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Global structure of one-sign solutions for a simply supported beam equation

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Abstract

In this paper, we consider the nonlinear eigenvalue problem

$$
\begin{aligned}
&u'''' = \lambda h(t)f(u), \quad 0 < t < 1,\\
&u(0) = u(1) = u''(0) = u''(1) = 0,
\end{aligned}
$$

where $h \in C([0,1],(0,\infty))$; $f \in C(\mathbb{R},\mathbb{R})$ and $sf(s) > 0$ for $s \neq 0$, and $f_0 = f_\infty = 0$, $f_0 = \lim_{|s|\to 0} f(s)/s$, $f_\infty = \lim_{|s|\to\infty} f(s)/s$. We investigate the global structure of one-sign solutions by using bifurcation techniques.

MSC: 34B27; 34C23; 74K10

Keywords: Connected component; Green function; One-sign solutions; Bifurcation; Simply supported beam

1 Introduction

The deformations of an elastic beam whose both end-points are simply supported are described by the fourth order problem

$$
\begin{aligned}
&u'''' = \lambda h(t)f(u), \quad 0 < t < 1,\\
&u(0) = u(1) = u''(0) = u''(1) = 0,
\end{aligned}
\tag{1.1}
$$

where $h \in C([0,1],(0,\infty))$; $f \in C(\mathbb{R},\mathbb{R})$ and $sf(s) > 0$ for $s \neq 0$.

Existence and multiplicity of positive solutions of (1.1) have been extensively studied by several authors, see [1, 2, 5–10, 13]. Cabada and Enguiça [2] developed the method of lower and upper solutions to show the existence and multiplicity of solutions, Jiang [6] and Li [7] proved the existence and multiplicity of solutions via the fixed point theorem in cone.

Bonanno and Di Bella [1] used variational method to obtain the following.

Theorem A ([1, Theorem 1.1]) *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that $xf(x)>0$ for all $x\neq 0$ and*

$$f_0=\lim_{|s|\to 0}f(s)/s=0,\qquad f_\infty=\lim_{|s|\to\infty}f(s)/s=0.$$

Then, for every

$$\lambda>\bar{\lambda}=:\left(\frac{8192}{27}+8\pi^2\right)\max\left\{\inf_{d>0}\frac{d^2}{\int_0^d f(x)\,dx},\inf_{d<0}\frac{d^2}{\int_0^d f(x)\,dx}\right\},$$

the problem

$$u''''=\lambda f(u),\quad 0<t<1,$$
$$u(0)=u(1)=u''(0)=u''(1)=0$$

has at least four nontrivial classical solutions.

In the present work, we attempt to give a direct and complete description of the global structure of one-sign solutions of (1.1) under the assumptions:
(A1) $h:[0,1]\to(0,\infty)$ is continuous;
(A2) $f\in C(\mathbb{R},\mathbb{R})$ and $sf(s)>0$ for $|s|>0$;
(A3) $f_0=0$;
(A4) $f_\infty=0$.
Let $Y=C[0,1]$ with the norm

$$\|u\|_\infty=\max_{t\in[0,1]}\bigl|u(t)\bigr|.$$

We shall use Dancer's bifurcation theorem and some properties of *superior limit* of certain infinity collection of connected sets to establish the following.

Theorem 1.1 *Let* (A1), (A2), (A3), *and* (A4) *hold. Then there exist a connected component $\mathcal{C}^+\subset\mathbb{R}^+\times C[0,1]$ of positive solutions of* (1.1) *and a connected component $\mathcal{C}^-\subset\mathbb{R}^+\times C[0,1]$ of negative solutions of* (1.1) *such that*
(1) *$\mathcal{C}^+$ is of $\subset$-shaped and joins $(+\infty,\mathbf{0})$ to $(+\infty,\infty)$;*
(2) *for every $\rho>0$, there exists $\Lambda_\rho>0$ such that*

$$(\lambda,u)\in\mathcal{C}^+\quad\text{with }\|u\|_\infty=\rho\quad\Rightarrow\quad\lambda>\Lambda_\rho;$$

