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On the ω -multiple Meixner polynomials of the first kind

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Abstract

In this study, we introduce a new family of discrete multiple orthogonal polynomials, namely ω -multiple Meixner polynomials of the first kind, where ω is a positive real number. Some structural properties of this family, such as the raising operator, Rodrigue's type formula and an explicit representation are derived. The generating function for ω -multiple Meixner polynomials of the first kind is obtained and by use of this generating function we find several consequences for these polynomials. One of them is a lowering operator which will be helpful for obtaining a difference equation. We give the proof of the lowering operator by use of new technique which is a more elementary proof than the proof of Lee in (J. Approx. Theory 150:132–152, 2008). By combining the lowering operator with the raising operator we obtain the difference equation which has the ω -multiple Meixner polynomials of the first kind as a solution. As a corollary we give a third order difference equation for the ω -multiple Meixner polynomials of the first kind. Also it is shown that, for the special case $\omega = 1$, the obtained results coincide with the existing results for multiple Meixner polynomials of the first kind. In the last section as an illustrative example we consider the special case when $\omega = 1/2$ and, for the $1/2$ -multiple Meixner polynomials of the first kind, we state the corresponding result for the main theorems.

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1 Introduction

Discrete multiple orthogonal polynomials are useful extension of discrete orthogonal polynomials, see [1–13]. The theory of discrete orthogonal polynomials on a linear lattice were extended to such polynomials by Arvesu, Coussement and Van Assche in [2]. Multiple Meixner polynomials are discrete multiple orthogonal polynomials. There are two kinds of multiple Meixner polynomials. In this paper, we concentrate on the multiple Meixner polynomials of the first kind.

Multiple Meixner polynomials of the first kind $M_{\vec{n}}^{(\beta, \vec{a})}(x)$, of degree $|\vec{n}|$, are orthogonal polynomials for the negative binomial distributions when $r > 1$. That is,

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{(\beta, \vec{a})}(x)(-x)_j w_i^{\beta}(x) = 0, \quad j = 0, 1, \dots, n_i - 1, i = 1, \dots, r, \quad (1)$$

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where $\beta > 0$ is the fixed parameter and different values for the parameter $\vec{a} = (a_1, \dots, a_r)$, ($a_i \neq a_j$ whenever $i \neq j$), where $0 < a < 1$ with multi-index $\vec{n} = (n_1, \dots, n_r)$ and $(x)_k = x(x + 1)(x + 2) \cdots (x + k - 1)$ is the Pochhammer symbol with $(x)_0 = 1$.

The functions

$$w_i^\beta(x) = \begin{cases} \frac{\Gamma(\beta+x)a_i^k}{\Gamma(\beta)\Gamma(x+1)} & \text{if } x \in \mathbb{R}/(\{-1, -2, \dots\} \cup \{-\beta, -\beta - 1, \dots\}), \\ 0 & \text{if } x \in \{-1, -2, \dots\}, \end{cases}$$

are weight functions of the multiple Meixner polynomials of the first kind where $\Gamma(x)$ is the gamma function.

The orthogonality conditions give us a linear system of $|\vec{n}| = n_1 + n_2 + \dots + n_r$ homogeneous equations for the $|\vec{n}| + 1$ unknown coefficients of polynomials which always has a nontrivial solution. If the given multi-index \vec{n} is normal, then the corresponding polynomials will be unique polynomials. For the uniqueness of the polynomials one can use a system called an AT system which was introduced by Nikishin and Sorokin [11].

Definition 1.1 (cf. [11]) A set of continuous real functions w_1, w_2, \dots, w_r defined on $[a, b]$ is called an AT system for the index $n \in \mathbb{Z}'_+, n \neq 0$, if

$$w_1(x), xw_1(x), \dots, x^{n_1-1}w_1(x), \dots, w_r(x), xw_r(x), \dots, x^{n_r-1}w_r(x)$$

is a Chebyshev system of order $|n| - 1$ on $[a, b]$.

