Guo and Zhou *Journal of Inequalities and Applications* (2020) 2020:161
https://doi.org/10.1186/s13660-020-02427-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the distance α-spectral radius of a connected graph

Haiyan Guo[1] and Bo Zhou[1*]

*Correspondence:
zhoubo@scnu.edu.cn
[1]School of Mathematical Sciences, South China Normal University, Guangzhou, P.R. China

Abstract

For a connected graph G and $\alpha \in [0,1)$, the distance α-spectral radius of G is the spectral radius of the matrix $D_\alpha(G)$ defined as $D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G)$, where $T(G)$ is a diagonal matrix of vertex transmissions of G and $D(G)$ is the distance matrix of G. We give bounds for the distance α-spectral radius, especially for graphs that are not transmission regular, propose local graft transformations that decrease or increase the distance α-spectral radius, and determine the graphs that minimize and maximize the distance α-spectral radius among several families of graphs.

MSC: 05C50; 05C12

Keywords: Distance spectral radius; Distance signless Laplacian spectral radius; Local graft transformation; Extremal graph

1 Introduction

We consider simple and undirected graphs. Let G be a connected graph of order n with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$ or simply d_{uv} if the graph G is clear from the context, is the length of a shortest path from u to v in G. The distance matrix of G is the $n \times n$ matrix $D(G) = (d_G(u,v))_{u,v\in V(G)}$. For $u \in V(G)$, the transmission of u in G, denoted by $T_G(u)$, is defined as the sum of distances from u to all other vertices of G, i.e., $T_G(u) = \sum_{v\in V(G)} d_G(u,v)$. The transmission matrix $T(G)$ of G is the diagonal matrix of transmissions of G. Then $Q(G) = T(G) + D(G)$ is the distance signless Laplacian matrix of G, proposed recently in [1]. Arisen from a data communication problem, the spectrum of the distance matrix was studied by Graham and Pollack [12] in 1971, early related work may be found also in [10, 11], and now it has been studied extensively, see the recent survey [2] and the very recent papers [4, 5, 17, 18, 26]. The distance signless Laplacian spectrum has also received much attention, see, e.g., [1, 3, 4, 7, 15, 16, 29].

Throughout this paper we assume that $\alpha \in [0,1)$. Motivated by the work of Nikiforov [22], we consider the convex combinations $D_\alpha(G)$ of $T(G)$ and $D(G)$, defined as

$$D_\alpha(G) = \alpha T(G) + (1-\alpha)D(G),$$

see [6]. Evidently, $D_0(G) = D(G)$ and $2D_{1/2}(G) = Q(G)$. We call the eigenvalues of $D_\alpha(G)$ the distance α-eigenvalues of G. As $D_\alpha(G)$ is a symmetric matrix, the distance α-eigenvalues of G are all real, which are denoted by $\mu_\alpha^{(1)}(G), \dots, \mu_\alpha^{(n)}(G)$, arranged in nonincreasing order, where $n = |V(G)|$. The largest distance α-eigenvalue $\mu_\alpha^{(1)}(G)$ of G is called the distance α-spectral radius of G, written as $\mu_\alpha(G)$. Obviously, $\mu_0^{(1)}(G), \dots, \mu_0^{(n)}(G)$ are the distance eigenvalues of G, and $2\mu_{1/2}^{(1)}(G), \dots, 2\mu_{1/2}^{(n)}(G)$ are the distance signless Laplacian eigenvalues of G. Particularly, $\mu_0(G)$ is just the distance spectral radius [2] and $2\mu_{1/2}(G)$ is just the distance signless Laplacian spectral radius of G [1].

In this paper, we give sharp bounds for the distance α-spectral radius, and particularly an upper bound for the distance α-spectral radius of connected graphs that are not transmission regular, and propose some types of graft transformations that decrease or increase the distance α-spectral radius. We also determine the unique graphs with minimum distance α-spectral radius among trees and unicyclic graphs, respectively, as well as the unique graphs (trees) with maximum and second maximum distance α-spectral radii, and the unique graph with maximum distance α-spectral radius among connected graphs with given clique number, and among odd-cycle unicyclic graphs, respectively.

2 Preliminaries

Let G be a connected graph with $V(G) = \{v_1, \dots, v_n\}$. A column vector $x = (x_{v_1}, \dots, x_{v_n})^\top \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_{v_i}, i.e., $x(v_i) = x_{v_i}$ for $i = 1, \dots, n$. Then

$$x^\top D_\alpha(G)x = \alpha \sum_{u \in V(G)} T_G(u)x_u^2 + 2 \sum_{\{u,v\} \subseteq V(G)} (1-\alpha)d_G(u,v)x_u x_v,$$

or equivalently,

$$x^\top D_\alpha(G)x = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)\big(\alpha\big(x_u^2 + x_v^2\big) + 2(1-\alpha)x_u x_v\big).$$

Since $D_\alpha(G)$ is a nonnegative irreducible matrix, by the Perron–Frobenius theorem, $\mu_\alpha(G)$ is simple and there is a unique positive unit eigenvector corresponding to $\mu_\alpha(G)$, which is called the distance α-Perron vector of G. If x is the distance α-Perron vector of G, then for each $u \in V(G)$,

$$\mu_\alpha(G)x_u = \sum_{v \in V(G)} d_G(u,v)\big(\alpha x_u + (1-\alpha)x_v\big),$$

which is called the α-equation of G at u. For a unit column vector $x \in \mathbb{R}^n$ with at least one nonnegative entry, by Rayleigh's principle, we have $\mu_\alpha(G) \ge x^\top D_\alpha(G)x$ with equality if and only if x is the distance α-Perron vector of G.

As in [27], we have the following result.

Lemma 2.1 *Suppose that G is a connected graph, η is an automorphism of G, and x is the distance α-Perron vector of G. Then for $u, v \in V(G)$, $\eta(u) = v$ implies that $x_u = x_v$.*

Proof Let $P = (p_{uv})_{u,v \in V(G)}$ be the permutation matrix such that $p_{vu} = 1$ if and only if $\eta(u) = v$ for $u, v \in V(G)$. We have $D_\alpha(G) = P^\top D_\alpha(G)P$ and Px is a positive unit vector. Thus

$\mu_\alpha(G) = x^\top D_\alpha(G)x = (Px)^\top D_\alpha(G)(Px)$, implying Px is also the distance α-Perron vector of G. Thus $Px = x$, and the result follows. □

Let G be a graph. For $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v in G, and $\deg_G(v)$ be the degree of v in G. Let $G - v$ be the subgraph of G obtained by deleting v and all edges containing v. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. For a subset E' of $E(G)$, $G - E'$ denotes the graph obtained from G by deleting all the edges in E', and in particular, we write $G - xy$ instead of $G - \{xy\}$ if $E' = \{xy\}$. Let $\overline{G}$ be the complement of G. For a subset E' of $E(\overline{G})$, denote $G + E'$ the graph obtained from G by adding all edges in E', and in particular, we write $G + xy$ instead of $G + \{xy\}$ if $E' = \{xy\}$.

For a nonnegative square matrix A, the Perron–Frobenius theorem implies that A has an eigenvalue that is equal the maximum modulus of all its eigenvalues; this eigenvalue is called the spectral radius of A, denoted by $\rho(A)$. Note that $\mu_\alpha(G) = \rho(D_\alpha(G))$ for a connected graph G.

Restating Corollary 2.2 in [20, p. 38], we have

Lemma 2.2 ([20]) *Suppose that A and B are square nonnegative matrices, A is irreducible, and $A - B$ is nonnegative but nonzero. Then $\rho(A) > \rho(B)$.*

By Lemma 2.2, we have

Lemma 2.3 *Suppose that G is a connected graph with $u, v \in V(G)$, and u and v are not adjacent. Then $\mu_\alpha(G + uv) < \mu_\alpha(G)$.*

The transmission of a connected graph G, denoted by $\sigma(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $\sigma(G) = \frac{1}{2}\sum_{v\in V(G)} T_G(v)$. A graph is said to be transmission regular if $T_G(v)$ is a constant for each $v \in V(G)$. By Rayleigh's principle, we have

Lemma 2.4 *Suppose that G is a connected graph of order n. Then $\mu_\alpha(G) \ge \frac{2\sigma(G)}{n}$ with equality if and only if G is transmission regular.*

For an $n \times n$ nonnegative matrix $A = (a_{ij})$, let r_i be the ith row sum of A, i.e., $r_i = \sum_{j=1}^n a_{ij}$ for $i = 1, \dots, n$, and let $r_{\min}$ and $r_{\max}$ be the minimum and maximum row sums of A, respectively.

Lemma 2.5 ([3]) *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, \dots, r_n$. Let $S = \{1, \dots, n\}$, $r_{\min} = r_p$, $r_{\max} = r_q$ for some p and q with $1 \le p, q \le n$, $\ell = \max\{r_i - a_{ip} : i \in S \setminus \{p\}\}$, $m = \min\{r_i - a_{iq} : i \in S \setminus \{q\}\}$, $s = \max\{a_{ip} : i \in S \setminus \{p\}\}$ and $t = \min\{a_{iq} : i \in S \setminus \{q\}\}$. Then*

$$\begin{aligned}
&\frac{a_{qq} + m + \sqrt{(m - a_{qq})^2 + 4t(r_{\max} - a_{qq})}}{2} \\
&\quad \le \rho(A) \\
&\quad \le \frac{a_{pp} + \ell + \sqrt{(\ell - a_{pp})^2 + 4s(r_{\min} - a_{pp})}}{2}.
\end{aligned}$$

Moreover, the first equality holds if $r_i - a_{iq} = m$ and $a_{iq} = t$ for all $i \in S \setminus \{q\}$, and the second equality holds if $r_i - a_{ip} = \ell$ and $a_{ip} = s$ for all $i \in S \setminus \{p\}$.