(3) *$\mathcal{C}^-$ is of $\subset$-shaped and joins $(+\infty,\mathbf{0})$ to $(+\infty,\infty)$;*
(4) *for every $\rho>0$, there exists $\Lambda_\rho>0$ such that*

$$(\lambda,u)\in\mathcal{C}^-\quad\text{with }\|u\|_\infty=\rho\quad\Rightarrow\quad\lambda>\Lambda_\rho.$$

The linear problem

$$\begin{cases}u''''(x)=y(x), & x\in(0,1),\\ u(0)=u(1)=u''(0)=u''(1)=0,\end{cases}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s)y(s)\,ds =: Ty(t),$$

where

$$G(t,s) = \frac{1}{6}\begin{cases} t(1-s)[2s-s^2-t^2], & 0 \le t \le s \le 1, \\ s(1-t)[2t-t^2-s^2], & 0 \le s \le t \le 1. \end{cases}$$

Let

$$q(t) = \frac{1}{2}t(1-t), \quad t \in [0,1],$$

$$j(s) = \begin{cases} 1 - \sqrt{\frac{1-s^2}{3}}, & s \in [0,1/2], \\ \sqrt{\frac{s(2-s)}{3}}, & s \in [1/2,1]. \end{cases}$$

Then

$$G\bigl(j(s),s\bigr) = \max_{0\le t\le 1} G(t,s),$$

$$G\bigl(j(s),s\bigr) = \frac{1}{9}\begin{cases} s(1-s)(1+s)\sqrt{\frac{1-s^2}{3}}, & s \in [0,1/2], \\ s(1-s)(2-s)\sqrt{\frac{s(2-s)}{3}}, & s \in [1/2,1], \end{cases}$$

$$G(t,s) \ge q(t)G\bigl(j(s),s\bigr), \quad t \in [0,1].$$

$$G(t,s) \ge \frac{3}{32}G\bigl(j(s),s\bigr), \quad t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

Let

$$K := \left\{ w \in C[0,1] : \min_{0\le t\le 1} w(t) \ge 0, \min_{1/4\le t\le 3/4} w(t) \ge \frac{3}{32}\|w\|_\infty \right\}. \tag{1.2}$$

Corollary 1.1 *Let* (A1), (A2), (A3), *and* (A4) *hold. Then* (1.1) *with* $h \equiv 1$ *has at least two positive solutions and at least two negative solutions* (*see Fig.* 1) *provided*

$$\lambda > \hat{\lambda} =: \left(\hat{m}_\rho \int_{1/4}^{3/4} G\bigl(j(s),s\bigr)h(s)\,ds\right)^{-1},$$

where

$$\hat{m}_\rho = \min_{3\rho/32\le x\le \rho}\bigl\{f(x)\bigr\}.$$

For other related results on the existence and multiplicity of positive solutions and nodal solutions of fourth order problems, see Rynne [13] and Ma [8, 9].

The rest of the paper is arranged as follows: In Sect. 2, we prove some properties of *superior limit* of certain infinity collection of connected sets. In Sect. 3, we state and prove some properties for the one-sign solutions (λ, u) of (1.1). Finally, in Sect. 4, we state and prove our main results.

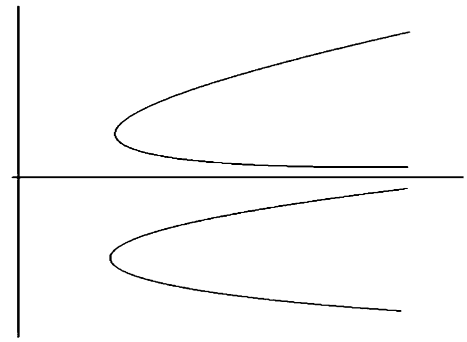

Figure 1 Components of one-sign solutions in Theorem 1.1

2 Superior limit and component

Definition 2.1 ([14]) Let X be a Banach space and $\{C_n \mid n = 1, 2, \dots\}$ be a family of subsets of X. Then the *superior limit* $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \bigl\{x \in X \mid \exists\{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x\bigr\}.$$

Definition 2.2 ([14]) A *component* of a set M means a maximal connected subset of M.

