y) := \begin{cases} 0, & 0 < |y| < 1, \\ |y|^{\sigma_1 - \frac{1}{qn}-1}, & y \geq 1, \end{cases}$$

and derive that

$$\begin{aligned} J_1 &:= \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left( 2 \int_0^1 x^{\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left( 2 \int_1^{\infty} y^{-\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = 2n. \end{aligned}$$

We obtain

$$\begin{aligned} I_1 &:= \int_{-\infty}^{\infty} g_n(y) \left[ \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} f_n(x) dx \right] dy \\ &= \int_{-\infty}^{-1} \left[ \int_{\frac{1}{y}}^{-\frac{1}{y}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} |x|^{\sigma + \frac{1}{pn}-1} dx \right] (-y)^{\sigma_1 - \frac{1}{qn}-1} dy \\ &\quad + \int_1^{\infty} \left[ \int_{-\frac{1}{y}}^{\frac{1}{y}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} |x|^{\sigma + \frac{1}{pn}-1} dx \right] y^{\sigma_1 - \frac{1}{qn}-1} dy \\ &= \int_1^{\infty} \left[ \int_{-\frac{1}{y}}^{\frac{1}{y}} (H(-xy) + H(xy)) |x|^{\sigma + \frac{1}{pn}-1} dx \right] y^{\sigma_1 - \frac{1}{qn}-1} dy \quad (u = xy) \\ &= 2 \int_1^{\infty} \left[ \int_0^1 (H(-u) + H(u)) u^{(\sigma + \frac{1}{pn})-1} du \right] y^{(\sigma_1 - \sigma) - \frac{1}{n}-1} dy, \end{aligned} \quad (6)$$

and then by (5) we get

$$2K^{(1)}\left(\sigma + \frac{1}{pn}\right) \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy = I_1 \leq M_1 J_1 = 2M_1 n. \quad (7)$$

Since  $(\sigma_1 - \sigma) - \frac{1}{n} \geq 0$ , it follows that

$$\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy = \infty.$$

By (7), for  $K^{(1)}(\sigma + \frac{1}{pn}) > 0$ , we have  $\infty \leq 2M_1 n < \infty$ , which is a contradiction.

If  $\sigma_1 < \sigma$ , then for  $n \geq \frac{1}{\sigma - \sigma_1}$  ( $n \in \mathbf{N}$ ) we consider the following functions:

$$\tilde{f}_n(x) := \begin{cases} 0, & 0 < |x| < 1, \\ |x|^{\sigma - \frac{1}{pn} - 1}, & |x| \geq 1, \end{cases} \quad \tilde{g}_n(y) := \begin{cases} |y|^{\sigma_1 + \frac{1}{qn} - 1}, & 0 < |y| \leq 1, \\ 0, & |y| > 1, \end{cases}$$

and derive that

$$\begin{aligned} \tilde{J}_1 &:= \left[ \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty y^{q(1-\sigma_1)-1} \tilde{g}_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left( 2 \int_1^\infty x^{-\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left( 2 \int_0^1 y^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = 2n. \end{aligned}$$

We obtain

$$\begin{aligned} \tilde{I}_1 &:= \int_{-\infty}^\infty \tilde{f}_n(x) \left[ \int_{-\frac{1}{|x|}}^{\frac{1}{|x|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} \tilde{g}_n(y) dy \right] dx \\ &= \int_{-\infty}^{-1} \left[ \int_{\frac{1}{x}}^{-\frac{1}{x}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} |y|^{\sigma_1 + \frac{1}{qn} - 1} dy \right] (-x)^{\sigma - \frac{1}{pn} - 1} dx \\ &\quad + \int_1^\infty \left[ \int_{-\frac{1}{x}}^{\frac{1}{x}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} |y|^{\sigma_1 + \frac{1}{qn} - 1} dy \right] x^{\sigma - \frac{1}{pn} - 1} dx \\ &= \int_1^\infty \left[ \int_{-\frac{1}{x}}^{\frac{1}{x}} (H(-xy) + H(xy)) |y|^{\sigma_1 + \frac{1}{qn} - 1} dy \right] x^{\sigma - \frac{1}{pn} - 1} dx \\ &= 2 \int_1^\infty \left[ \int_0^1 (H(-u) + H(u)) u^{\sigma_1 + \frac{1}{qn} - 1} du \right] x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx, \end{aligned} \quad (8)$$

and thus, by Fubini's theorem (cf. [36]) and (5), it follows that

$$\begin{aligned} &2K^{(1)}\left(\sigma_1 + \frac{1}{qn}\right) \int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx \\ &= \tilde{I}_1 = \int_{-\infty}^\infty \tilde{g}_n(y) \left( \int_{\frac{1}{|y|}}^{\frac{1}{|y|}} H(xy) \tilde{f}_n(x) dx \right) dy \leq M_1 \tilde{J}_1 \\ &= 2M_1 n. \end{aligned} \quad (9)$$

Since  $(\sigma - \sigma_1) - \frac{1}{n} \geq 0$ , it follows that

$$\int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx = \infty.$$

By (9), for  $K^{(1)}(\sigma_1 + \frac{1}{qn}) > 0$ , we get that  $\infty \leq 2M_1 n < \infty$ , which is a contradiction.

Hence, we conclude that  $\sigma_1 = \sigma$ .

For  $\sigma_1 = \sigma$ , we reduce (9) as follows:

$$K^{(1)}\left(\sigma_1 + \frac{1}{qn}\right) = \int_0^1 (H(-u) + H(u)) u^{\sigma_1 + \frac{1}{qn} - 1} du \leq M_1.$$

Since  $\{(H(-u) + H(u)) u^{\sigma + \frac{1}{qn} - 1}\}_{n=1}^\infty$  is increasing in  $(0, 1)$ , by Levi's theorem (cf. [36]), we obtain that

$$\begin{aligned} K^{(1)}(\sigma) &= \int_0^1 \lim_{n \rightarrow \infty} (H(-u) + H(u)) u^{\sigma + \frac{1}{qn} - 1} du \\ &= \lim_{n \rightarrow \infty} \int_0^1 (H(-u) + H(u)) u^{\sigma + \frac{1}{qn} - 1} du \leq M_1 < \infty. \end{aligned}$$

By Remark 1, it follows that  $\sigma > -\alpha - 1$  and  $\beta > 0$ .