Let $J_{s\times t}$ be the $s \times t$ matrix of all 1's, $0_{s\times t}$ the $s \times t$ matrix of all 0's, and I_s the identity matrix of order s.

Let K_n, P_n, and S_n be the complete graph, the path, and the star of order n, respectively. Let C_n denote the cycle of order $n \geq 3$.

For a connected graph G, let $T_{\min}(G)$ and $T_{\max}(G)$ be the minimum and maximum transmissions of G, respectively.

3 Bounds for the distance α-spectral radius

Let G be a connected graph of order n. Note that $D_\alpha(K_n) = \alpha(n-1)I_n + (1-\alpha)(J_{n\times n} - I_n)$, and thus $\mu_\alpha(K_n) = n-1$. By Lemma 2.3, we have $\mu_\alpha(G) \geq n-1$ with equality if and only if $G \cong K_n$.

If $(d_1, \dots, d_n)$ is the nonincreasing degree sequence of a graph G of order at least 2, then d_1 (resp. d_2) is the maximum (resp. second maximum) degree, d_n (resp. d_{n-1}) is the minimum (resp. second minimum) degree of G. The diameter of G is the maximum distance between all vertex pairs of G. Using techniques from [33] by considering the first two minima or maxima of the entries of the distance α-Perron vector, we may prove the following lower and upper bounds: If G is a connected graph of order $n \geq 2$ with maximum degree Δ and second maximum degree Δ', then

$$\mu_\alpha(G) \geq \frac{1}{2}\Big(\alpha\big(4n-4-\Delta-\Delta'\big) + \sqrt{\alpha^2\big(4n-4-\Delta-\Delta'\big)^2 - 4(2\alpha-1)(2n-2-\Delta)\big(2n-2-\Delta'\big)}\Big)$$

with equality if and only if G is regular with diameter at most 2. If G is a connected graph of order $n \geq 2$ with minimum degree δ and second minimum degree δ', then

$$\mu_\alpha(G) \leq \frac{1}{2}\Big(\alpha\big(2dn-2-(d-1)\big(d+\delta+\delta'\big)\big) + \sqrt{\alpha^2\big(2dn-2-(d-1)\big(d+\delta+\delta'\big)\big)^2 - 4(2\alpha-1)SS'}\Big)$$

with equality if and only if G is regular with $d \leq 2$, where d is the diameter of G, $S = dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)$ and $S' = dn - \frac{d(d-1)}{2} - 1 - \delta'(d-1)$. The proof of the above bounds may be found in the early version of this paper at arXiv:1901.10180.

Similarly, bounds for the distance α-spectral radius for connected bipartite graphs may be obtained as in [33].

A connected graph G of order n is distinguished vertex deleted regular (DVDR) if there is a vertex v of degree $n-1$ such that $G - v$ is regular. By the techniques in [3], we have the following bounds. For completeness, we include a proof here.

Theorem 3.1 *Let G be a connected graph and u and v be vertices such that $T_G(u) = T_{\min}(G)$ and $T_G(v) = T_{\max}(G)$. Let $m_1 = \max\{T_G(w) - (1-\alpha)d(u,w) : w \in V(G) \setminus \{u\}\}$, $m_2 = \min\{T_G(w) - (1-\alpha)d(v,w) : w \in V(G) \setminus \{v\}\}$, and $e(w) = \max\{d(w,z) : z \in V(G)\}$ for*

$w \in V(G)$. Then

$$\begin{aligned}
&\frac{m_2 + \alpha T_{\max}(G) + \sqrt{(m_2 - \alpha T_{\max}(G))^2 + 4(1-\alpha)^2 T_{\max}(G)}}{2} \\
&\quad \le \mu_\alpha(G) \\
&\quad \le \frac{m_1 + \alpha T_{\min}(G) + \sqrt{(m_1 - \alpha T_{\min}(G))^2 + 4(1-\alpha)^2 e(u) T_{\min}(G)}}{2}.
\end{aligned}$$

The first equality holds if and only if G is a complete graph and the second equality holds if and only if G is a DVDR graph.

Proof Let M be the submatrix of $D_\alpha(G)$ obtained by deleting the row and column corresponding to vertex v. Let M' be the matrix obtained from M by reducing some nondiagonal entries of each row with row sum greater than m_2 in M such that M' is nonnegative and each row sum in M' is m_2.

Let $D^{(1)}$ be the matrix obtained from $D_\alpha(G)$ by replacing all (w,v)-entries by $1-\alpha$ for $w \in V(G) \setminus \{v\}$, and replacing the submatrix M by M'. Obviously, $D_\alpha(G)$ and $D^{(1)}$ are nonnegative and irreducible, and $D_\alpha(G) \ge D^{(1)}$. By Lemma 2.2, we have $\mu_\alpha(G) \ge \rho(D^{(1)})$ with equality if and only if $D_\alpha(G) = D^{(1)}$. By applying Lemma 2.5 to $D^{(1)}$, we obtain the lower bound for $\mu_\alpha(G)$. Suppose that this lower bound is attained. Then $D_\alpha(G) = D^{(1)}$. As all (w,v)-entries are equal to $1-\alpha$ for $w \in V(G) \setminus \{v\}$, implying $\deg_G(v) = n-1$. As $T_G(v) = T_{\max}(G)$, G is a complete graph. Conversely, if G is a complete graph, then it is obvious that the lower bound for $\mu_\alpha(G)$ is attained.

Let C be the submatrix of $D_\alpha(G)$ obtained by deleting the row and column corresponding to vertex u. Let C' be the matrix obtained from C by adding positive numbers to nondiagonal entries of each row with row sum less than m_1 in C such that each row sum in C' is m_1. Let $D^{(2)}$ be the matrix obtained from $D_\alpha(G)$ by replacing all (w,u)-entries by $(1-\alpha)e(u)$ for $w \in V(G) \setminus \{u\}$, and replacing the submatrix C by C'. Note that $D_\alpha(G)$ and $D^{(2)}$ are nonnegative and irreducible, and $D^{(2)} \ge D_\alpha(G)$. By Lemma 2.2, $\mu_\alpha(G) \le \rho(D^{(2)})$ with equality if and only if $D_\alpha(G) = D^{(2)}$. By applying Lemma 2.5 to $D^{(2)}$, we obtain the upper bound for $\mu_\alpha(G)$.

Suppose that this upper bound is attained. By Lemma 2.2, $D_\alpha(G) = D^{(2)}$. As all (w,u)-entries are equal to $(1-\alpha)e(u)$ for $w \in V(G) \setminus \{u\}$, implying $e(u) = 1$, i.e., $\deg_G(u) = n-1$. Note that $T_G(w) = m_1 + 1 - \alpha$ for all $w \in V(G) \setminus \{u\}$ and $T_{\min}(G) = T_G(u) = n-1$. If $m_1 + 1 - \alpha = n-1$, then G is a complete graph, which is a DVDR graph. Otherwise, $m_1 + 1 - \alpha > n-1$.

Recall from [3] that an incomplete connected graph of order n is a DVDR graph if and only if except one vertex of degree $n-1$ each other vertex has the same transmission. Thus, the upper bound for $\mu_\alpha(G)$ is attained if and only if G is a DVDR graph. □

We mention that more bounds for $\mu_\alpha(G)$ may be derived even from some known bounds for nonnegative matrices, see, e.g., [9].

Let G be a connected graph of order n. Let $\Lambda = T_{\max}(G)$. As $\mu_\alpha(G) \le \Lambda$ with equality if and only if G is transmission regular. For a connected non-transmission-regular graph G of order n, Liu et al. [19] showed that

$$\mu_0(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(n\Lambda - 2\sigma(G) + 1)n}$$

and

$$\mu_{1/2}(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(2(n\Lambda - 2\sigma(G)) + 1)n}.$$

Note that $4\sigma(G) < n^2\Lambda$. We show new bounds as follows:

$$\mu_0(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(n\Lambda - 2\sigma(G))\frac{4\sigma(G)}{n\Lambda} + n}$$

and

$$\mu_{1/2}(G) < \Lambda - \frac{n\Lambda - 2\sigma(G)}{(n\Lambda - 2\sigma(G))\frac{8\sigma(G)}{n\Lambda} + n}.$$

Instead of proving the two inequalities, we prove the following somewhat general result.

Theorem 3.2 *Let G be a connected non-transmission-regular graph of order n. Then*

$$\mu_\alpha(G) < \Lambda - \frac{(1-\alpha)n\Lambda(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1-\alpha)n^2\Lambda},$$

where $\Lambda = T_{\max}(G)$.