In an AT system, all the multi-index \vec{n} are normal. By using the following example it will be easy to show that the weight functions for the multiple Meixner polynomials of the first kind form an AT system.

Example 1.2 (cf. [2]) The functions $w(x)a_1^x, xw(x)a_1^x, \dots, x^{n_1-1}w(x)a_1^x, \dots, w(x)a_r^x, xw(x)a_r^x, \dots, x^{n_r-1}w(x)a_r^x$, with all the $a_i > 0, i = 1, \dots, r$, different and $w(x)$ a continuous function which has no zeros on \mathbb{R}^+ , form a Chebyshev system on \mathbb{R}^+ for every index $\vec{n} \in \mathbb{N}^r$.

In [2], Arvesu, Coussement and Van Assche investigated the raising operator and the Rodrigues formula for multiple Meixner polynomials of the first kind. Also, via the Rodrigues formula an explicit formula for these polynomials is obtained by these authors. They investigated these properties for multiple orthogonal polynomials of discrete variables by extending the classical orthogonal polynomials of discrete variables.

Van Assche in [12] obtained a lowering operator for multiple Meixner polynomials of the first kind for the case $r = 2$ and then by combining lowering and raising operators he gave the third order difference equation for these polynomials. Later, Lee in [7] obtained a lowering operator for the case r and then by combining lowering and raising operators Lee gave the $(r + 1)$ th order difference equation for these polynomials.

Ndayiragije and Van Assche in [4] gave generating functions and explicit expressions for the coefficients in the nearest neighbor recurrence relation for multiple Meixner polynomials of the first kind.

In this paper we introduce a new class for discrete multiple polynomials called ω -multiple Meixner polynomials of the first kind. The aim is to obtain some structural properties for these newly introduced polynomials. Firstly we define the orthogonality for the

ω -multiple Meixner polynomials of the first kind on the linear lattice by using two important operators, namely the ω -forward and ω -backward difference operators, where

$$\Delta_{\omega}f(x) = f(x + \omega) - f(x)$$

is the ω -forward operator and

$$\nabla_{\omega}f(x) = f(x) - f(x - \omega)$$

is the ω -backward operator.

In Sect. 2 we obtain some properties of the ω -multiple Meixner polynomials of the first kind, such as the raising operator, Rodrigues' formula and an explicit form. The generating function for the ω -multiple Meixner polynomials of the first kind is given in Sect. 3 and we obtain some results from the generating function such as connection and addition formulas. Section 4 includes the lowering operator and difference equation for the ω -multiple Meixner polynomials of the first kind. We also show that when $\omega = 1$, the results obtained in Sects. 2, 3 and 4, coincide with the existing results for multiple Meixner polynomials of the first kind. One of the main results of this paper is in Sect. 5 where we give some results for the $1/2$ -multiple Meixner polynomials of the first kind.

2 Orthogonality for ω -multiple Meixner polynomials of the first kind

In this section, we first give the definition of the orthogonality of ω -multiple Meixner polynomials of the first kind.

Definition 2.1 The monic discrete ω -multiple Meixner polynomial of the first kind, corresponding to the multi-index $\vec{n} = (n_1, \dots, n_r)$, the fixed parameter $\beta > 0$ and the parameter $\vec{a} = (a_1, \dots, a_r)$, ($a_i \neq a_j$ whenever $i \neq j$), is the unique polynomial of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{(\omega; \beta; \vec{a})}(\omega x)(-\omega x)_{j, \omega} W_i^{(\omega; \beta)}(\omega x) = 0, \quad j = 0, 1, \dots, n_i - 1, i = 1, 2, \dots, r, \tag{2}$$

where $(a)_{n, \omega} = a(a + \omega)(a + 2\omega) \cdots (a + (n - 1)\omega)$ is the ω -Pochhammer symbol for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ and $\omega > 0$.