This completes the proof of the lemma.  $\square$

**Lemma 2** *If  $\sigma_1 \in \mathbf{R}$  and there exists a constant  $M_2$  such that, for any nonnegative measurable functions  $f(x)$  and  $g(y)$  in  $\mathbf{R}$ , the following inequality*

$$\begin{aligned} &\int_{-\infty}^\infty g(y) \left[ \int_{\{x; |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} f(x) dx \right] dy \\ &\leq M_2 \left[ \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \quad (10)$$

*holds true, then we have  $\sigma_1 = \sigma$ ,  $\mu > -\alpha - 1$ , and  $\beta > 0$ .*

*Proof* If  $\sigma_1 < \sigma$ , then for  $n \geq \frac{1}{\sigma - \sigma_1}$  ( $n \in \mathbf{N}$ ) we consider the functions  $\tilde{f}_n(x)$  and  $\tilde{g}_n(y)$  as in Lemma 1 and get

$$\tilde{I}_1 = \left[ \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1} \tilde{g}_n^q(y) dy \right]^{\frac{1}{q}} = 2n.$$

We obtain

$$\begin{aligned} \tilde{I}_2 &:= \int_{-\infty}^\infty \tilde{g}_n(y) \left[ \int_{\{x; |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} \tilde{f}_n(x) dx \right] dy \\ &= \int_{-1}^0 \left[ \int_{\{x; |x| \geq \frac{1}{y}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} |x|^{\sigma - \frac{1}{pn} - 1} dx \right] (-y)^{\sigma_1 + \frac{1}{qn} - 1} dy \\ &\quad + \int_0^1 \left[ \int_{\{x; |x| \geq \frac{1}{y}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} |x|^{\sigma - \frac{1}{pn} - 1} dx \right] y^{\sigma_1 + \frac{1}{qn} - 1} dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left[ \int_{\{x; |x| \geq \frac{1}{y}\}} (H(-xy) + H(xy)) |x|^{\sigma - \frac{1}{pn} - 1} dx \right] y^{\sigma_1 + \frac{1}{qn} - 1} dy \\
&= 2 \int_0^1 \left[ \int_1^\infty (H(-u) + H(u)) u^{\sigma - \frac{1}{pn} - 1} du \right] y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy,
\end{aligned}$$

and then by (10) it follows that

$$2K^{(2)} \left( \sigma - \frac{1}{pn} \right) \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \tilde{I}_2 \leq M_2 \tilde{J}_1 = 2M_2 n. \quad (11)$$

Since  $(\sigma_1 - \sigma) + \frac{1}{n} \leq 0$ , it follows that

$$\int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \infty.$$

By (11), for  $K^{(2)}(\sigma - \frac{1}{pn}) > 0$ , we have  $\infty \leq 2M_2 n < \infty$ , which is a contradiction.

If  $\sigma_1 > \sigma$ , then for  $n \geq \frac{1}{\sigma_1 - \sigma}$  ( $n \in \mathbf{N}$ ) we consider the functions  $f_n(x)$  and  $g_n(y)$  as in Lemma 1 and derive that

$$J_1 = \left[ \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} = 2n.$$

We obtain

$$\begin{aligned}
I_2 &:= \int_{-\infty}^\infty f_n(x) \left[ \int_{\{y; |y| \geq \frac{1}{|x|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} g_n(y) dy \right] dx \\
&= \int_{-1}^0 \left[ \int_{\{y; |y| \geq \frac{1}{x}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} |y|^{\sigma_1 - \frac{1}{qn} - 1} dy \right] (-x)^{\sigma + \frac{1}{pn} - 1} dx \\
&\quad + \int_0^1 \left[ \int_{\{y; |y| \geq \frac{1}{x}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} |y|^{\sigma_1 - \frac{1}{qn} - 1} dy \right] x^{\sigma + \frac{1}{pn} - 1} dx \\
&= \int_0^1 \left[ \int_{\{y; |y| \geq \frac{1}{x}\}} (H(-xy) + H(xy)) |y|^{\sigma_1 - \frac{1}{qn} - 1} dy \right] x^{\sigma + \frac{1}{pn} - 1} dx \\
&= 2 \int_1^\infty (H(-u) + H(u)) u^{(\sigma_1 - \frac{1}{qn}) - 1} du \int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx,
\end{aligned}$$

and then, by Fubini's theorem (cf. [36]) and (8), it follows that

$$\begin{aligned}
2K_2 \left( \sigma_1 - \frac{1}{qn} \right) \int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx &= I_2 \\
&= \int_0^\infty g_n(y) \left( \int_{\{x; |x| \geq \frac{1}{|y|}\}} H(xy) f_n(x) dx \right) dy \leq M_2 J_1 = 2M_2 n.
\end{aligned} \quad (12)$$

Since  $(\sigma - \sigma_1) + \frac{1}{n} \leq 0$ , we get that

$$\int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx = \infty.$$

By (12), for  $K^{(2)}(\sigma_1 - \frac{1}{qn}) > 0$ , we deduce that  $\infty \leq 2M_2 n < \infty$ , which is a contradiction.

Hence, we conclude the fact that  $\sigma_1 = \sigma$ .