Proof Let x be the α-Perron vector of G. Denote by $x_u = \max\{x_w : w \in V(G)\}$ and $x_v = \min\{x_w : w \in V(G)\}$. Since G is not transmission regular, we have $x_u > x_v$, and thus

$$\begin{aligned}\mu_\alpha(G) &= x^\top D_\alpha(G)x \\ &= \alpha \sum_{w\in V(G)} T_G(w)x_w^2 + 2(1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}x_wx_z \\ &< 2\alpha\sigma(G)x_u^2 + 2(1-\alpha)\sigma(G)x_u^2,\end{aligned}$$

implying that $x_u^2 > \frac{\mu_\alpha(G)}{2\sigma(G)}$. Note that

$$\begin{aligned}&\Lambda - \mu_\alpha(G) \\ &\quad = \Lambda - \alpha\sum_{w\in V(G)} T_G(w)x_w^2 - 2(1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}x_wx_z \\ &\quad = \sum_{w\in V(G)} \bigl(\Lambda - T_G(w)\bigr)x_w^2 + (1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 \\ &\quad \ge \sum_{w\in V(G)} \bigl(\Lambda - T_G(w)\bigr)x_v^2 + (1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 \\ &\quad = \bigl(n\Lambda - 2\sigma(G)\bigr)x_v^2 + (1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2.\end{aligned}$$

We need to estimate $\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2$. Let $P = w_0w_1\dots w_\ell$ be a shortest path connecting u and v, where $w_0 = u$, $w_\ell = v$, and $\ell \ge 1$. Obviously,

$$\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 \ge N_1 + N_2,$$

where $N_1 = \sum_{w\in V(G)\setminus V(P)} \sum_{z\in V(P)} d_{wz}(x_w - x_z)^2$ and $N_2 = \sum_{\{w,z\}\subseteq V(P)} d_{wz}(x_w - x_z)^2$. For $w \in V(G) \setminus V(P)$, by the Cauchy–Schwarz inequality, we have

$$d_{wu}(x_w - x_u)^2 + d_{wv}(x_w - x_v)^2 \geq (x_w - x_u)^2 + (x_w - x_v)^2 \geq \frac{1}{2}(x_u - x_v)^2,$$

and thus

$$\begin{aligned} N_1 &\geq \sum_{w\in V(G)\setminus V(P)} \bigl(d_{wu}(x_w - x_u)^2 + d_{wv}(x_w - x_v)^2\bigr) \\ &\geq \sum_{w\in V(G)\setminus V(P)} \frac{1}{2}(x_u - x_v)^2 \\ &= \frac{n-\ell-1}{2}(x_u - x_v)^2. \end{aligned}$$

For $1 \leq i \leq \ell - 1$ and $\ell \geq 2$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} &d_{w_0w_i}(x_{w_0} - x_{w_i})^2 + d_{w_iw_\ell}(x_{w_i} - x_{w_\ell})^2 \\ &\quad\geq \min\{i, \ell - i\}\bigl((x_{w_0} - x_{w_i})^2 + (x_{w_i} - x_{w_\ell})^2\bigr) \\ &\quad\geq \min\{i, \ell - i\} \cdot \frac{1}{2}(x_{w_0} - x_{w_\ell})^2 \\ &\quad= \frac{1}{2}\min\{i, \ell - i\}(x_u - x_v)^2, \end{aligned}$$

and thus

$$\begin{aligned} N_2 &\geq d_{uv}(x_u - x_v)^2 + \sum_{i=1}^{\ell-1}\bigl(d_{w_0w_i}(x_{w_i} - x_{w_0})^2 + d_{w_iw_\ell}(x_{w_i} - x_{w_\ell})^2\bigr) \\ &\geq \ell(x_u - x_v)^2 + \sum_{i=1}^{\ell-1}\frac{1}{2}\min\{i, \ell - i\}(x_u - x_v)^2 \\ &= \left(\ell + \frac{1}{2}\sum_{i=1}^{\ell-1}\min\{i, \ell - i\}\right)(x_u - x_v)^2 \\ &= \begin{cases} \frac{\ell^2+8\ell}{8}(x_u - x_v)^2 & \text{if } \ell \text{ is even,} \\ \frac{\ell^2+8\ell-1}{8}(x_u - x_v)^2 & \text{if } \ell \text{ is odd.} \end{cases} \end{aligned}$$

Case 1. u and v are adjacent, i.e., $\ell = 1$.

In this case, we have

$$\begin{aligned} \sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 &\geq N_1 + N_2 \\ &\geq \frac{n-1-1}{2}(x_u - x_v)^2 + (x_u - x_v)^2 \\ &= \frac{n}{2}(x_u - x_v)^2. \end{aligned}$$

Thus

$$\begin{aligned}\Lambda - \mu_\alpha(G) &\geq \bigl(n\Lambda - 2\sigma(G)\bigr)x_v^2 + (1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 \\ &\geq \bigl(n\Lambda - 2\sigma(G)\bigr)x_v^2 + (1-\alpha)\frac{n}{2}(x_u - x_v)^2.\end{aligned}$$

Viewed as a function of x_v, $(n\Lambda - 2\sigma(G))x_v^2 + (1-\alpha)\frac{n}{2}(x_u - x_v)^2$ achieves its minimum value $\frac{(1-\alpha)n(n\Lambda-2\sigma(G))}{2(n\Lambda-2\sigma(G))+(1-\alpha)n}x_u^2$. Recall that $x_u^2 > \frac{\mu_\alpha(G)}{2\sigma(G)}$. Then we have

$$\begin{aligned}\Lambda - \mu_\alpha(G) &> \frac{(1-\alpha)n(n\Lambda - 2\sigma(G))}{2(n\Lambda - 2\sigma(G)) + (1-\alpha)n}\cdot\frac{\mu_\alpha(G)}{2\sigma(G)} \\ &= \frac{(1-\alpha)n(n\Lambda - 2\sigma(G))\Lambda}{2\sigma(G)(2(n\Lambda - 2\sigma(G)) + (1-\alpha)n)} \\ &\quad - \frac{(1-\alpha)n(n\Lambda - 2\sigma(G))(\Lambda - \mu_\alpha(G))}{2\sigma(G)(2(n\Lambda - 2\sigma(G)) + (1-\alpha)n)},\end{aligned}$$

which implies that

$$\Lambda - \mu_\alpha(G) > \frac{(1-\alpha)n\Lambda(n\Lambda - 2\sigma(G))}{4\sigma(G)(n\Lambda - 2\sigma(G)) + (1-\alpha)n^2\Lambda}.$$

Case 2. u and v are not adjacent, i.e., $\ell \geq 2$.

Suppose first that ℓ is even. Then

$$\begin{aligned}\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 &\geq N_1 + N_2 \\ &\geq \frac{n-\ell-1}{2}(x_u - x_v)^2 + \frac{\ell^2 + 8\ell}{8}(x_u - x_v)^2 \\ &= \frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2.\end{aligned}$$

Thus

$$\begin{aligned}\Lambda - \mu_\alpha(G) &\geq \bigl(n\Lambda - 2\sigma(G)\bigr)x_v^2 + (1-\alpha)\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 \\ &\geq \bigl(n\Lambda - 2\sigma(G)\bigr)x_v^2 + (1-\alpha)\frac{\ell^2 + 4\ell + 4n - 4}{8}(x_u - x_v)^2.\end{aligned}$$

Viewed as a function of x_v, $(n\Lambda - 2\sigma(G))x_v^2 + (1-\alpha)\frac{\ell^2+4\ell+4n-4}{8}(x_u - x_v)^2$ achieves its minimum value $\frac{(1-\alpha)(n\Lambda-2\sigma(G))(\ell^2+4\ell+4n-4)}{8(n\Lambda-2\sigma(G))+(1-\alpha)(\ell^2+4\ell+4n-4)}x_u^2$. As $x_u^2 > \frac{\mu_\alpha(G)}{2\sigma(G)}$, we have

$$\Lambda - \mu_\alpha(G) > \frac{(1-\alpha)(n\Lambda - 2\sigma(G))(\ell^2 + 4\ell + 4n - 4)}{(1-\alpha)(\ell^2 + 4\ell + 4n - 4) + 8(n\Lambda - 2\sigma(G))}\cdot\frac{\mu_\alpha(G)}{2\sigma(G)},$$

i.e.,

$$\Lambda - \mu_\alpha(G) > \frac{(1-\alpha)(n\Lambda - 2\sigma(G))(\ell^2 + 4\ell + 4n - 4)\Lambda}{16\sigma(G)(n\Lambda - 2\sigma(G)) + (1-\alpha)(\ell^2 + 4\ell + 4n - 4)n\Lambda}.$$

As a function of ℓ, the expression on the right-hand side in the above inequality is strictly increasing for $\ell \geq 2$. Thus we have

$$\begin{aligned} \varLambda - \mu_\alpha(G) &> \frac{(1-\alpha)(n\varLambda - 2\sigma(G))(n+2)\varLambda}{4\sigma(G)(n\varLambda - 2\sigma(G)) + (1-\alpha)(n+2)n\varLambda} \\ &> \frac{(1-\alpha)n\varLambda(n\varLambda - 2\sigma(G))}{4\sigma(G)(n\varLambda - 2\sigma(G)) + (1-\alpha)n^2\varLambda}. \end{aligned}$$