Lemma 2.1 ([14]) *Suppose that Y is a compact metric space, A and B are non-intersecting closed subsets of Y, and no component of Y intersects both A and B. Then there exist two disjoint compact subsets Y_A and Y_B such that $Y = Y_A \cup Y_B$, $A \subset Y_A$, $B \subset Y_B$.*

Lemma 2.2 ([11]) *Let X be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of X. Assume that*

(i) *there exist $z_n \in C_n$, $n = 1, 2, \dots$, and $z^* \in X$ such that $z_n \to z^*$;*
(ii) $r_n = \sup\{\|x\| \mid x \in C_n\} = \infty$;
(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of X, where*

$$B_R = \bigl\{x \in X \mid \|x\| \le R\bigr\}.$$

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ and $z^* \in \mathcal{C}$.

Let $E = \{u \in C^3[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$ with the norm

$$\|u\| = \max\bigl\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty, \|u'''\|_\infty\bigr\}.$$

It is well known that the linear eigenvalue problem

$$\begin{cases} u'''' = \mu h(x)u(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has an infinite sequence of simple eigenvalues

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots, \quad k \to \infty,$$

and the eigenfunction ϕ_k corresponding to μ_k has exactly $k - 1$ simple zeros in $(0, 1)$, see [13].

3 Some preliminary results

Let us define an operator $T_\lambda : Y \to Y$ by

$$T_\lambda u := \lambda \int_0^1 G(t,s)h(s)f\big(u(s)\big)\,ds.$$

Lemma 3.1 *Assume that* (A1)–(A4) *hold. Then* $T_\lambda : K \to K$ *is completely continuous.*

Lemma 3.2 *Let* $\Omega_r := \{u \in K : \|u\|_\infty < r\}$. *Let* (A1)–(A4) *hold. If* $u \in \partial\Omega_r, r > 0$, *then*

$$\|T_\lambda u\|_\infty \le \lambda \hat{M}_r \int_0^1 G\big(j(s),s\big)h(s)\,ds, \tag{3.1}$$

where $\hat{M}_r = \max_{0\le s\le r}\{f(s)\}$.

Proof Since $f(u(t)) \le \hat{M}_r$ for $t \in [0,1]$, it follows that

$$\begin{aligned}\|T_\lambda u\|_\infty &\le \lambda \int_0^1 G\big(j(s),s\big)h(s)f\big(u(s)\big)\,ds \\ &\le \lambda \hat{M}_r \int_0^1 G\big(j(s),s\big)h(s)\,ds.\end{aligned}$$ □

Lemma 3.3 *Let* (A1)–(A4) *hold. Assume that* $\{(\mu_k, y_k)\} \subset (0,+\infty) \times K$ *is a sequence of positive solutions of* (1.1). *Assume that* $\mu_k \le C_0$ *for some constant* $C_0 > 0$, *and*

$$\lim_{k\to\infty} \|y_k'''\|_\infty = \infty. \tag{3.2}$$

Then

$$\lim_{k\to\infty} \|y_k\|_\infty = \infty. \tag{3.3}$$

Proof Assume on the contrary that $\{\|y_k\|_\infty\}$ is bounded. Then

$$\big\|\mu_k h(x) f\big(y_k(x)\big)\big\|_\infty \le M$$

for some constant M that is independent of k. Thus, it follows from the relation

$$y_k''''(x) = \mu_k h(x) f\big(y_k(x)\big)$$

that $\{y_k''''\}$ is uniformly bounded in $C[0,1]$, and subsequently $\{y_k'''\}$ is uniformly bounded in $C[0,1]$. However, this contradicts (3.2). □