For  $\sigma_1 = \sigma$ , we reduce (12) as follows:

$$K^{(2)}\left(\sigma - \frac{1}{qn}\right) = \int_1^\infty (H(-u) + H(u))u^{\sigma - \frac{1}{qn} - 1} du \leq M_2. \quad (13)$$

Since  $\{(H(-u) + H(u))u^{\sigma - \frac{1}{qn} - 1}\}_{n=1}^\infty$  is increasing in  $[1, \infty)$ , applying again Levi's theorem (cf. [36]), we have that

$$\begin{aligned} K^{(2)}(\sigma) &= \int_1^\infty \lim_{n \rightarrow \infty} (H(-u) + H(u))u^{\sigma - \frac{1}{qn} - 1} du \\ &= \lim_{n \rightarrow \infty} \int_1^\infty (H(-u) + H(u))u^{\sigma - \frac{1}{qn} - 1} du \leq M_2 < \infty. \end{aligned}$$

By Remark 1, we get that  $\mu > -\alpha - 1$  and  $\beta > 0$ .

This completes the proof of the lemma.  $\square$

### 3 Main results and some corollaries

**Theorem 1** *If  $\sigma_1 \in \mathbf{R}$ , then the following statements (i), (ii), and (iii) are equivalent:*

(i) *There exists a constant  $M_1$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we have the following Hardy-type integral inequality of the first kind with the nonhomogeneous kernel:*

$$\begin{aligned} J &:= \left\{ \int_{-\infty}^\infty |y|^{p\sigma_1-1} \left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M_1 \left[ \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (14)$$

(ii) *There exists a constant  $M_1$  such that, for any  $f(x), g(y) \geq 0$  satisfying*

$$0 < \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

*we have the following inequality:*

$$\begin{aligned} I &:= \int_{-\infty}^\infty g(y) \left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right] dy \\ &< M_1 \left[ \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (15)$$

(iii)  $\sigma_1 = \sigma > -\alpha - 1$  and  $\beta > 0$ .

*If statement (iii) holds true, then the constant  $M_1 = K^{(1)}(\sigma) (\in \mathbf{R}_+)$  in (14) and (15) (for  $\sigma_1 = \sigma$ ) is the best possible.*



*Proof* (i)  $\Rightarrow$  (ii). By Hölder's inequality (cf. [37]), we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left( |y|^{\sigma_1 - \frac{1}{p}} \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy) f(x) dx \right) (|y|^{\frac{1}{p} - \sigma_1} g(y)) dy \\ &\leq J \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (16)$$

Then by (14) we deduce (15).

(ii)  $\Rightarrow$  (iii). By Lemma 1, we have  $\sigma_1 = \sigma > -\alpha - 1$  and  $\beta > 0$ .

(iii)  $\Rightarrow$  (i). We obtain the following weight function:

For  $y \neq 0$ ,

$$\begin{aligned} \omega_1(\sigma, y) &:= |y|^\sigma \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} |x|^{\sigma-1} dx \\ &= |y|^\sigma \int_{\frac{-1}{|y|}}^0 H(xy) (-x)^{\sigma-1} dx + |y|^\sigma \int_0^{\frac{1}{|y|}} H(xy) x^{\sigma-1} dx \\ &= |y|^\sigma \int_0^{\frac{1}{|y|}} H(-xy) x^{\sigma-1} dx + |y|^\sigma \int_0^{\frac{1}{|y|}} H(xy) x^{\sigma-1} dx \\ &= |y|^\sigma \int_0^{\frac{1}{|y|}} (H(-x|y|) + H(x|y|)) x^{\sigma-1} dx \\ &= \int_0^1 (H(-u) + H(u)) u^{\sigma-1} du \\ &= K^{(1)}(\sigma). \end{aligned} \quad (17)$$

By the weighted Hölder inequality and (17), we obtain

$$\begin{aligned} &\left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p \\ &= \left\{ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy) \left[ \frac{|y|^{(\sigma-1)/p}}{|x|^{(\sigma-1)/q}} f(x) \right] \left[ \frac{|x|^{(\sigma-1)/q}}{|y|^{(\sigma-1)/p}} \right] dx \right\}^p \\ &\leq \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy) \frac{|y|^{\sigma-1} f^p(x)}{|x|^{(\sigma-1)p/q}} dx \left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} h(xy) \frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} dx \right]^{p-1} \\ &= \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx \cdot [\omega_1(\sigma, y) |y|^{q(1-\sigma)-1}]^{p-1} \\ &= (K^{(1)}(\sigma))^{p-1} |y|^{-p\sigma+1} \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx. \end{aligned} \quad (18)$$

If (18) takes the form of equality for some  $y \in \mathbf{R} \setminus \{0\}$ , then (cf. [37]) there exist constants  $A$  and  $B$  such that they are not both zero and

$$A \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) = B \frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} \quad \text{a.e. in } \mathbf{R}.$$

Let us assume that  $A \neq 0$  (otherwise  $B = A = 0$ ). It follows that

$$|x|^{p(1-\sigma)-1}f^p(x) = |y|^{q(1-\sigma)} \frac{B}{A|x|} \quad \text{a.e. in } \mathbf{R},$$

which contradicts the fact that

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1}f^p(x) dx < \infty.$$

Hence, (18) takes the form of strict inequality.

For  $\sigma_1 = \sigma > -\alpha - 1$  and  $\beta > 0$ , we have  $K^{(1)}(\sigma) \in \mathbf{R}_+$ . In view of Fubini's theorem (cf. [36]), we obtain

$$\begin{aligned} J &< (K^{(1)}(\sigma))^{\frac{1}{q}} \left\{ \int_{-\infty}^{\infty} \left[ \int_{\frac{1}{|y|}}^{\frac{1}{|x|}} H(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx \right] dy \right\}^{\frac{1}{p}} \\ &= (K^{(1)}(\sigma))^{\frac{1}{q}} \left\{ \int_{-\infty}^{\infty} \left[ \int_{\frac{1}{|x|}}^{\frac{1}{|y|}} H(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (K^{(1)}(\sigma))^{\frac{1}{q}} \left[ \int_{-\infty}^{\infty} \omega_1(\sigma, x) |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= K^{(1)}(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Setting  $M_1 \geq K^{(1)}(\sigma)$ , we have

$$J < K^{(1)}(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \leq M_1 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}},$$

namely, (14) follows.