Now suppose that ℓ is odd. Then

$$\begin{aligned} &\sum_{\{w,z\}\subseteq V(G)} d_{wz}(x_w - x_z)^2 \\ &\geq N_1 + N_2 \\ &\geq \frac{n-\ell-1}{2}(x_u - x_v)^2 + \frac{\ell^2 + 8\ell - 1}{8}(x_u - x_v)^2 \\ &= \frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2. \end{aligned}$$

Thus, as early, we have

$$\begin{aligned} &\varLambda - \mu_\alpha(G) \\ &\geq \bigl(n\varLambda - 2\sigma(G)\bigr)x_v^2 + (1-\alpha)\frac{\ell^2 + 4\ell + 4n - 5}{8}(x_u - x_v)^2 \\ &\geq \frac{(1-\alpha)(\ell^2 + 4\ell + 4n - 5)(n\varLambda - 2\sigma(G))}{8(n\varLambda - 2\sigma(G)) + (1-\alpha)(\ell^2 + 4\ell + 4n - 5)}x_u^2 \\ &> \frac{(1-\alpha)(\ell^2 + 4\ell + 4n - 5)(n\varLambda - 2\sigma(G))}{8(n\varLambda - 2\sigma(G)) + (1-\alpha)(\ell^2 + 4\ell + 4n - 5)} \cdot \frac{\mu_\alpha(G)}{2\sigma(G)}, \end{aligned}$$

implying

$$\varLambda - \mu_\alpha(G) > \frac{(1-\alpha)(n\varLambda - 2\sigma(G))(\ell^2 + 4\ell + 4n - 5)\varLambda}{16\sigma(G)(n\varLambda - 2\sigma(G)) + (1-\alpha)(\ell^2 + 4\ell + 4n - 5)n\varLambda}.$$

As a function of ℓ, the expression on the right-hand side in the above inequality is strictly increasing for $\ell \geq 3$. Thus we have

$$\begin{aligned} \varLambda - \mu_\alpha(G) &> \frac{(1-\alpha)(n\varLambda - 2\sigma(G))(4+n)\varLambda}{4\sigma(G)(n\varLambda - 2\sigma(G)) + (1-\alpha)(4+n)n\varLambda} \\ &> \frac{(1-\alpha)n\varLambda(n\varLambda - 2\sigma(G))}{4\sigma(G)(n\varLambda - 2\sigma(G)) + (1-\alpha)n^2\varLambda}. \end{aligned}$$

The result follows by combining Cases 1 and 2. □

4 Effect of graft transformations on distance α-spectral radius

In this section, we study the effect of some local graft transformations on distance α-spectral radius.

A path $u_0 \cdots u_r$ (with $r \geq 1$) in a graph G is called a pendant path (of length r) at u_0 if $\deg_G(u_0) \geq 3$, the degrees of $u_1, \dots, u_{r-1}$ (if any exists) are all equal to 2 in G, and $\deg_G(u_r) = 1$. A pendant path of length 1 at u_0 is called a pendant edge at u_0.

A vertex of a graph is a pendant vertex if its degree is 1. A cut edge of a connected graph is an edge whose removal yields a disconnected graph.

If P is a pendant path of G at u with length $r \geq 1$, then we say G is obtained from H by attaching a pendant path P of length r at u with $H = G[V(G) \setminus (V(P) \setminus \{u\})]$. If the pendant path of length 1 is attached to a vertex u of H, then we also say that a pendant vertex is attached to u.

Theorem 4.1 *Suppose that G is a connected graph, uv is a cut edge with $\deg_G(u) \geq 2$, and v is adjacent to a pendant vertex v'. Let*

$$G_{uv} = G - \{uw : w \in N_G(u) \setminus \{v\}\} + \{vw : w \in N_G(u) \setminus \{v\}\}.$$

Then $\mu_\alpha(G) > \mu_\alpha(G_{uv})$.

Proof Let G_1 and G_2 be the components of $G - uv$ containing u and v, respectively. Let x be the distance α-Perron vector of G_{uv}. By Lemma 2.1, $x_u = x_{v'}$. As we pass from G to G_{uv}, the distance between a vertex in $V(G_1) \setminus \{u\}$ and a vertex in $V(G_2)$ is decreased by 1, the distance between a vertex $V(G_1) \setminus \{u\}$ and u is increased by 1, and the distances between all other vertex pairs remain unchanged. Thus

$$\begin{aligned}
&\mu_\alpha(G) - \mu_\alpha(G_{uv}) \\
&\quad \geq x^\top \big(D_\alpha(G) - D_\alpha(G_{uv})\big)x \\
&\quad = \sum_{w \in V(G_1)\setminus\{u\}} \sum_{z \in V(G_2)} \big(\alpha\big(x_w^2 + x_z^2\big) + 2(1-\alpha)x_w x_z\big) \\
&\qquad - \sum_{w \in V(G_1)\setminus\{u\}} \big(\alpha\big(x_w^2 + x_u^2\big) + 2(1-\alpha)x_w x_u\big) \\
&\quad \geq \sum_{w \in V(G_1)\setminus\{u\}} \big(\alpha\big(x_w^2 + x_v^2\big) + 2(1-\alpha)x_w x_v\big) \\
&\qquad + \sum_{w \in V(G_1)\setminus\{u\}} \big(\alpha\big(x_w^2 + x_{v'}^2\big) + 2(1-\alpha)x_w x_{v'}\big) \\
&\qquad - \sum_{w \in V(G_1)\setminus\{u\}} \big(\alpha\big(x_w^2 + x_u^2\big) + 2(1-\alpha)x_w x_u\big) \\
&\quad = \sum_{w \in V(G_1)\setminus\{u\}} \big(\alpha\big(x_w^2 + x_v^2\big) + 2(1-\alpha)x_w x_v\big) \\
&\quad > 0,
\end{aligned}$$

implying $\mu_\alpha(G) - \mu_\alpha(G_{uv}) > 0$, i.e., $\mu_\alpha(G) > \mu_\alpha(G_{uv})$. □

The previous theorem has been established for $\alpha = 0, \frac{1}{2}$ in [16, 25].

Theorem 4.2 *Suppose that G is a connected graph with k edge-disjoint nontrivial induced subgraphs $G_1, \dots, G_k$ such that $V(G_i) \cap V(G_j) = \{u\}$ for $1 \leq i < j \leq k$ and $\bigcup_{i=1}^k V(G_i) = V(G)$,* *where* $k \geq 3$. *Let* $\emptyset \neq K \subseteq \{3, \dots, k\}$ *and let* $N_K = \bigcup_{i \in K} N_{G_i}(u)$. *For* $v' \in V(G_1) \setminus \{u\}$ *and* $v'' \in V(G_2) \setminus \{u\}$, *let*

$$G' = G - \{uw : w \in N_K\} + \{v'w : w \in N_K\}$$

and

$$G'' = G - \{uw : w \in N_K\} + \{v''w : w \in N_K\}.$$

Then $\mu_\alpha(G) < \max\{\mu_\alpha(G'), \mu_\alpha(G'')\}$.

Proof Let x be the distance α-Perron vector of G. Let $V_K = (\bigcup_{i \in K} V(G_i)) \setminus \{u\}$. Let

$$\begin{aligned} \Gamma = & \sum_{w \in V(G_2) \setminus \{u\}} \sum_{z \in V_K} \left(\alpha\left(x_w^2 + x_z^2\right) + 2(1-\alpha)x_w x_z\right) \\ & - \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} \left(\alpha\left(x_w^2 + x_z^2\right) + 2(1-\alpha)x_w x_z\right). \end{aligned}$$

As we pass from G to G', the distance between a vertex in $V(G_2)$ and a vertex in V_K is increased by $d_G(u, v')$, the distance between a vertex w in $V(G_1) \setminus \{u\}$ and a vertex in V_K is decreased by $d_G(w, u) - d_G(w, v')$, which is at most $d_G(u, v')$, and the distances between all other vertex pairs are increased or remain unchanged. Thus

$$\begin{aligned} & \mu_\alpha(G') - \mu_\alpha(G) \\ & \quad \geq x^\top \left(D_\alpha(G') - D_\alpha(G)\right)x \\ & \quad \geq \sum_{w \in V(G_2)} \sum_{z \in V_K} \left(d_G(u, v')\left(\alpha\left(x_w^2 + x_z^2\right) + 2(1-\alpha)x_w x_z\right)\right) \\ & \qquad - \sum_{w \in V(G_1) \setminus \{u\}} \sum_{z \in V_K} \left(d_G(u, v')\left(\alpha\left(x_w^2 + x_z^2\right) + 2(1-\alpha)x_w x_z\right)\right) \\ & \quad = d_G(u, v')\left(\Gamma + \sum_{z \in V_K} \left(\alpha\left(x_u^2 + x_z^2\right) + 2(1-\alpha)x_u x_z\right)\right) \\ & \quad > d_G(u, v')\Gamma. \end{aligned}$$