The weight functions for ω -multiple Meixner polynomials of the first kind are defined as

$$w_i^{(\omega; \beta)}(x) = \begin{cases} \frac{\Gamma_{\omega}(\beta+x)a_i^x}{\Gamma_{\omega}(\beta)\Gamma_{\omega}(x+\omega)} & \text{if } x \in \mathbb{R}/(\{-1, -2, \dots\} \cup \{-\beta, -\beta - 1, \dots\}), \\ 0 & \text{if } x \in \{-1, -2, \dots\}, \end{cases}$$

where $0 < a_i < 1$, for $i = 1, 2, \dots, r$, with all the a_i different and Γ_{ω} is the ω -gamma function given by

$$\Gamma_{\omega}(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t\omega}{\omega}} dt = \omega^{\frac{x}{\omega}-1} \Gamma\left(\frac{x}{\omega}\right).$$

By Example 1.2 it is easy to conclude that the weight functions form an AT system which implies the uniqueness of such polynomials. When $\omega = 1$, the given orthogonality conditions coincide with the orthogonality conditions in (1).

In the rest of this paper the properties of multiple Meixner polynomials of the first kind are extended to ω -multiple Meixner polynomials of the first kind.

Theorem 2.2 *Let ω be a positive real number. The raising operator for ω -multiple Meixner polynomials of the first kind is given as*

$$\nabla_{\omega} \left[M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) w_i^{(\omega; \beta)}(x) \right] = \frac{a_i^{\omega} - 1}{a_i^{\omega}(\beta - \omega)} M_{\vec{n} + \vec{e}_i}^{(\omega; \beta - \omega; \vec{a})}(x) w_i^{(\omega; \beta - \omega)}(x). \tag{3}$$

Proof By using the product rule for the ω -backward operator

$$\nabla_{\omega} [f(x)g(x)] = f(x)\nabla_{\omega}g(x) + g(x - \omega)\nabla_{\omega}f(x),$$

we obtain

$$\nabla_{\omega} \left[M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) w_i^{(\omega; \beta)}(x) \right] = \frac{a_i^{\omega} - 1}{a_i^{\omega}(\beta - \omega)} (x) w_i^{(\omega; \beta - \omega)}(x) P_{\vec{n} + \vec{e}_i}^{(\omega)}.$$

Using the ω -summation by parts formula,

$$\sum_{k=0}^{\infty} \Delta_{\omega} [f(\omega k)]g(\omega k) = - \sum_{k=0}^{\infty} \nabla_{\omega} [g(\omega k)]f(\omega k) \quad \text{where } g(-\omega) = 0,$$

and the orthogonality conditions,

$$\sum_{k=0}^{\infty} P_{\vec{n} + \vec{e}_i}^{(\omega)}(\omega x)(-\omega x)_{i, \omega} w_i^{(\omega; \beta - \omega)}(\omega x) = 0,$$

we find the result with $P_{\vec{n} + \vec{e}_i}^{(\omega)} = M_{\vec{n} + \vec{e}_i}^{(\omega; \beta - \omega; \vec{a})}$, which was guaranteed from the uniqueness of the orthogonal polynomials. □

Theorem 2.3 *Let ω be a positive real number. The Rodrigues formula for ω -multiple Meixner polynomials of the first kind are introduced by*

$$\begin{aligned} M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) &= (\beta)_{|\vec{n}|, \omega} \left[\prod_{k=1}^r \left(\frac{a_k^{\omega}}{a_k^{\omega} - 1} \right)^{n_k} \frac{\Gamma_{\omega}(\beta) \Gamma_{\omega}(x + \omega)}{\Gamma_{\omega}(\beta + x)} \right] \\ &\times \prod_{i=1}^r \frac{1}{a_i^x} \nabla_{\omega}^{n_i} \left[\frac{\Gamma_{\omega}(\beta + x + |\vec{n}| \omega) a_i^x}{\Gamma_{\omega}(x + \omega) \Gamma_{\omega}(\beta + |\vec{n}| \omega)} \right]. \end{aligned} \tag{4}$$

Proof Replacing \vec{n} by $\vec{n} - \vec{e}_i$ and β by $\beta + \omega$ in the raising operator formula, we obtain

$$w_i^{(\omega; \beta)}(x) M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) = \frac{\beta a_i^{\omega}}{a_i^{\omega} - 1} \nabla_{\omega} \left[w_i^{(\omega; \beta + \omega)}(x) M_{\vec{n} - \vec{e}_i}^{(\omega; \beta + \omega; \vec{a})}(x) \right].$$

Then for $r = 2$ the multi-index will be $\vec{n} = (n_1, n_2)$.