Therefore, statements (i), (ii), and (iii) are equivalent.

When statement (iii) is satisfied, if there exists a constant  $M_1 \leq K^{(1)}(\sigma)$  such that (15) is valid, then by the proof of Lemma 1, we have  $K^{(1)}(\sigma) \leq M_1$ . It follows that the constant factor  $M_1 = K^{(1)}(\sigma)$  in (15) is the best possible. The constant factor  $M_1 = K^{(1)}(\sigma)$  in (14) is still the best possible. Otherwise, by (16) (for  $\sigma_1 = \sigma$ ), we would conclude that the constant factor  $M_1 = K^{(1)}(\sigma)$  in (15) was not the best possible.

This completes the proof of the theorem.  $\square$

In particular, for  $\sigma = \sigma_1 = \frac{1}{p} > -\alpha - 1$  in Theorem 1, the following corollary holds true.

**Corollary 1** *The following statements (i), (ii), and (iii) are equivalent:*

(i) *There exists a constant  $M_1$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx < \infty,$$

the following inequality is satisfied:

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^{\beta}}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_1 \left( \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx \right)^{\frac{1}{p}}. \quad (19)$$

(ii) There exists a constant  $M_1$  such that, for any  $f(x), g(y) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty} g(y) \left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^{\beta}}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right] dy < M_1 \left( \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_{-\infty}^{\infty} g^q(y) dy \right)^{\frac{1}{q}}. \quad (20)$$

(iii)  $\alpha > -\frac{1}{p} - 1$  and  $\beta > 0$ .

If statement (iii) holds true, then the constant  $M_1 = K^{(1)}(\frac{1}{p}) \in \mathbf{R}_+$  in (19) and (20) is the best possible.

Setting  $y = \frac{1}{x}$ ,  $G(Y) = g(\frac{1}{Y}) \frac{1}{Y^2}$  in Theorem 1, and then replacing  $Y$  by  $y$ , we obtain the following corollary.

**Corollary 2** If  $\sigma_1 \in \mathbf{R}$ , then the following statements (i), (ii), and (iii) are equivalent:

(i) There exists a constant  $M_1$  such that, for any  $f(x) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left\{ \int_{-\infty}^{\infty} |y|^{-p\sigma_1-1} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x/y|, 1\})^{1+\alpha} |\ln |x/y||^{\beta}}{(\max\{|x/y|, 1\})^{\lambda+\alpha} |x/y-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_1 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (21)$$

(ii) There exists a constant  $M_1$  such that, for any  $f(x), G(y) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} y^{q(1+\sigma_1)-1} G^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} & \int_{-\infty}^{\infty} G(y) \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x/y|, 1\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x/y|, 1\})^{\lambda+\alpha} |x/y - 1|} f(x) dx \right] dy \\ & < M_1 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (22)$$

(iii)  $\sigma_1 = \sigma > -\alpha - 1$  and  $\beta > 0$ .

If statement (iii) holds true, then the constant  $M_1 = K^{(1)}(\sigma) (\in \mathbf{R}_+)$  in (21) and (22) (for  $\sigma_1 = \sigma$ ) is the best possible.

For  $g(y) = y^\lambda G(y)$  and  $\mu_1 = \lambda - \sigma_1$  in Corollary 2, we deduce the following corollary.

**Corollary 3** If  $\mu_1 \in \mathbf{R}$ , then the following statements (i), (ii), and (iii) are equivalent:

(i) There exists a constant  $M_1$  such that, for any  $f(x) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following Hardy-type integral inequality of the first kind with homogeneous kernel:

$$\begin{aligned} & \left\{ \int_{-\infty}^{\infty} y^{p\mu_1-1} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ & < M_1 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (23)$$

(ii) There exists a constant  $M_1$  such that, for any  $f(x), g(y) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} |y|^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} & \int_{-\infty}^{\infty} g(y) \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \right] dy \\ & < M_1 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\mu_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (24)$$

(iii)  $\mu_1 = \mu < \lambda + \alpha + 1$  and  $\beta > 0$ .

If statement (iii) holds true, then the constant  $M_1 = K^{(1)}(\sigma) (\in \mathbf{R}_+)$  in (23) and (24) (for  $\mu_1 = \mu$ ) is the best possible.

In particular, for  $\lambda = 1$ ,  $\sigma = \frac{1}{q}$ ,  $\mu = \frac{1}{p}$  in Corollary 3, we get the following corollary.

**Corollary 4** *The following statements (i), (ii), and (iii) are equivalent:*

(i) *There exists a constant  $M_1$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} f^p(x) dx < \infty,$$

*the following inequality holds true:*

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{-|y|}^{|y|} \left( \frac{\min\{|x|, |y|\}}{\max\{|x|, |y|\}} \right)^{1+\alpha} \frac{|\ln |x/y||^\beta}{|x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_1 \left( \int_{-\infty}^{\infty} f^p(x) dx \right)^{\frac{1}{p}}. \quad (25)$$

(ii) *There exists a constant  $M_1$  such that, for any  $f(x), g(y) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} g^q(y) dy < \infty,$$

*we have the following inequality:*

$$\int_{-\infty}^{\infty} g(y) \left[ \int_{-|y|}^{|y|} \left( \frac{\min\{|x|, |y|\}}{\max\{|x|, |y|\}} \right)^{1+\alpha} \frac{|\ln |x/y||^\beta}{|x-y|} f(x) dx \right] dy < M_1 \left( \int_{-\infty}^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_{-\infty}^{\infty} g^q(y) dy \right)^{\frac{1}{q}}. \quad (26)$$

(iii)  $\alpha > -\frac{1}{q} - 1$  and  $\beta > 0$ .