If $\Gamma \geq 0$, then $\mu_\alpha(G') - \mu_\alpha(G) > d_G(u, v')\Gamma \geq 0$, implying $\mu_\alpha(G) < \mu_\alpha(G')$. Suppose that $\Gamma < 0$. As we pass from G to G'', the distance between a vertex in $V(G_1)$ and a vertex in V_K is increased by $d_G(u, v'')$, the distance between a vertex w in $V(G_2) \setminus \{u\}$ and a vertex in V_K is decreased by $d_G(w, u) - d_G(w, v'')$, which is at most $d_G(u, v'')$, and the distances between all other vertex pairs are increased or remain unchanged. Thus

$$\begin{aligned} & \mu_\alpha(G'') - \mu_\alpha(G) \\ & \quad \geq x^\top \left(D_\alpha(G'') - D_\alpha(G)\right)x \\ & \quad \geq \sum_{w \in V(G_1)} \sum_{z \in V_K} \left(d_G(u, v'')\left(\alpha\left(x_w^2 + x_z^2\right) + 2(1-\alpha)x_w x_z\right)\right) \end{aligned}$$

$$\begin{aligned}
&- \sum_{w\in V(G_2)\setminus\{u\}} \sum_{z\in V_K} \left(d_G\left(u,v''\right)\left(\alpha\left(x_w^2+x_z^2\right)+2(1-\alpha)x_w x_z\right)\right)\\
&= d_G\left(u,v''\right)\left(-\Gamma + \sum_{z\in V_K}\left(\alpha\left(x_u^2+x_z^2\right)+2(1-\alpha)x_u x_z\right)\right)\\
&> d_G\left(u,v''\right)(-\Gamma)\\
&> 0,
\end{aligned}$$

implying $\mu_\alpha(G'')-\mu_\alpha(G)>0$, i.e., $\mu_\alpha(G)<\mu_\alpha(G'')$. □

Weak versions of previous theorem for $\alpha = 0$ have been given in [28, 30] and a weak version for $\alpha = \frac{1}{2}$ may be found in [16].

For positive integer p and a graph G with $u \in V(G)$, let $G(u;p)$ be the graph obtained from G by attaching a pendant path of length p at u. Let $G(u;0) = G$, and in this case a pendant path of length 0 is understood the trivial path consisting of a single vertex u.

For nonnegative integers p, q and a graph G, let $G_u(p,q)$ be the graph $H(u;q)$ with $H = G(u;p)$. The following corollary has been known for $\alpha = 0$ in [24, 28] and $\alpha = \frac{1}{2}$ in [15, 16].

Corollary 4.1 *Let H be a nontrivial connected graph with $u \in V(H)$. If $p \ge q \ge 1$, then $\mu_\alpha(H_u(p,q)) < \mu_\alpha(H_u(p+1,q-1))$.*

Proof Let $G = H_u(p,q)$. Let $P = uu_1\cdots u_p$ and $Q = uv_1\cdots v_q$ be two pendant paths of lengths p and q, respectively, in G. Using the notations in Theorem 4.2 with $k = 3$, $G_1 = P$, $G_2 = Q$, $G_3 = H$, $v' = u_{p-q+1}$ and $v'' = v_1$, we have $G' \cong G'' \cong H_u(p+1,q-1)$, and thus by Theorem 4.2, we have $\mu_\alpha(H_u(p,q)) < \mu_\alpha(H_u(p+1,q-1))$. □

Theorem 4.3 *Suppose that G is a connected graph with three edge-disjoint induced subgraphs G_1, G_2 and G_3 such that $V(G_1)\cap V(G_3) = \{u\}$, $V(G_2)\cap V(G_3) = \{v\}$, $\bigcup_{i=1}^3 V(G_i) = V(G)$, and $G_1 - u$, $G_2 - v$, and $G_3 - u - v$ are all nontrivial. Suppose that $uv \in E(G_3)$. For $u' \in N_{G_1}(u)$ and $v' \in N_{G_2}(v)$, let*

$$G' = H + \left\{u'w : w \in N_{G_3-uv}(u)\right\} + \left\{uw : w \in N_{G_3-uv}(v)\right\}$$

and

$$G'' = H + \left\{vw : w \in N_{G_3-uv}(u)\right\} + \left\{v'w : w \in N_{G_3-uv}(v)\right\},$$

where $H = G - \{uw : w \in N_{G_3-uv}(u)\} - \{vw : w \in N_{G_3-uv}(v)\}$. Then $\mu_\alpha(G) < \mu_\alpha(G')$ or $\mu_\alpha(G) < \mu_\alpha(G'')$.

Proof Let x be the distance α-Perron vector of G. Let

$$\begin{aligned}
\Gamma = &\sum_{w\in V(G_2)} \sum_{z\in V(G_3)\setminus\{u,v\}} \left(\alpha\left(x_w^2+x_z^2\right)+2(1-\alpha)x_w x_z\right)\\
&- \sum_{w\in V(G_1)} \sum_{z\in V(G_3)\setminus\{u,v\}} \left(\alpha\left(x_w^2+x_z^2\right)+2(1-\alpha)x_w x_z\right).
\end{aligned}$$

As we pass from G to G', the distance between a vertex in $V(G_2)$ and a vertex in $V(G_3)\setminus\{u,v\}$ is increased by 1, the distance between a vertex in $V(G_1)$ and a vertex in $V(G_3)\setminus\{u,v\}$ may be increased, unchanged, or decreased by 1, and the distances between any other vertex pairs remain unchanged. Thus

$$\begin{aligned}\mu_\alpha(G') - \mu_\alpha(G) &\geq x^\top\big(D_\alpha(G') - D_\alpha(G)\big)x\\ &\geq \sum_{w\in V(G_2)}\sum_{z\in V(G_3)\setminus\{u,v\}}\big(\alpha\big(x_w^2+x_z^2\big)+2(1-\alpha)x_wx_z\big)\\ &\quad - \sum_{w\in V(G_1)}\sum_{z\in V(G_3)\setminus\{u,v\}}\big(\alpha\big(x_w^2+x_z^2\big)+2(1-\alpha)x_wx_z\big)\\ &= \Gamma.\end{aligned}$$

If $\Gamma \geq 0$, then $\mu_\alpha(G') - \mu_\alpha(G) \geq 0$, i.e., $\mu_\alpha(G) \leq \mu_\alpha(G')$. If $\mu_\alpha(G) = \mu_\alpha(G')$, then $\mu_\alpha(G') = x^\top D_\alpha(G')x$, implying x is the distance α-Perron vector of G'. By the α-equations of G and G' at v, we have

$$\begin{aligned}0 &= \mu_\alpha(G')x_v - \mu_\alpha(G)x_v\\ &= \sum_{w\in V(G_3)\setminus\{u,v\}}\big(d_{G'}(v,w)-d_G(v,w)\big)\big(\alpha x_v + (1-\alpha)x_w\big)\\ &= \sum_{w\in V(G_3)\setminus\{u,v\}}\big(\alpha x_v + (1-\alpha)x_w\big)\\ &> 0,\end{aligned}$$

a contradiction. Thus, if $\Gamma \geq 0$, then $\mu_\alpha(G) < \mu_\alpha(G')$.

Suppose that $\Gamma < 0$. As earlier, we have

$$\begin{aligned}\mu_\alpha(G'') - \mu_\alpha(G) &\geq x^\top\big(D_\alpha(G'') - D_\alpha(G)\big)x\\ &\geq \sum_{w\in V(G_1)}\sum_{z\in V(G_3)\setminus\{u,v\}}\big(\alpha\big(x_w^2+x_z^2\big)+2(1-\alpha)x_wx_z\big)\\ &\quad - \sum_{w\in V(G_2)}\sum_{z\in V(G_3)\setminus\{u,v\}}\big(\alpha\big(x_w^2+x_z^2\big)+2(1-\alpha)x_wx_z\big)\\ &= -\Gamma\\ &> 0,\end{aligned}$$

and thus $\mu_\alpha(G) < \mu_\alpha(G'')$. □

A weak version of previous theorem for $\alpha = \frac{1}{2}$ has been established in [16].

For nonnegative integers p, q and a graph G with $u, v \in V(G)$, let $G_{u,v}(p,q)$ be the graph $H(v;q)$ with $H = G(u;p)$. The following corollary has been known for $\alpha = 0, \frac{1}{2}$ in [15, 32].

Corollary 4.2 *Let H be a connected graph of order at least 3 with $uv \in E(H)$. Suppose that $\eta(u) = v$ for some automorphism η of G. For $p \geq q \geq 1$, we have $\mu_\alpha(H_{u,v}(p,q)) < \mu_\alpha(H_{u,v}(p+1, q-1))$.*