Lemma 3.4 *Assume that* (A1)–(A4) *hold. If* $u \in \partial\Omega_r$, $r > 0$, *then*

$$\|T_\lambda u\|_\infty \ge \lambda \hat{m}_r \int_{1/4}^{3/4} G\big(j(s),s\big)h(s)\,ds, \tag{3.4}$$

where

$$\hat{m}_r = \min_{3r/32\le x\le r}\big\{f(x)\big\}. \tag{3.5}$$

Proof Since $f(u(t)) \geq \hat{m}_r$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, it follows that

$$\begin{aligned} \|T_\lambda u\|_\infty &\geq \lambda \int_0^1 G\big(j(s), s\big) h(s) f\big(u(s)\big)\, ds \\ &\geq \lambda \hat{m}_r \int_{1/4}^{3/4} G\big(j(s), s\big) h(s)\, ds. \end{aligned}$$

□

4 Proof of the main results

We only deal with the global behavior of positive solutions of (1.1). The global behavior of negative solutions of (1.1) can be treated by a similar method.

Let Σ^+ be the closure of the set of positive solutions of (1.1) in E. To prove Theorem 1.1, we will develop a bifurcation approach to treat the case $f_0 = 0$. Crucial to this approach is to construct a sequence of functions $\{f^{[n]}\}$ that is asymptotic linear at 0 and satisfies

$$f^{[n]} \to f, \qquad \big(f^{[n]}\big)_0 \to 0.$$

By means of the corresponding auxiliary equations, we obtain a sequence of unbounded components $\{C_+^{[n]}\}$ via nonlinear Krein–Rutman bifurcation theorem, see Dancer [3] and Zeidler [15], and this enables us to find unbounded components $\hat{\zeta}$ satisfying

$$\hat{\zeta} \subset \limsup_{n\to\infty} C_+^{[n]}$$

and joining $(+\infty, \mathbf{0})$ with $(+\infty, \infty)$.

Define $g^{[n]} : \mathbb{R} \to \mathbb{R}$ by

$$g^{[n]}(s) = \begin{cases} f(s), & s \in (\frac{1}{n}, \infty) \cup (-\infty, -\frac{1}{n}), \\ nf(\frac{1}{n})s, & s \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases} \tag{4.1}$$

Then $g^{[n]} \in C(\mathbb{R}, \mathbb{R})$ with

$$sg^{[n]}(s) > 0, \quad \forall |s| \in (0, \infty), \quad \text{and} \quad \big(g^{[n]}\big)_0 = nf\left(\frac{1}{n}\right). \tag{4.2}$$

By (A3), it follows that

$$\lim_{n\to\infty} \big(g^{[n]}\big)_0 = 0.$$

To apply the nonlinear Krein–Rutman theorem [4], let us consider the auxiliary family of the equations

$$u'''' = \lambda h(t) g^{[n]}(u), \quad t \in (0,1), \tag{4.3}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{4.4}$$

Let $\xi^{[n]} \in C(\mathbb{R})$ be such that

$$g^{[n]}(u) = \big(g^{[n]}\big)_0 u + \xi^{[n]}(u) = nf\left(\frac{1}{n}\right) u + \xi^{[n]}(u). \tag{4.5}$$

Then

$$\lim_{|u|\to 0} \frac{\xi^{[n]}(u)}{u} = 0. \tag{4.6}$$

Let $D := \{u \in C^4[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$. Let $L : D \to Y$ be the linear operator defined by

$$Lu := u'''', \quad u \in D.$$

Let us consider

$$Lu - \lambda h(t)\big(g^{[n]}\big)_0 u = \lambda h(t)\xi^{[n]}(u) \tag{4.7}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (4.7) can be converted to the equivalent equation

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\big[\lambda h(s)\big(g^{[n]}\big)_0 u(s) + \lambda h(s)\xi^{[n]}\big(u(s)\big)\big]\,ds \\ &:= \lambda L^{-1}\big[h(\cdot)\big(g^{[n]}\big)_0 u(\cdot)\big](t) + \lambda L^{-1}\big[h(\cdot)\xi^{[n]}\big(u(\cdot)\big)\big](t). \end{aligned} \tag{4.8}$$

Further we note that $\|L^{-1}[h(\cdot)\xi^{[n]}(u(\cdot))]\|_\infty = o(\|u\|_\infty)$ for u near θ in E.