*If statement (iii) holds true, then the constant factor  $M_1 = K^{(1)}(\frac{1}{q}) \in (\mathbf{R}_+)$  in (25) and (26) is the best possible.*

**Remark 2**

(i) For  $\sigma_1 = \sigma = -\alpha + 1$  in (14), we have the following inequality with the best possible constant factor  $\frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1)$  ( $\beta > 0$ ):

$$\left\{ \int_{-\infty}^{\infty} |y|^{p(1-\alpha)-1} \left[ \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1) \left[ \int_{-\infty}^{\infty} |x|^{p\alpha-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (27)$$

(ii) For  $\mu_1 = \mu = \lambda + \alpha - 1$  in (23), we have the following inequality with the best possible constant factor  $\frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1)$  ( $\beta > 0$ ):

$$\left\{ \int_{-\infty}^{\infty} y^{p(\lambda+\alpha-1)-1} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1) \left[ \int_{-\infty}^{\infty} |x|^{p\alpha-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (28)$$

- (iii) For  $\alpha = -1$  in (25), we have the following inequality with the best possible constant factor  $\frac{\Gamma(\beta+1)}{2^\beta} \zeta\left(\beta + 1, \frac{1}{2q}\right)$  ( $\beta > 0$ ):

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{-|y|}^{|y|} \frac{|\ln |x/y||^\beta}{|x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\beta+1)}{2^\beta} \zeta\left(\beta + 1, \frac{1}{2q}\right) \left( \int_{-\infty}^{\infty} f^p(x) dx \right)^{\frac{1}{p}}. \quad (29)$$

Similarly, in view of Lemma 2, we obtain the following weight function:

For  $y \neq 0$ ,

$$\begin{aligned} \omega_2(\sigma, y) &:= |y|^\sigma \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} |x|^{\sigma-1} dx \\ &= \int_1^\infty (H(-u) + H(u)) u^{\sigma-1} du = K^{(2)}(\sigma), \end{aligned}$$

and then similarly, we derive the following results.

**Theorem 2** *If  $\sigma_1 \in \mathbf{R}$ , then the following statements (i), (ii), and (iii) are equivalent:*

- (i) *There exists a constant  $M_2$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*the following Hardy-type integral inequality of the second kind with nonhomogeneous kernel is satisfied:*

$$\begin{aligned} &\left\{ \int_{-\infty}^{\infty} y^{p\sigma_1-1} \left[ \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M_2 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (30)$$

- (ii) *There exists a constant  $M_2$  such that, for any  $f(x), g(y) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

*we have the following inequality:*

$$\begin{aligned} &\int_{-\infty}^{\infty} g(y) \left[ \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right] dy \\ &< M_2 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (31)$$

- (iii)  $\sigma_1 = \sigma < \lambda + \alpha + 1$  and  $\beta > 0$ .

*If statement (iii) holds true, then the constant  $M_2 = K^{(2)}(\sigma)$  ( $\in \mathbf{R}_+$ ) in (30) and (31) (for  $\sigma_1 = \sigma$ ) is the best possible.*

In particular, for  $\sigma = \sigma_1 = \frac{1}{p}$  in Theorem 2, we obtain the following corollary.

**Corollary 5** *The following statements (i), (ii), and (iii) are equivalent:*

(i) *There exists a constant  $M_2$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx < \infty,$$

*we have the following inequality:*

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_2 \left( \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx \right)^{\frac{1}{p}}. \quad (32)$$

(ii) *There exists a constant  $M_2$  such that, for any  $f(x), g(y) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} g^q(y) dy < \infty,$$

*we have the following inequality:*

$$\int_{-\infty}^{\infty} g(y) \left[ \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right] dy < M_2 \left( \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_{-\infty}^{\infty} g^q(y) dy \right)^{\frac{1}{q}}. \quad (33)$$

(iii)  $\alpha > -\lambda - \frac{1}{q}$  and  $\beta > 0$ .

*If statement (iii) holds true, then the constant  $M_2 = K^{(2)}(\frac{1}{p}) \in \mathbf{R}_+$  in (32) and (33) is the best possible.*

Setting  $y = \frac{1}{Y}$ ,  $G(Y) = g(\frac{1}{Y}) \frac{1}{Y^2}$  in Theorem 2, and then replacing  $Y$  by  $y$ , we deduce the following corollary.

**Corollary 6** *If  $\sigma_1 \in \mathbf{R}$ , then the following statements (i), (ii), and (iii) are equivalent:*

(i) *There exists a constant  $M_2$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we have the following inequality:*

$$\left\{ \int_{-\infty}^{\infty} y^{-p\sigma_1-1} \left[ \int_{\{x: |x| \geq |y|\}} \frac{(\min\{|x/y|, 1\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x/y|, 1\})^{\lambda+\alpha} |x/y-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_2 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (34)$$

(ii) There exists a constant  $M_2$  such that, for any  $f(x), G(y) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} & \int_{-\infty}^{\infty} G(y) \left[ \int_{\{x; |x| \geq |y|\}} \frac{(\min\{|x/y|, 1\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x/y|, 1\})^{\lambda+\alpha} |x/y - 1|} f(x) dx \right] dy \\ & < M_2 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (35)$$

(iii)  $\sigma_1 = \sigma < \lambda + \alpha + 1$  and  $\beta > 0$ .

If statement (iii) holds true, then the constant  $M_2 = K^{(2)}(\sigma) (\in \mathbf{R}_+)$  in (34) and (35) (for  $\sigma_1 = \sigma$ ) is the best possible.

For  $g(y) = y^\lambda G(y)$  and  $\mu_1 = \lambda - \sigma_1$  in Corollary 6, we deduce the following corollary.