Proof Let $G = H_{u,v}(p,q)$. Let $P = uu_1 \cdots u_p$ and $Q = vv_1 \cdots v_q$ be two pendant paths of lengths p and q in G at u and v, respectively. Using the notations of Theorem 4.3 with $G_1 = P$, $G_2 = Q$, $G_3 = H$, $u' = u_1$ and $v' = v_1$, we have $G' \cong H_{u,v}(p-1,q+1)$ and $G'' \cong H_{u,v}(p+1,q-1)$, and thus by Theorem 4.3, we have $\mu_\alpha(H_{u,v}(p,q)) < \max\{\mu_\alpha(H_{u,v}(p-1,q+1)), \mu_\alpha(H_{u,v}(p+1,q-1))\}$. If $p = q$ ($p = q+1$, respectively), then $H_{u,v}(p-1,q+1) \cong H_{u,v}(p+1,q-1)$ ($H_{u,v}(p,q) \cong H_{u,v}(p-1,q+1)$, respectively) as $\eta(u) = v$ for some automorphism η of G, and thus from the above inequality, we have $\mu_\alpha(G) < \mu_\alpha(H_{u,v}(p+1,q-1))$. Suppose that $p \geq q+2$ and $\mu_\alpha(G) < \mu_\alpha(H_{u,v}(p-1,q+1))$. If $p \not\equiv q \pmod 2$, then we have

$$\begin{aligned}\mu_\alpha(G) &\leq \mu_\alpha\left(H_{u,v}\left(\frac{p+q+3}{2}, \frac{p+q-3}{2}\right)\right)\\ &< \mu_\alpha\left(H_{u,v}\left(\frac{p+q+1}{2}, \frac{p+q-1}{2}\right)\right)\\ &< \mu_\alpha\left(H_{u,v}\left(\frac{p+q+3}{2}, \frac{p+q-3}{2}\right)\right),\end{aligned}$$

which is impossible. If $p \equiv q \pmod 2$, then we have

$$\begin{aligned}\mu_\alpha(G) &\leq \mu_\alpha\left(H_{u,v}\left(\frac{p+q}{2}+1, \frac{p+q}{2}-1\right)\right)\\ &< \mu_\alpha\left(H_{u,v}\left(\frac{p+q}{2}, \frac{p+q}{2}\right)\right)\\ &< \mu_\alpha\left(H_{u,v}\left(\frac{p+q}{2}-1, \frac{p+q}{2}+1\right)\right),\end{aligned}$$

which is also impossible. Therefore $\mu_\alpha(H_{u,v}(p,q)) < \mu_\alpha(H_{u,v}(p+1,q-1))$. □

5 Graphs with small or large distance α-spectral radius

First we determine the graphs with minimum distance α-spectral radius among trees and unicyclic graphs.

Theorem 5.1 *Let G be a tree of order n. Then $\mu_\alpha(G) \geq \mu_\alpha(S_n)$ with equality if and only if $G \cong S_n$.*

Proof The result is trivial if $n = 1,2,3$. Suppose that $n \geq 4$. Let G be a tree of order n such that $\mu_\alpha(G)$ is as small as possible. Let d be the diameter of G. Evidently, $d \geq 2$. Suppose that $d \geq 3$. Let $v_0v_1 \cdots v_d$ be a diametral path of G. By Theorem 4.1, $\mu_\alpha(G_{v_1v_2}) < \mu_\alpha(G)$, a contradiction. Thus $d = 2$, i.e., $G \cong S_n$. □

In Theorem 5.1, the case $\alpha = 0$ has been known in [24] and the case $\alpha = \frac{1}{2}$ has been known in [16, 29].

For $n-1 \geq 3$ and $1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor$, let $D_{n,a}$ be the tree obtained from vertex-disjoint S_{a+1} with center u and S_{n-a-1} with center v by adding an edge uv. Let T be a tree of order n with minimum distance α-spectral radius, where $T \ncong S_n$. Let d be the diameter of T. Then $d \geq 3$. Suppose that $d \geq 4$. Let $v_0v_1 \cdots v_d$ be a diametral path of T. Note that $T_{v_1v_2} \ncong S_n$. By Theorem 4.1, $\mu_\alpha(T_{v_1v_2}) < \mu_\alpha(T)$, a contradiction. Thus $d = 3$, implying $T \cong D_{n,a}$ for some a with $1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor$.

Let S_n^+ is the graph obtained from S_n by adding an edge between two vertices of degree one.

Lemma 5.1 ([29]) *Let G be a unicyclic graph of order $n \geq 6$. If $G \ncong S_n^+$, then*

$$\sigma(G) \geq n^2 - n - 4 > \sigma\left(S_n^+\right) = n^2 - 2n.$$

Note that for $n = 5$, we have $\sigma(C_n) = \sigma(S_n^+)$. So, in the above lemma, the condition $n \geq 6$ is necessary.

Theorem 5.2 *Let G be a unicyclic graph of order $n \geq 8$. Then $\mu_\alpha(G) \geq \mu_\alpha(S_n^+)$ with equality if and only if $G \cong S_n^+$.*

Proof Suppose that $G \ncong S_n^+$. We only need to show that $\mu_\alpha(G) > \mu_\alpha(S_n^+)$.

By Lemmas 2.4 and 5.1, we have

$$\mu_\alpha(G) \geq \frac{2\sigma(G)}{n} \geq \frac{2(n^2 - n - 4)}{n}.$$

By [20, p. 24, Theorem 1.1] or by Theorem 3.2, we have

$$\mu_\alpha\left(S_n^+\right) < T_{\max}\left(S_n^+\right) = 2n - 3.$$

Since $n \geq 8$, we have

$$\mu_\alpha(G) \geq \frac{2(n^2 - n - 4)}{n} \geq 2n - 3 > \mu_\alpha\left(S_n^+\right),$$

as desired. □

The result in Theorem 5.2 for $\alpha = 0, \frac{1}{2}$ has been known in [29, 31].

In the following, we determine the graphs with maximum distance α-spectral radius among some classes of graphs.

For $2 \leq \Delta \leq n - 1$, let $B_{n,\Delta}$ be a tree obtained by attaching $\Delta - 1$ pendant vertices to a terminal vertex of the path $P_{n-\Delta+1}$. In particular, $B_{n,2} = P_n$ and $B_{n,n-1} = S_n$. The following theorem for $\alpha = 0, \frac{1}{2}$ was given in [16, 24] for trees.

Theorem 5.3 *Let G be a connected graph of order n with maximum degree Δ, where $2 \leq \Delta \leq n - 1$. Then $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta})$ with equality if and only if $G \cong B_{n,\Delta}$.*

Proof Let G be a graph among connected graphs of order n with maximum degree Δ such that $\mu_\alpha(G)$ is as large as possible. Then G has a spanning tree T with maximum degree Δ. By Lemma 2.3, $\mu_\alpha(G) \leq \mu_\alpha(T)$ with equality if and only if $G \cong T$. Thus G is a tree.

The result is trivial if $n = 3, 4$ and if $\Delta = 2, n - 1$. Suppose that $3 \leq \Delta \leq n - 2$. We only need to show that $G \cong B_{n,\Delta}$.

Let $u \in V(G)$ with $\deg_G(u) = \Delta$. Suppose that there exists a vertex different from u with degree at least 3. Then we may choose such a vertex w of degree at least 3 such that $d_G(u, w)$ is as large as possible. Obviously, there are two pendant paths, say P and Q, at w of lengths at least 1. Let p and q be the lengths of P and Q, respectively. Assume that $p \geq q$. Let

$H = G[V(G) \setminus ((V(P) \cup V(Q)) \setminus \{w\})]$. Then $G \cong H_w(p,q)$. Note that $G' = H_w(p+1, q-1)$ is a tree of order n with maximum degree Δ. By Corollary 4.1, $\mu_\alpha(G) < \mu_\alpha(G')$, a contradiction. Then u is the unique vertex of G with degree at least 3, and thus G consists of Δ pendant paths, say $Q_1, \dots, Q_\Delta$ at u. If two of them, say Q_i and Q_j with $i \neq j$ are of lengths at least 2, then $G \cong H'_u(r,s)$, where $H' = G[V(G) \setminus ((V(Q_i) \cup V(Q_j)) \setminus \{u\})]$, and r and s are the lengths of Q_i and Q_j, respectively. Assume that $r \geq s$. Obviously, $G'' = H'_u(r+1, s-1)$ is a tree of order n with maximum degree Δ. By Corollary 4.1, $\mu_\alpha(G) < \mu_\alpha(G'')$, also a contradiction. Thus there is exactly one pendant path at u of length at least 2, implying $G \cong B_{n,\Delta}$. □

If G is a connected graph of order 1 or 2, then $G \cong P_n$. If G is a connected graph of order 3, then $G \cong P_3, K_3$, and by Lemma 2.3, $\mu_\alpha(K_3) < \mu_\alpha(P_3)$.

Ruzieh and Powers [23] showed that P_n is the unique connected graph of order n with maximum distance 0-spectral radius, and it was proved in [25] that $B_{n,3}$ is the unique tree of order n different from P_n with maximum distance 0-spectral radius. For $\alpha = \frac{1}{2}$, the following theorem was given in [16].

Theorem 5.4 *Let G be a connected graph of order $n \geq 4$, where $G \ncong P_n$. Then $\mu_\alpha(G) \leq \mu_\alpha(B_{n,3}) < \mu_\alpha(P_n)$ with equality if and only if $G \cong B_{n,3}$.*

Proof First suppose that G is a tree. If $n = 4$, then the result follows from Theorem 4.1. Suppose that $n \geq 5$. Let Δ be the maximum degree of G. Since $G \ncong P_n$, we have $\Delta \geq 3$. By Theorem 5.3, $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta})$ with equality if and only if $G \cong B_{n,\Delta}$. By Corollary 4.1, $\mu_\alpha(G) \leq \mu_\alpha(B_{n,\Delta}) \leq \mu_\alpha(B_{n,3}) < \mu_\alpha(P_n)$ with equalities if and only if $\Delta = 3$ and $G \cong B_{n,\Delta}$, i.e., $G \cong B_{n,3}$.

Now suppose that G is not a tree. Then G contains at least one cycle. If there is a spanning tree T with $T \ncong P_n$, then by Lemma 2.3 and the above argument, we have $\mu_\alpha(G) < \mu_\alpha(T) \leq \mu_\alpha(B_{n,3})$. If any spanning tree of G is a path, then G is a cycle C_n. Now we only need to show that $\mu_\alpha(C_n) < \mu_\alpha(B_{n,3})$.