By the fact $(g^{[n]})_0 > 0$, the results of nonlinear Krein–Rutman theorem (see Dancer [3] and Zeidler [15, Corollary 15.12]) for (4.7) can be stated as follows: there exists a continuum $C_+^{[n]}$ of positive solutions of (4.7) joining $(\frac{\lambda_1}{(g^{[n]})_0}, \theta)$ to infinity in $[0,\infty) \times K$. Moreover, $C_+^{[n]} \setminus \{(\frac{\lambda_1}{(g^{[n]})_0}, \theta)\} \subset ([0,\infty) \times \operatorname{int} K)$ and $(\frac{\lambda_1}{(g^{[n]})_0}, \theta)$ is the only positive bifurcation point of (4.7) lying on the trivial solutions line $u \equiv \theta$.

Lemma 4.1 *Let* (A1)–(A4) *hold. Then, for each fixed* n, $C_+^{[n]}$ *joins* $(\frac{\lambda_1}{(g^{[n]})_0}, \theta)$ *to* (∞, ∞) *in* $[0,\infty) \times K$.

Proof We divide the proof into two steps.

Step 1. We show that $\sup\{\lambda \mid (\lambda, u) \in C_+^{[n]}\} = \infty$.

Assume on the contrary that $\sup\{\lambda \mid (\lambda, u) \in C_+^{[n]}\} =: c_0 < \infty$. Let $\{(\mu_k, y_k)\} \subset C_+^{[n]}$ be such that

$$|\mu_k| + \|y_k\|_\infty \to \infty.$$

Then $\|y_k\|_\infty \to \infty$. This together with the fact

$$\min_{\sigma \le t \le 1-\sigma} y_k(t) \ge q(\sigma)\|y_k\|_\infty, \quad \forall 0 < \sigma < \frac{1}{2} \tag{4.9}$$

implies that, for arbitrary $\sigma \in (0, \frac{1}{2})$,

$$\lim_{k\to\infty} y_k(t) = \infty, \quad \text{uniformly for } t \in [\sigma, 1-\sigma]. \tag{4.10}$$

Since $(\mu_k, y_k) \in C_+^{[n]}$, we have that

$$y_k''''(t) = \mu_k h(t) g^{[n]}\big(y_k(t)\big), \quad t \in (0,1), \tag{4.11}$$

$$y_k(0) = y_k(1) = y_k''(0) = y_k''(1) = 0. \tag{4.12}$$

Set $v_k(t) = \frac{y_k(t)}{\|y_k\|_\infty}$. Then

$$\|v_k\|_\infty = 1,$$

$$v_k''''(t) = \mu_k h(t) \frac{g^{[n]}(y_k(t))}{y_k(t)} v_k(t), \quad t \in (0,1), \tag{4.13}$$

$$v_k(0) = v_k(1) = v_k''(0) = v_k''(1) = 0. \tag{4.14}$$

From (4.13) and the fact that $(g^{[n]})_\infty = 0$, we conclude that

$$\big\|v_k''''\big\|_\infty \le M$$

for some constant $M > 0$ independent of k.