For $i = 1$ we have

$$w_1^{(\omega; \beta)}(x) M_{n_1, n_2}^{(\omega; \beta; a_1, a_2)}(x) = \frac{\beta a_1^{\omega}}{a_1^{\omega} - 1} \nabla_{\omega} \left[w_1^{(\omega; \beta + \omega)}(x) M_{n_1 - 1, n_2}^{(\omega; \beta + \omega; a_1, a_2)}(x) \right],$$

and iterating it n_1 times we get

$$w_1^{(\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x) = (\beta)_{n_1,\omega} \left(\frac{a_1^\omega}{a_1^\omega - 1}\right)^{n_1} \times \nabla_\omega^{n_1} [w_1^{(\omega;\beta+n_1\omega)}(x)M_{0,n_2}^{(\omega;\beta+n_1\omega;a_1,a_2)}(x)].$$

For $i = 2$ we have

$$w_2^{(\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x) = \frac{\beta a_2^\omega}{a_2^\omega - 1} \nabla_\omega [w_2^{(\omega;\beta+\omega)}(x)M_{n_1,n_2-1}^{(\omega;\beta+\omega;a_1,a_2)}(x)],$$

and iterating it n_2 times we get

$$w_2^{(\omega;\beta)}(x)M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x) = (\beta)_{n_2,\omega} \left(\frac{a_2^\omega}{a_2^\omega - 1}\right)^{n_2} \times \nabla_\omega^{n_2} [w_1^{(\omega;\beta+n_2\omega)}(x)M_{n_1,0}^{(\omega;\beta+n_2\omega;a_1,a_2)}(x)].$$

By combining these two equations, we obtain the expression for $M_{n_1,n_2}^{(\omega;\beta;a_1,a_2)}(x)$ and if we continue the iteration for r , we derive the Rodrigues formula for ω -multiple Meixner polynomials of the first kind. □

The explicit form can easily be obtained from the Rodrigues formula using the Leibniz rule for the ω -backward operator,

$$\nabla_\omega^n f(x) = \sum_{i=1}^n (-1)^i \binom{n}{i} f(x - i\omega).$$

Theorem 2.4 *Let ω be a positive real number. The explicit form for ω -multiple Meixner polynomials of the first kind is given by*

$$M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \dots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \binom{n_2}{k_2} \dots \binom{n_r}{k_r} (-x)_{|\vec{k}|,\omega} \times \prod_{j=1}^r \left[\frac{(a_j^\omega)^{n_j-k_j}}{(a_j^\omega - 1)^{n_j}} (\beta + x)_{|\vec{n}|-|\vec{k}|,\omega} \right]. \tag{5}$$

Corollary 2.5 *The special cases of Theorem 2.2, Theorem 2.3 and Theorem 2.4 when $\omega = 1$ are easily reduced to results obtained in [2] and [4]. For instance, when $\omega = 1$ in Theorem 2.2, our result coincides with the raising operator for multiple Meixner polynomials of the first kind,*

$$\nabla [M_{\vec{n}}^{(\beta,\vec{a})}(x)w_i^\beta(x)] = \frac{a_i - 1}{a_i(\beta - 1)} M_{\vec{n} + \vec{e}_i}^{(\beta-1,\vec{a})}(x)w_i^{\beta-1}(x),$$

which is exactly the same formula as in [2, equation (4.6), p. 33].