**Corollary 7** If  $\mu_1 \in \mathbf{R}$ , then the following statements (i), (ii), and (iii) are equivalent:

(i) There exists a constant  $M_2$  such that, for any  $f(x) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$\begin{aligned} & \left\{ \int_{-\infty}^{\infty} y^{p\mu_1-1} \left[ \int_{\{x; |x| \geq |y|\}} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x - y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ & < M_2 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (36)$$

(ii) There exists a constant  $M_2$  such that, for any  $f(x), g(y) \geq 0$  satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} |y|^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} & \int_{-\infty}^{\infty} g(y) \left[ \int_{\{x; |x| \geq |y|\}} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x - y|} f(x) dx \right] dy \\ & < M_2 \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\mu_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (37)$$

(iii)  $\mu_1 = \mu > -\alpha - 1$  and  $\beta > 0$ .

If statement (iii) holds true, then the constant  $M_2 = K^{(2)}(\sigma) (\in \mathbf{R}_+)$  in (36) and (37) (for  $\mu_1 = \mu$ ) is the best possible.

In particular, for  $\lambda = 1$ ,  $\sigma = \frac{1}{q}$ ,  $\mu = \frac{1}{p}$  in Corollary 7, we obtain the following corollary.



**Corollary 8** *The following statements (i), (ii), and (iii) are equivalent:*

(i) *There exists a constant  $M_2$  such that, for any  $f(x) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} f^p(x) dx < \infty,$$

*we have the following inequality:*

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{\{x; |x| \geq |y|\}} \left( \frac{\min\{|x|, |y|\}}{\max\{|x|, |y|\}} \right)^{1+\alpha} \frac{|\ln |x/y||^\beta}{|x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_2 \left( \int_{-\infty}^{\infty} f^p(x) dx \right)^{\frac{1}{p}}. \quad (38)$$

(ii) *There exists a constant  $M_2$  such that, for any  $f(x), g(y) \geq 0$  satisfying*

$$0 < \int_{-\infty}^{\infty} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^{\infty} g^q(y) dy < \infty,$$

*we have the following inequality:*

$$\int_{-\infty}^{\infty} g(y) \left[ \int_{\{x; |x| \geq |y|\}} \left( \frac{\min\{|x|, |y|\}}{\max\{|x|, |y|\}} \right)^{1+\alpha} \frac{|\ln |x/y||^\beta}{|x-y|} f(x) dx \right] dy < M_2 \left( \int_{-\infty}^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_{-\infty}^{\infty} g^q(y) dy \right)^{\frac{1}{q}}. \quad (39)$$

(iii)  $\alpha > -\frac{1}{p} - 1$  and  $\beta > 0$ .

*If statement (iii) holds true, then the constant  $M_2 = K^{(2)}(\frac{1}{q}) \in \mathbf{R}_+$  in (38) and (39) is the best possible.*

**Remark 3**

(i) For  $\sigma_1 = \sigma = \lambda + \alpha - 1$  in (30), we have the following inequality with the best possible constant factor  $\frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1)$  ( $\beta > 0$ ):

$$\left\{ \int_{-\infty}^{\infty} y^{p(\lambda+\alpha-1)-1} \left[ \int_{\{x; |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1) \left[ \int_{-\infty}^{\infty} |x|^{p(2-\lambda-\alpha)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (40)$$

(ii) For  $\mu_1 = \mu = -\alpha + 1$  in (36), we have the following inequality with the best possible constant factor  $\frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1)$  ( $\beta > 0$ ):

$$\left\{ \int_{-\infty}^{\infty} y^{p(1-\alpha)-1} \left[ \int_{\{x; |x| \geq |y|\}} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1) \left[ \int_{-\infty}^{\infty} |x|^{p(2-\lambda-\alpha)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (41)$$

(iii) For  $\alpha = -1$  in (38), we have the following inequality with the best possible constant factor  $\frac{\Gamma(\beta+1)}{2^\beta} \zeta(\beta+1, \frac{1}{2p})$  ( $\beta > 0$ ):

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{\{x: |x| \geq |y|\}} \frac{|\ln |x/y||^\beta}{|x-y|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\beta+1)}{2^\beta} \zeta\left(\beta+1, \frac{1}{2p}\right) \left( \int_{-\infty}^{\infty} f^p(x) dx \right)^{\frac{1}{p}}. \quad (42)$$

#### 4 Operator expressions

We set the following functions:

$$\varphi(x) := |x|^{p(1-\sigma)-1}, \quad \psi(y) := |y|^{q(1-\sigma)-1}, \quad \phi(y) := |y|^{q(1-\mu)-1},$$

wherefrom

$$\varphi^{1-p}(y) = |y|^{p\sigma-1}, \quad \phi^{1-p}(y) = |y|^{p\mu-1} \quad (x, y \in \mathbb{R}).$$

We also define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbb{R}) := \left\{ f : \|f\|_{p,\varphi} := \left( \int_{-\infty}^{\infty} \varphi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom

$$\begin{aligned} L_{q,\psi}(\mathbb{R}) &= \left\{ g : \|g\|_{q,\psi} := \left( \int_{-\infty}^{\infty} \psi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{q,\phi}(\mathbb{R}) &= \left\{ g : \|g\|_{q,\phi} := \left( \int_{-\infty}^{\infty} \phi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{p,\psi^{1-p}}(\mathbb{R}) &= \left\{ h : \|h\|_{p,\psi^{1-p}} = \left( \int_{-\infty}^{\infty} \psi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\phi^{1-p}}(\mathbb{R}) &= \left\{ h : \|h\|_{q,\phi^{1-p}} = \left( \int_{-\infty}^{\infty} \phi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 1, for  $\sigma_1 = \sigma$  and  $f \in L_{p,\varphi}(\mathbb{R})$ , setting

$$h_1(y) := \int_{\frac{1}{|y|}}^{\frac{1}{|x|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \quad (y \in \mathbb{R}),$$

by (14), we obtain that

$$\|h_1\|_{p,\psi^{1-p}} = \left[ \int_{-\infty}^{\infty} \psi^{1-p}(y) |h_1(y)|^p dy \right]^{\frac{1}{p}} < M_1 \|f\|_{p,\varphi} < \infty. \quad (43)$$

**Definition 1** We define a Hardy-type integral operator of the first kind with nonhomogeneous kernel

$$T_1^{(1)} : L_{p,\varphi}(\mathbb{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbb{R})$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R})$ , there exists a unique representation

$$T_1^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$$

satisfying  $T_1^{(1)}f(y) = h_1(y)$  for any  $y \in \mathbf{R}$ .