Let $C_n = u_1u_2\cdots u_nu_1$ and $T' = C_n - \{u_1u_2, u_2u_3\} + u_2u_n$. Then $T' \cong B_{n,3}$. Let x be the distance α-Perron vector of C_n. By Lemma 2.3, we have $x_{u_1} = \cdots = x_{u_n}$. As we pass from C_n to T', the distance between u_2 and u_1 is increased by 1, the distance between u_2 and u_i with $3 \leq i \leq \lceil \frac{n+1}{2} \rceil$ is increased by $n-2i+3$, the distance between u_2 and u_i with $\lfloor \frac{n+1}{2} \rfloor + 2 \leq i \leq n$ is decreased by 1, and the distances between all other vertex pairs are increased or remain unchanged. Thus

$$
\begin{aligned}
&\mu_\alpha(T') - \mu_\alpha(C_n) \\
&\quad = x^\top \left(D_\alpha(T') - D_\alpha(G)\right)x \\
&\quad \geq \alpha\left(x_{u_2}^2 + x_{u_1}^2\right) + 2(1-\alpha)x_{u_2}x_{u_1} \\
&\qquad - \sum_{i=\lfloor \frac{n+1}{2} \rfloor + 2}^{n} \left(\alpha\left(x_{u_2}^2 + x_{u_i}^2\right) + 2(1-\alpha)x_{u_2}x_{u_i}\right) \\
&\qquad + \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (n-2i+3)\left(\alpha\left(x_{u_2}^2 + x_{u_i}^2\right) + 2(1-\alpha)x_{u_2}x_{u_i}\right) \\
&\quad = 2x_{u_1}^2 \left(1 - \left(n - \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right) + \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (n-2i+3)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2x_{u_1}^2\left(1+\left(n-1-\left\lceil\frac{n+1}{2}\right\rceil\right)\left(\left\lceil\frac{n+1}{2}\right\rceil-2\right)\right)\\
&\geq 2x_{u_1}^2\\
&> 0,
\end{aligned}
$$

and therefore $\mu_\alpha(C_n) < \mu_\alpha(B_{n,3})$, as desired. □

A clique of G is a subset of vertices whose induced subgraph is a complete graph, and the clique number of G is the maximum number of vertices in a clique of G. For $2 \leq \omega \leq n$. Let $Ki_{n,\omega}$ be the graph obtained from a complete graph K_ω and a path $P_{n-\omega}$ by adding an edge between a vertex of K_ω and a terminal vertex of $P_{n-\omega}$ if $\omega < n$ and let $Ki_{n,\omega} = K_n$ if $\omega = n$. In particular, $Ki_{n,2} \cong P_n$ for $n \geq 2$. The following result for $\alpha = 0, \frac{1}{2}$ was given in [15, 21].

Theorem 5.5 *Let G be a connected graph of order $n \geq 2$ with clique number $\omega \geq 2$. Then $\mu_\alpha(G) \leq \mu_\alpha(Ki_{n,\omega})$ with equality if and only if $G \cong Ki_{n,\omega}$.*

Proof It is trivial if $\omega = n$ and it follows from Theorem 5.4 if $\omega = 2$.

Suppose that $3 \leq \omega \leq n-1$. Let G be a graph among connected graphs of order n with clique number ω such that $\mu_\alpha(G)$ is as large as possible. We only need to show that $G \cong Ki_{n,\omega}$.

Let $S = \{v_1, \dots, v_\omega\}$ be a clique of G. By Lemma 2.3, $G - E(G[S])$ is a forest. Let T_i be the component of $G - E(G[S])$ containing v_i, where $1 \leq i \leq \omega$. For $1 \leq i \leq \omega$, by Corollary 4.1, if T_i is nontrivial, then T_i is a pendant path at v_i. Note that any two distinct vertices in $G[S]$ are adjacent. By Corollary 4.2, there is only one nontrivial T_i, and thus $G \cong Ki_{n,\omega}$. □

Recall that $Ki_{n,3}$ is the unique unicyclic graph of order $n \geq 3$ with maximum distance 0-spectral radius [31], and the unique odd-cycle unicyclic graph of order $n \geq 3$ with maximum distance $\frac{1}{2}$-spectral radius [15].

Theorem 5.6 *Let G be a unicyclic odd-cycle graph of order $n \geq 3$. Then $\mu_\alpha(G) \leq \mu(Ki_{n,3})$ with equality if and only if $G \cong Ki_{n,3}$.*

Proof If $n = 3, 4$, the result is trivial. Suppose that $n \geq 5$. Let G be a graph with maximum distance α-spectral radius among unicyclic odd-cycle graphs of order n. We only need to show that $G \cong Ki_{n,3}$.

Let $C = v_1 \cdots v_{2k+1} v_1$ be the unique cycle of G, where $k \geq 1$. Let T_i be the component of $G - E(C)$ containing v_i for $1 \leq i \leq 2k+1$. Let $U_1 = V(T_{2k}) \cup V(T_{2k+1})$, $U_2 = \bigcup_{k+1 \leq i \leq 2k-1} V(T_i)$ and $U_3 = \bigcup_{1 \leq i \leq k-1} V(T_i)$. Let x be the distance α-Perron vector of G. Let

$$
\begin{aligned}
\Gamma = {} & \sum_{u \in U_1} \sum_{v \in U_3} \left(\alpha\left(x_u^2 + x_v^2\right) + 2(1-\alpha)x_u x_v\right)\\
& - \sum_{u \in U_1} \sum_{v \in U_2} \left(\alpha\left(x_u^2 + x_v^2\right) + 2(1-\alpha)x_u x_v\right).
\end{aligned}
$$

Suppose that $k \geq 2$. Let $G' = G - v_1 v_{2k+1} + v_{2k+1} v_{2k-1}$. Note that the length of C is odd. As we pass from G to G', the distance between a vertex in S_1 and a vertex in S_3 is increased by at least 1, the distance between S_2 and $V(T_{2k+1})$ is decreased by 1, and the distance between all other vertex pairs are increased or remain unchanged. Thus

$$\begin{aligned}\mu_\alpha(G') - \mu_\alpha(G) &\geq x^\top\left(D_\alpha(G') - D_\alpha(G)\right)x\\ &\geq \sum_{u\in U_1}\sum_{v\in U_3}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &\quad - \sum_{u\in V(T_{2k+1})}\sum_{v\in U_2}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &> \sum_{u\in U_1}\sum_{v\in U_3}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &\quad - \sum_{u\in U_1}\sum_{v\in U_2}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right).\end{aligned}$$

If $\Gamma \geq 0$, then $\mu_\alpha(G') > \mu_\alpha(G)$, a contradiction. Thus $\Gamma < 0$. Let $G'' = G - v_{2k}v_{2k-1} + v_{2k}v_1$. As we pass from G to G'', the distance between a vertex in S_1 and a vertex in U_2 is increased by at least 1, the distance between U_3 and $V(T_{2k})$ is decreased by 1, and the distance between all other vertex pairs are increased or remain unchanged. As above, we have

$$\begin{aligned}\mu_\alpha(G'') - \mu_\alpha(G) &\geq x^\top\left(D_\alpha(G'') - D_\alpha(G)\right)x\\ &\geq \sum_{u\in U_1}\sum_{v\in U_2}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &\quad - \sum_{u\in V(T_{2k})}\sum_{v\in U_3}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &> \sum_{u\in U_1}\sum_{v\in U_2}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &\quad - \sum_{u\in U_1}\sum_{v\in U_3}\left(\alpha\left(x_u^2+x_v^2\right)+2(1-\alpha)x_ux_v\right)\\ &> 0.\end{aligned}$$

Thus $\mu_\alpha(G'') > \mu_\alpha(G)$, also a contradiction. It follows that $k = 1$, i.e., the unique cycle of G is of length 3.

Obviously, T_i is a tree for $1 \leq i \leq 3$. For $1 \leq i \leq 3$, by Corollary 4.1, if T_i is nontrivial, then it is a path with a terminal vertex v_i. Then by Corollary 4.2, only one T_i is nontrivial. Thus $G \cong Ki_{n,3}$. □

Let G be a unicyclic graph of order $n \geq 4$ with maximum distance α-spectral radius. By Corollary 4.1, the maximum degree of G is 3 and all vertices of degree 3 lie on the unique cycle. Let u be a vertex of degree 3 and P be the pendant path at u. Let v and w be the two neighbors of u on the cycle, and z the neighbor of u on P. Let $G_1 = G - uw + vw$ and $G_2 = G - uw + wz$. Then $\mu_\alpha(G) < \max\{\mu_\alpha(G_1), \mu_\alpha(G_2)\}$ if the length of the cycle of G is odd, see [4, Lemma 6.11]. Note that the argument does not work when the length of the cycle of G is even. So we need other ways to determine the unicyclic graph(s) with maximum distance α-spectral radius even for $\alpha = \frac{1}{2}$.

6 Remarks

In this paper, we study the distance α-spectral radius of a connected graph. We consider bounds for the distance α-spectral radius, local transformations to change the distance α-spectral radius, and the characterizations for graphs with minimum and/or maximum distance α-spectral radius in some classes of connected graphs.