When $\omega = 1$ in Theorem 2.3, we find the Rodrigues formula for multiple Meixner polynomials of the first kind,

$$M_{\vec{n}}^{(\beta,\vec{a})}(x) = (\beta)_{|\vec{n}|} \left[\prod_{k=1}^r \left(\frac{a_k}{a_k - 1}\right)^{n_k} \frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)} \right]$$

$$\times \prod_{i=1}^r \frac{1}{a_i^x} \nabla^{n_i} \left[\frac{\Gamma(\beta + x + |\vec{n}|) a_i^x}{\Gamma(x + 1) \Gamma(\beta + |\vec{n}|)} \right],$$

which coincides with the formula in [2, equation (4.7), p. 33].

When $\omega = 1$ in Theorem 2.4, we have an explicit form for multiple Meixner polynomials of the first kind,

$$M_{\vec{n}}^{(\beta, \vec{a})}(x) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \binom{n_2}{k_2} \cdots \binom{n_r}{k_r} (-x)_{|\vec{k}|} \\ \times \prod_{j=1}^r \left[\frac{(a_j)^{n_j - k_j}}{(a_j - 1)^{n_j}} (\beta + x)_{|\vec{n}| - |\vec{k}|} \right],$$

which coincides with the formula in [4, equation (3), p. 3]

3 Generating function for ω -multiple Meixner polynomials of the first kind

ω -Multiple Meixner polynomials of the first kind have a multivariate generating function with r variables. The following lemma will be useful for the proof of the theorem for the generating function.

The relation between Pochhammer symbol and ω -Pochhammer symbol is given as follows:

$$(a)_{n, \omega} = \omega^n \left(\frac{a}{\omega} \right)_n. \tag{6}$$

Lemma 3.1 (cf. [4, Lemma 1, p. 4]) *The generating function for the multinomial coefficients is given as follows:*

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(-x)_{|\vec{m}|}}{m_1! \cdots m_r!} t_1^{m_1} \cdots t_r^{m_r} = (1 - t_1 \cdots - t_r)^x.$$

This series converges absolutely and uniformly for $|t_1| + \cdots + |t_r| < 1$ when $x \in \mathbb{N}$ and contains a finite number of terms if $x \in \mathbb{N}$.

Theorem 3.2 *Let ω be a positive real number. The generating function for the ω -multiple Meixner polynomials of the first kind is*

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} = \left(1 - \frac{\omega t_1}{a_1^\omega - 1} - \cdots - \frac{\omega t_r}{a_r^\omega - 1} \right)^{\frac{x}{\omega}} \\ \times \left(1 - \frac{\omega t_1 a_1^\omega}{a_1^\omega - 1} - \cdots - \frac{\omega t_r a_r^\omega}{a_r^\omega - 1} \right)^{\frac{-(\beta+x)}{\omega}}. \tag{7}$$

Proof Replacing $M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x)$ with the explicit form in left hand side of (7) we obtain

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \binom{n_2}{k_2} \cdots \binom{n_r}{k_r} (-x)_{|\vec{k}|, \omega}$$

$$\times \prod_{j=1}^r \left[\frac{(a_k^\omega)^{n_j-k_j}}{(a_k^\omega - 1)^{n_j}} (\beta + x) \right]_{|\vec{n}|-|\vec{k}|, \omega} \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!}.$$

Changing the order of the summation gives

$$\sum_{k_1=0}^\infty \cdots \sum_{k_r=0}^\infty \sum_{n_1=k_1}^\infty \cdots \sum_{n_r=k_r}^\infty \frac{(-x)_{|\vec{k}|, \omega}}{k_1!(n_1 - k_1)! \cdots k_r!(n_r - k_r)!} \frac{a_1^{\omega(n_1-k_1)} \cdots a_r^{\omega(n_r-k_r)}}{(a_1^\omega - 1)^{n_1} \cdots (a_r^\omega - 1)^{n_r}} \times x t_1^{n_1} \cdots t_r^{n_r} (\beta + x)_{|\vec{n}|-|\vec{k}|, \omega}.$$