Now, choosing a subsequence and relabeling if necessary, it follows that there exists $(\mu_*, v_*) \in [0, c_0] \times E$ with

$$\|v_*\|_\infty = 1, \tag{4.15}$$

such that

$$\lim_{k\to\infty} (\mu_k, v_k) = (\mu_*, v_*), \quad \text{in } [0, c_0] \times E. \tag{4.16}$$

Notice that (4.13), (4.14) is equivalent to

$$v_k(t) = \mu_k \int_0^1 G(t,s) h(s) \frac{g^{[n]}(y_k(s))}{y_k(s)} v_k(s)\, ds, \quad t \in (0,1).$$

Combining this with (4.16) and using (4.10) and the Lebesgue dominated convergence theorem, we conclude that

$$v_*(t) = \mu_* \int_0^1 G(t,s) h(s) 0 v_*(s)\, ds = 0, \quad t \in (0,1).$$

This contradicts (4.15). Therefore

$$\sup\big\{\lambda \mid (\lambda, y) \in C_+^{[n]}\big\} = \infty.$$

Step 2. We show that $\sup\{\|u\|_\infty \mid (\lambda, u) \in C_+^{[n]}\} = \infty$.

Assume on the contrary that $\sup\{\|u\|_\infty \mid (\lambda, u) \in C_+^{[n]}\} =: M_\infty < \infty$. Let $\{(\mu_k, y_k)\} \subset C_+^{[n]}$ be such that

$$\mu_k \to \infty, \qquad \|y_k\|_\infty \le M_\infty. \tag{4.17}$$

Since $(\mu_k, y_k) \in C_+^{[n]}$, for any $t \in [\sigma, 1-\sigma]$, we have from (1.2) that

$$\begin{aligned} y_k(t) &= \mu_k \int_0^1 G(t,s)h(s)g^{[n]}\big(y_k(s)\big)\,ds \\ &\geq \mu_k \int_\sigma^{1-\sigma} q(\sigma)G\big(j(s),s\big)h(s)\frac{g^{[n]}(y_k(s))}{y_k(s)}y_k(s)\,ds \\ &\geq \mu_k \int_\sigma^{1-\sigma} q(\sigma)G\big(j(s),s\big)h(s)\frac{g^{[n]}(y_k(s))}{y_k(s)}q(\sigma)\,ds\|y_k\|_\infty \\ &\geq \mu_k \int_\sigma^{1-\sigma} q^2(\sigma)G\big(j(s),s\big)h(s)b_*\,ds\|y_k\|_\infty \end{aligned}$$

(where $b_* := \inf\{\frac{g^{[n]}(x)}{x} \mid x \in (0, M_\infty]\} > 0$), which yields that $\{\mu_k\}$ is bounded. However, this contradicts (4.17).

Therefore, $C_+^{[n]}$ joins $(\frac{\lambda_1}{(g^{[n]})_0}, \theta)$ to (∞, ∞) in K. □

Lemma 4.2 *Let* (A1)–(A4) *hold and let* $I \subset (0,\infty)$ *be a closed interval. Then there exists a positive constant* M *such that*

$$\sup\big\{\|y\|_\infty \mid (\mu, y) \in C_+^{[n]} \textit{ and } \mu \in I\big\} \leq M.$$

Proof Assume on the contrary that there exists a sequence $\{(\mu_k, y_k)\} \subset C_+^{[n]} \cap (I \times K)$ such that

$$\|y_k\|_\infty \to \infty.$$

Then, (4.9), (4.10), (4.11), and (4.12) hold. Set $v_k(t) = \frac{y_k(t)}{\|y_k\|_\infty}$. Then

$$\|v_k\|_\infty = 1.$$

Now, choosing a subsequence and relabeling if necessary, it follows that there exists $(\mu_*, v_*) \in I \times Y$ with

$$\|v_*\|_\infty = 1 \tag{4.18}$$

such that

$$\lim_{k\to\infty} (\mu_k, v_k) = (\mu_*, v_*) \quad \text{in } \mathbb{R} \times Y.$$