Besides the distance α-spectral radius, we may concern other eigenvalues of $D_\alpha(G)$ for a connected graph G. We give examples.

For an $n \times n$ Hermitian matrix C, let $\lambda_1(C), \dots, \lambda_n(C)$ be the eigenvalues of C, arranged in a nonincreasing order. Let A, B be $n \times n$ Hermitian matrices. Weyl's inequalities [13, p. 181] state that

$$\lambda_j(A+B) \le \lambda_i(A) + \lambda_{j-i+1}(B) \quad \text{for } 1 \le i \le j \le n,$$

and

$$\lambda_j(A+B) \ge \lambda_i(A) + \lambda_{j-i+n}(B) \quad \text{for } 1 \le j \le i \le n.$$

Using these inequalities, and as in the recent work of Atik and Panigrahi [3], we have

Theorem 6.1 *Let G be a connected graph and λ be any eigenvalue of $D_\alpha(G)$ other than the distance α-spectral radius. Then*

$$2\alpha T_{\min}(G) - T_{\max}(G) + (1-\alpha)(n-2) \le \lambda \le T_{\max}(G) - (1-\alpha)n.$$

Proof Let $D_\alpha(G) = A + B$, where $A = (\alpha T_{\min}(G) - (1-\alpha))I_n + (1-\alpha)J_{n\times n}$. Then B is a nonnegative symmetric matrix with maximum row sum $T_{\max}(G) - \alpha T_{\min}(G) - (1-\alpha)(n-1)$. Thus $|\lambda_n(B)| \le \lambda_1(B) \le T_{\max}(G) - \alpha T_{\min}(G) - (1-\alpha)(n-1)$.

For matrix A, we have $\lambda_1(A) = \alpha T_{\min}(G) + (1-\alpha)(n-1)$ and $\lambda_j(A) = \alpha T_{\min}(G) - 1 + \alpha$ for $j = 2, \dots, n$. Thus, for $j = 2, \dots, n$, we have by the above Weyl's inequalities that

$$\begin{aligned}
\lambda_j\bigl(D_\alpha(G)\bigr) &\le \lambda_1(B) + \lambda_j(A) \\
&\le T_{\max}(G) - \alpha T_{\min}(G) - (1-\alpha)(n-1) + \alpha T_{\min}(G) - 1 + \alpha \\
&= T_{\max}(G) - (1-\alpha)n
\end{aligned}$$

and

$$\begin{aligned}
\lambda_j\bigl(D_\alpha(G)\bigr) &\ge \lambda_n(B) + \lambda_j(A) \\
&\ge -T_{\max}(G) + \alpha T_{\min}(G) + (1-\alpha)(n-1) + \alpha T_{\min}(G) - 1 + \alpha \\
&= 2\alpha T_{\min}(G) - T_{\max}(G) + (1-\alpha)(n-2).
\end{aligned}$$

This completes the proof. □

Let G be a connected graph and λ be any eigenvalue of $D_\alpha(G)$ other than the distance α-spectral radius. By previous theorem, we have

$$|\lambda| \le T_{\max}(G) - (1-\alpha)(n-2).$$

The distance α-energy of a connected graph G of order n is defined as

$$\mathcal{E}_\alpha(G) = \sum_{i=1}^{n} \left| \mu_\alpha^{(i)}(G) - \frac{2\alpha\sigma(G)}{n} \right|.$$

Then $\mathcal{E}_0(G)$ is the distance energy of G [14, 33], while

$$\mathcal{E}_{1/2}(G) = \frac{1}{2} \sum_{i=1}^{n} \left| 2\mu_{1/2}^{(i)}(G) - \frac{2\sigma(G)}{n} \right|$$

is half of the distance signless Laplacian energy of G [8]. Thus, it is possible to study the distance energy and the distance signless Laplacian energy in a unified way.

Acknowledgements
We would like to thank the referees and the editor for their suggestions and comments.

Funding
This work was supported by the National Natural Science Foundation of China (No. 11671156).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions. They read and approved the final manuscript.

Publisher's Note

Received: 9 August 2019 Accepted: 1 June 2020 Published online: 11 June 2020

References
1. Aouchiche, M., Hansen, P.: Two Laplacians for the distance matrix of a graph. Linear Algebra Appl. **439**, 21–33 (2013)
2. Aouchiche, M., Hansen, P.: Distance spectra of graphs: a survey. Linear Algebra Appl. **458**, 301–386 (2014)
3. Atik, F., Panigrahi, P.: On the distance and distance signless Laplacian eigenvalues of graphs and the smallest Geršgorin disc. Electron. J. Linear Algebra **34**, 191–204 (2018)
4. Bapat, R.B., Kalita, D., Nath, M., Sarma, D.: Convex and quasiconvex functions on trees and their applications. Linear Algebra Appl. **533**, 210–234 (2017)
5. Cardoso, D.M., Díaz, R.C., Rojo, O.: Distance matrices on the *H*-join of graphs: a general result and applications. Linear Algebra Appl. **559**, 34–53 (2018)
6. Cui, S., He, J., Tian, G.: The generalized distance matrix. Linear Algebra Appl. **563**, 1–23 (2019)
7. Das, K.C.: Proof of conjectures on the distance signless Laplacian eigenvalues of graphs. Linear Algebra Appl. **467**, 100–115 (2015)
8. Das, K.C., Aouchiche, M., Hansen, P.: On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs. Discrete Appl. Math. **243**, 172–185 (2018)
9. Duan, X., Zhou, B.: Sharp bounds on the spectral radius of a nonnegative matrix. Linear Algebra Appl. **439**, 2961–2970 (2013)
10. Edelberg, M., Garey, M.R., Graham, R.L.: On the distance matrix of a tree. Discrete Math. **14**, 23–39 (1976)
11. Graham, R.L., Lovász, L.: Distance matrix polynomials of trees. Adv. Math. **29**, 60–88 (1978)
12. Graham, R.L., Pollack, H.O.: On the addressing problem for loop switching. Bell Syst. Tech. J. **50**, 2495–2519 (1971)
13. Horn, R., Johnson, C.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
14. Indulal, G.: Sharp bounds on the distance spectral radius and the distance energy of graphs. Linear Algebra Appl. **430**, 106–113 (2009)
15. Lin, H., Lu, X.: Bounds on the distance signless Laplacian spectral radius in terms of clique number. Linear Multilinear Algebra **63**, 1750–1759 (2015)
16. Lin, H., Zhou, B.: The effect of graft transformation on the distance signless Laplacian spectral radius. Linear Algebra Appl. **504**, 433–461 (2016)
17. Lin, H., Zhou, B.: The distance spectral radius of graphs with given number of odd vertices. Electron. J. Linear Algebra **31**, 286–305 (2016)
18. Lin, H., Zhou, B.: On least distance eigenvalue of uniform hypergraphs. Taiwan. J. Math. **22**, 1289–1307 (2018)
19. Liu, S., Shu, J., Xue, J.: On the largest distance (signless Laplacian) eigenvalue of non-transmission-regular graphs. Electron. J. Linear Algebra **34**, 459–471 (2018)

20. Minc, H.: Nonnegative Matrices. Wiley, New York (1988)
21. Nath, M., Paul, S.: A note on the distance spectral radius of some graphs. Discrete Math. Algorithms Appl. **6**, 1450015 (2014)
22. Nikiforov, V.: Merging the *A*- and *Q*-spectral theories. Appl. Anal. Discrete Math. **11**, 81–107 (2017)
23. Ruzieh, S.N., Powers, D.L.: The distance spectrum of the path P_n and the first distance eigenvector of connected graphs. Linear Multilinear Algebra **28**, 75–81 (1990)
24. Stevanović, D., Ilić, A.: Distance spectral radius of trees with fixed maximum degree. Electron. J. Linear Algebra **20**, 168–179 (2010)
25. Wang, Y., Zhou, B.: On distance spectral radius of graphs. Linear Algebra Appl. **438**, 3490–3503 (2013)
26. Wang, Y., Zhou, B.: On distance spectral radius of hypergraphs. Linear Multilinear Algebra **66**, 2232–2246 (2018)
27. Xing, R., Zhou, B.: On the distance and distance signless Laplacian spectral radii of bicyclic graphs. Linear Algebra Appl. **439**, 3955–3963 (2013)
28. Xing, R., Zhou, B., Dong, F.: The effect of a graft transformation on distance spectral radius. Linear Algebra Appl. **457**, 261–275 (2014)
29. Xing, R., Zhou, B., Li, J.: On the distance signless Laplacian spectral radius of graphs. Linear Multilinear Algebra **62**, 1377–1387 (2014)
30. Yu, G., Jia, H., Zhang, H., Shu, J.: Some graft transformations and its applications on the distance spectral radius of a graph. Appl. Math. Lett. **25**, 315–319 (2012)
31. Yu, G., Wu, Y., Zhang, Y., Shu, J.: Some graft transformations and its application on a distance spectrum. Discrete Math. **311**, 2117–2123 (2011)
32. Zhang, X., Godsil, C.: Connectivity and minimal distance spectral radius of graphs. Linear Multilinear Algebra **59**, 745–754 (2011)
33. Zhou, B., Ilić, A.: On distance spectral radius and distance energy of graphs. MATCH Commun. Math. Comput. Chem. **64**, 261–280 (2010)