By setting $m_i = n_i - k_i$ and putting the factors in m_i and k_i together we obtain

$$\sum_{k_1=0}^\infty \cdots \sum_{k_r=0}^\infty \frac{(-x)_{|\vec{k}|, \omega}}{k_1! \cdots k_r!} \left(\frac{t_1}{a_1^\omega - 1} \right)^{k_1} \cdots \left(\frac{t_r}{a_r^\omega - 1} \right)^{k_r} \times \sum_{m_1=0}^\infty \cdots \sum_{m_r=0}^\infty \frac{(\beta + x)_{|\vec{m}|, \omega}}{m_1! \cdots m_r!} \left(\frac{t_1 a_1^\omega}{a_1^\omega - 1} \right)^{m_1} \cdots \left(\frac{t_r a_r^\omega}{a_r^\omega - 1} \right)^{m_r}.$$

Using the relation between the Pochhammer symbol and the ω -Pochhammer symbol (6), the above equation becomes

$$\sum_{k_1=0}^\infty \cdots \sum_{k_r=0}^\infty \frac{\left(\frac{-x}{\omega}\right)_{|\vec{k}|, \omega}}{k_1! \cdots k_r!} \left(\frac{\omega t_1}{a_1^\omega - 1} \right)^{k_1} \cdots \left(\frac{\omega t_r}{a_r^\omega - 1} \right)^{k_r} \times \sum_{m_1=0}^\infty \cdots \sum_{m_r=0}^\infty \frac{\left(\frac{\beta+x}{\omega}\right)_{|\vec{m}|, \omega}}{m_1! \cdots m_r!} \left(\frac{\omega t_1 a_1^\omega}{a_1^\omega - 1} \right)^{m_1} \cdots \left(\frac{\omega t_r a_r^\omega}{a_r^\omega - 1} \right)^{m_r}.$$

Now using Lemma 3.1, we obtain the generating function for ω -multiple Meixner polynomials of the first kind, which is the desired result. □

Corollary 3.3 *When $\omega = 1$, the special case of Theorem 3.2 reduces to the result of [4]. For instance, when $\omega = 1$ in Theorem 3.2, we have a generating function for multiple Meixner polynomials of the first kind,*

$$\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \cdots \sum_{n_r=0}^\infty M_{\vec{n}}^{(\beta, \vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} = \left(1 - \frac{t_1}{a_1 - 1} - \cdots - \frac{t_r}{a_r - 1} \right)^x \times \left(1 - \frac{t_1 a_1}{a_1 - 1} - \cdots - \frac{t_r a_r}{a_r - 1} \right)^{-(\beta+x)},$$

which coincides with the formula in [4, equation (7), p. 4].

The generating function will be used to establish the connection formula and the addition formula for ω -multiple Meixner polynomials of the first kind.

Theorem 3.4 *Let ω be a positive real number. ω -Multiple Meixner polynomials of the first kind $M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x)$ and $M_{\vec{n}-\vec{k}}^{(\omega;\beta+\gamma;\vec{a})}(x)$ satisfy the following connection formula:*

$$M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} (-\gamma)_{|\vec{k}|,\omega} \times \left(\frac{a_1^\omega}{a_1^{\omega-1}}\right)^{k_1} \cdots \left(\frac{a_r^\omega}{a_r^{\omega-1}}\right)^{k_r} M_{\vec{n}-\vec{k}}^{(\omega;\beta+\gamma;\vec{a})}(x). \tag{8}$$

Proof Replacing β by $\beta - \gamma + \gamma$ in the generating function (7) we obtain

$$\begin{aligned} \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} &= \left(1 - \frac{\omega t_1}{a_1^\omega - 1} - \cdots - \frac{\omega t_r}{a_r^\omega - 1}\right)^{\frac{x}{\omega}} \\ &\times \left(1 - \frac{\omega t_1 a_1^\omega}{a_1^\omega - 1} - \cdots - \frac{\omega t_r a_r^\omega}{a_r^\omega - 1}\right)^{\frac{-(\beta+\gamma+x)}{\omega}} \\ &\times \left(1 - \frac{\omega t_1 a_1^\omega}{a_1^\omega - 1} - \cdots - \frac{\omega t_r a_r^\omega}{a_r^\omega - 1}\right)^{\frac{\gamma}{\omega}}. \end{aligned}$$