In view of (43), it follows that

$$\|T_1^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M_1 \|f\|_{p,\varphi},$$

and thus the operator  $T_1^{(1)}$  is bounded satisfying

$$\|T_1^{(1)}\| = \sup_{f \neq 0, f \in L_{p,\varphi}(\mathbf{R})} \frac{\|T_1^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of  $T_1^{(1)}f$  and  $g$  as follows:

$$(T_1^{(1)}f, g) := \int_{-\infty}^{\infty} \left[ \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} f(x) dx \right] g(y) dy,$$

we can then rewrite Theorem 1 (for  $\sigma_1 = \sigma$ ) as follows.

**Theorem 3** *The following statements (i), (ii), and (iii) are equivalent:*

- (i) *There exists a constant  $M_1$  such that, for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), \|f\|_{p,\varphi} > 0$ , we have the following inequality:*

$$\|T_1^{(1)}f\|_{p,\psi^{1-p}} < M_1 \|f\|_{p,\varphi}. \quad (44)$$

- (ii) *There exists a constant  $M_1$  such that, for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\psi}(\mathbf{R}), \|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$ , we have the following inequality:*

$$(T_1^{(1)}f, g) < M_1 \|f\|_{p,\varphi} \|g\|_{q,\psi}. \quad (45)$$

- (iii)  $\sigma > -\alpha - 1$  and  $\beta > 0$ .

*If statement (iii) holds true, then it holds that  $\|T_1^{(1)}\| = K^{(1)}(\sigma)$ .*

- (b) In view of Corollary 3, for  $\mu_1 = \mu$  and for  $f \in L_{p,\varphi}(\mathbf{R})$ , setting

$$h_2(y) := \int_{-|y|}^{|y|} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x - y|} f(x) dx \quad (y \in \mathbf{R}),$$

by (23), we have

$$\|h_2\|_{p,\phi^{1-p}} = \left[ \int_{-\infty}^{\infty} \phi^{1-p}(y) h_2^p(y) dy \right]^{\frac{1}{p}} < M_1 \|f\|_{p,\varphi} < \infty. \quad (46)$$

**Definition 2** We define a Hardy-type integral operator of the first kind with homogeneous kernel

$$T_1^{(2)} : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\phi^{1-p}}(\mathbf{R})$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R})$ , there exists a unique representation

$$T_1^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R})$$

satisfying  $T_1^{(2)}f(y) = h_2(y)$  for any  $y \in \mathbf{R}$ .

In view of (46), it follows that

$$\|T_1^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M_1 \|f\|_{p,\varphi},$$

and thus the operator  $T_1^{(2)}$  is bounded satisfying

$$\|T_1^{(2)}\| = \sup_{f(\neq 0) \in L_{p,\varphi}(\mathbf{R})} \frac{\|T_1^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of  $T_1^{(2)}f$  and  $g$  in the following manner:

$$(T_1^{(2)}f, g) := \int_{-\infty}^{\infty} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \right] g(y) dy,$$

then we can rewrite Corollary 3 (for  $\mu_1 = \mu$ ) as follows.

**Corollary 9** The following statements (i), (ii), and (iii) are equivalent:

- (i) There exists a constant  $M_1$  such that, for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$ ,  $\|f\|_{p,\varphi} > 0$ , we have the following inequality:

$$\|T_1^{(2)}f\|_{p,\phi^{1-p}} < M_1 \|f\|_{p,\varphi}. \quad (47)$$

- (ii) There exists a constant  $M_1$  such that, for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\phi}(\mathbf{R})$ ,  $\|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$ , we have the following inequality:

$$(T_1^{(2)}f, g) < M_1 \|f\|_{p,\varphi} \|g\|_{q,\phi}. \quad (48)$$

- (iii)  $\mu < \lambda + \alpha + 1$  and  $\beta > 0$ .

If statement (iii) holds true, then we have  $\|T_1^{(2)}\| = K^{(1)}(\sigma)$ .

- (c) In view of Theorem 2, for  $\sigma_1 = \sigma$  and for  $f \in L_{p,\varphi}(\mathbf{R})$ , setting

$$H_1(y) := \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy-1|} f(x) dx \quad (y \in \mathbf{R}),$$

by (30) we obtain that

$$\|H_1\|_{p,\psi^{1-p}} = \left[ \int_{-\infty}^{\infty} \psi^{1-p}(y) H_1^p(y) dy \right]^{\frac{1}{p}} < M_2 \|f\|_{p,\varphi} < \infty. \quad (49)$$

**Definition 3** We define a Hardy-type integral operator of the second kind with nonhomogeneous kernel

$$T_2^{(1)} : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R})$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R})$ , there exists a unique representation

$$T_2^{(1)} f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$$

satisfying  $T_2^{(1)} f(y) = H_1(y)$  for any  $y \in \mathbf{R}$ .