Moreover, from (4.11), (4.12), (4.10) and the assumption $f_\infty = 0$, it follows that

$$\begin{aligned} &v_*''''(t) = \mu_* h(t) \cdot 0, \quad t \in (0,1), \\ &v_*(0) = v_*(1) = v_*''(0) = v_*''(1) = 0, \end{aligned}$$

and subsequently, $v_*(t) \equiv 0$ for $t \in [0,1]$. This contradicts (4.18). Therefore

$$\sup\big\{\|y\|_\infty \mid (\mu, y) \in C_+^{[n]} \text{ and } \mu \in I\big\} \leq M.$$ □

Lemma 4.3 *Let* (A1)–(A4) *hold. Then there exists* $\rho^* > 0$ *such that*

$$\left(\bigcup_{n=1}^{\infty} C_+^{[n]}\right) \cap \bigl((0,\rho^*) \times K\bigr) = \emptyset.$$

Proof Assume on the contrary that there exists $\{(\mu_k, y_k)\} \subset (\bigcup_{n=1}^{\infty} C_+^{[n]}) \cap ((0,+\infty) \times K)$ such that $\mu_k \to 0$. Then

$$y_k(t) = \mu_k \int_0^1 G(t,s)h(s)g^{[n]}\bigl(y_k(s)\bigr)\,ds, \quad t \in (0,1).$$

Set $v_k(t) = \frac{y_k(t)}{\|y_k\|_\infty}$. Then

$$\|v_k\|_\infty = 1,$$

and for all $t \in (0,1)$,

$$\begin{aligned} v_k(t) &= \mu_k \int_0^1 G(t,s)h(s)\frac{g^{[n]}(y_k(s))}{y_k(s)}\frac{y_k(s)}{\|y_k\|_\infty}\,ds \\ &\le \mu_k \int_0^1 G\bigl(j(s),s\bigr)h(s)B_n^*\|v_k\|_\infty\,ds, \end{aligned}$$

where $B_n^* = \sup\{\frac{g^{[n]}(x)}{x} \mid x \in (0,\infty), n \in \mathbb{N}\}$. Let

$$B^* = \sup\bigl\{B_n^* \mid n \in \mathbb{N}\bigr\}.$$

Then $B^* < \infty$, and

$$v_k(t) \le \mu_k \int_0^1 G\bigl(j(s),s\bigr)h(s)B^*\|v_k\|_\infty\,ds \to 0,$$

which contradicts the fact $\|v_k\|_\infty = 1$. Therefore, there exists $\rho^* > 0$, such that

$$\left(\bigcup_{n=1}^{\infty} C_+^{[n]}\right) \cap \bigl((0,\rho^*) \times K\bigr) = \emptyset. \qquad \square$$

Proof of Theorem 1.1 By Lemmas 4.1–4.3 and the similar method to prove Ma and An [12, Theorem 4.1], with obvious changes, we may get a desired connected component $\mathcal{C}^+ \subset \limsup C_+^{[n]}$ of positive solutions of (1.1) and a connected component $\mathcal{C}^- \subset \limsup C_-^{[n]}$ of negative solutions of (1.1) such that

(1) $\mathcal{C}^+$ is of $\subset$-shaped and joins $(+\infty,\theta)$ to $(+\infty,\infty)$;
(2) for every $\rho > 0$, there exists $\Lambda_\rho > 0$ such that

$$(\lambda,u) \in \mathcal{C}^+ \quad \text{with } \|u\|_\infty = \rho \quad \Rightarrow \quad \lambda > \Lambda_\rho;$$

(3) $\mathcal{C}^-$ is of $\subset$-shaped and joins $(+\infty,\theta)$ to $(+\infty,\infty)$;
(4) for every $\rho > 0$, there exists $\Lambda_\rho > 0$ such that

$$(\lambda,u) \in \mathcal{C}^- \quad \text{with } \|u\|_\infty = \rho \quad \Rightarrow \quad \lambda > \Lambda_\rho. \qquad \square$$

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Authors' contributions
The authors claim that the research was realized in collaboration with the same responsibility. All authors read and approved the last version of the manuscript.

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