In view of (49), it follows that

$$\|T_2^{(1)} f\|_{p,\psi^{1-p}} = \|H_1\|_{p,\psi^{1-p}} \leq M_2 \|f\|_{p,\varphi},$$

and then the operator  $T_2^{(1)}$  is bounded satisfying

$$\|T_2^{(1)}\| = \sup_{f \neq 0, f \in L_{p,\varphi}(\mathbf{R})} \frac{\|T_2^{(1)} f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of  $T_2^{(1)} f$  and  $g$  in the following manner:

$$(T_2^{(1)} f, g) := \int_{-\infty}^{\infty} \left[ \int_{\{x: |x| \geq \frac{1}{|y|}\}} \frac{(\min\{|xy|, 1\})^{1+\alpha} |\ln |xy||^\beta}{(\max\{|xy|, 1\})^{\lambda+\alpha} |xy - 1|} f(x) dx \right] g(y) dy,$$

then we can rewrite Theorem 2 (for  $\sigma_1 = \sigma$ ) as follows.

**Theorem 4** The following statements (i), (ii), and (iii) are equivalent:

- (i) There exists a constant  $M_2$  such that, for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), \|f\|_{p,\varphi} > 0$ , we have the following inequality:

$$\|T_2^{(1)} f\|_{p,\psi^{1-p}} < M_2 \|f\|_{p,\varphi}. \quad (50)$$

- (ii) There exists a constant  $M_2$  such that, for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\psi}(\mathbf{R}), \|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$ , we have the following inequality:

$$(T_2^{(1)} f, g) < M_2 \|f\|_{p,\varphi} \|g\|_{q,\psi}. \quad (51)$$

- (iii)  $\sigma < \lambda + \alpha + 1$  and  $\beta > 0$ .

If statement (iii) holds true, then we have  $\|T_2^{(1)}\| = K^{(2)}(\sigma)$ .

(d) In view of Corollary 7 ( $\mu_1 = \mu$ ), for  $f \in L_{p,\varphi}(\mathbf{R})$ , setting

$$H_2(y) := \int_{\{x; |x| \geq |y|\}} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \quad (y \in \mathbf{R}),$$

by (36) we obtain that

$$\|H_2\|_{p,\phi^{1-p}} = \left[ \int_{-\infty}^{\infty} \phi^{1-p}(y) H_2^p(y) dy \right]^{\frac{1}{p}} < M_2 \|f\|_{p,\varphi} < \infty. \quad (52)$$

**Definition 4** We define a Hardy-type integral operator of the second kind with homogeneous kernel

$$T_2^{(2)} : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\phi^{1-p}}(\mathbf{R})$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R})$ , there exists a unique representation

$$T_2^{(2)} f = H_2 \in L_{p,\phi^{1-p}}(\mathbf{R})$$

satisfying  $T_2^{(2)} f(y) = H_2(y)$  for any  $y \in \mathbf{R}$ .

In view of (52), it follows that

$$\|T_2^{(2)} f\|_{p,\phi^{1-p}} = \|H_2\|_{p,\phi^{1-p}} \leq M_2 \|f\|_{p,\varphi},$$

and thus the operator  $T_2^{(2)}$  is bounded satisfying

$$\|T_2^{(2)}\| = \sup_{f(\neq 0) \in L_{p,\varphi}(\mathbf{R})} \frac{\|T_2^{(2)} f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of  $T_1^{(2)} f$  and  $g$  as follows:

$$(T_2^{(2)} f, g) := \int_{-\infty}^{\infty} \left[ \int_{\{x; |x| \geq |y|\}} \frac{(\min\{|x|, |y|\})^{1+\alpha} |\ln |x/y||^\beta}{(\max\{|x|, |y|\})^{\lambda+\alpha} |x-y|} f(x) dx \right] g(y) dy,$$

then we can rewrite Corollary 7 (for  $\mu_1 = \mu$ ) as follows.

**Corollary 10** The following statements (i), (ii), and (iii) are equivalent:

- (i) There exists a constant  $M_2$  such that, for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), \|f\|_{p,\varphi} > 0$ , we have the following inequality:

$$\|T_2^{(2)} f\|_{p,\phi^{1-p}} < M_2 \|f\|_{p,\varphi}. \quad (53)$$

- (ii) There exists a constant  $M_2$  such that, for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\phi}(\mathbf{R}), \|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$ , we have the following inequality:

$$(T_2^{(2)} f, g) < M_2 \|f\|_{p,\varphi} \|g\|_{q,\phi}. \quad (54)$$

(iii)  $\mu > -\alpha - 1$  and  $\beta > 0$ .

If statement (iii) holds true, then we have  $\|T_2^{(2)}\| = K^{(2)}(\sigma)$ .

## 5 Conclusions

In the present paper, using weight functions we obtain in Theorems 1, 2 a few equivalent statements of two kinds of Hardy-type integral inequalities with nonhomogeneous kernel and multi-parameters in the whole plane. The constant factors related to the extended Hurwitz-zeta function are proved to be the best possible. In the form of applications, a few equivalent statements of two kinds of Hardy-type integral inequalities with the homogeneous kernel in the whole plane are also deduced in Corollaries 3, 7. We also consider some particular cases in Corollaries 1, 4, 5, 8 and in Remarks 2, 3. We additionally consider operator expressions in Theorems 3, 4 and Corollaries 9, 10. The lemmas and theorems within the present work provide an extensive account of this type of inequalities.

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## Authors' contributions

The authors have contributed equally to the preparation of the present paper. All authors read and approved the final manuscript.

## Author details

<sup>1</sup>Institute of Mathematics, University of Zurich, Zurich, Switzerland. <sup>2</sup>Program in Interdisciplinary Studies, Institute for Advanced Study, Princeton, USA. <sup>3</sup>Department of Mathematics, Guangdong University of Education, Guangzhou, P.R. China. <sup>4</sup>Moscow Institute of Physics and Technology, Dolgoprudny, Russia. <sup>5</sup>Moscow State University, Moscow, Russia. <sup>6</sup>Buryat State University, Ulan-Ude, Russia. <sup>7</sup>Caucasus Mathematical Center, Adyghe State University, Maykop, Russia.

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