From the generating function (7) and Lemma 3.1, the above equation gets the following form:

$$\begin{aligned} \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(\beta+\gamma;\vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} \\ &\times \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \frac{\left(\frac{-\gamma}{\omega}\right)_{|\vec{k}|}}{k_1! \cdots k_r!} \left(\frac{\omega a_1^\omega}{a_1^\omega - 1}\right)^{k_1} \cdots \left(\frac{\omega a_r^\omega}{a_r^\omega - 1}\right)^{k_r} \\ &\times t_1^{k_1} \cdots t_r^{k_r}. \end{aligned}$$

Changing the order of summations and using the relation between the Pochhammer symbol and the ω -Pochhammer symbol (6) we get

$$\begin{aligned} \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(\omega;\beta;\vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \\ &\times M_{\vec{n}-\vec{k}}^{(\omega;\beta+\gamma;\vec{a})}(x) (-\gamma)_{|\vec{k}|,\omega} \left(\frac{a_1^\omega}{a_1^{\omega-1}}\right)^{k_1} \cdots \\ &\times \left(\frac{a_r^\omega}{a_r^{\omega-1}}\right)^{k_r} \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!}. \end{aligned}$$

Finally, comparing the coefficients of $\frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!}$, appearing on both sides of the above equation, we obtain the desired result. □

Theorem 3.5 *Let ω be a positive real number. The addition formula for ω -multiple Meixner polynomials of the first kind is given by*

$$M_{\vec{n}}^{(\omega; \beta + \gamma; \vec{a})}(x + y) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} M_{\vec{n}-\vec{k}}^{(\omega; \beta; \vec{a})}(x) M_{\vec{k}}^{(\omega; \gamma; \vec{a})}(y). \tag{9}$$

Proof Changing x with $x + y$ and β with $\beta + \gamma$ in the generating function (7), together with changing the order of summations we obtain

$$\begin{aligned} \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(\beta + \gamma; \vec{a})}(x + y) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \frac{t_1^{k_1} \cdots t_r^{k_r}}{k_1! \cdots k_r!} M_{\vec{k}}^{(\gamma; \vec{a})}(y) \\ &\quad \times \frac{t_1^{n_1-k_1} \cdots t_r^{n_r-k_r}}{(n_1 - k_1)! \cdots (n_r - k_r)!} M_{\vec{n}-\vec{k}}^{(\beta; \vec{a})}(x). \end{aligned}$$

Finally, comparing the coefficients of $\frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!}$, appearing on both sides of the above equation we get the result. □

Note that in the case when $\omega = 1$, the connection and addition formula for multiple Meixner polynomials are also new. So, here we have new relations for multiple Meixner polynomials of the first kind, which are given below.

Corollary 3.6 *Multiple Meixner polynomials of the first kind satisfy the following connection and addition formulas, respectively:*

$$\begin{aligned} M_{\vec{n}}^{(\beta; \vec{a})}(x) &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} a_1^{k_1} \cdots a_r^{k_r} (-\gamma)_{|\vec{k}|} M_{\vec{n}-\vec{k}}^{(\beta + \gamma; \vec{a})}(x), \\ M_{\vec{n}}^{(\beta + \gamma; \vec{a})}(x + y) &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} M_{\vec{n}-\vec{k}}^{(\beta; \vec{a})}(x) M_{\vec{k}}^{(\gamma; \vec{a})}(y). \end{aligned}$$

4 Difference equation for ω -multiple Meixner polynomials of the first kind

In this section, the aim is to introduce the difference equation for the ω -multiple Meixner polynomials of the first kind by combining the lowering and raising operators.

Theorem 4.1 *Let ω be a positive real number. The raising operator for ω -multiple Meixner polynomials of the first kind is given as*

$$L_{a_i}^{(\beta)} [M_{\vec{n}}^{(\omega; \beta; \vec{a})}] = -M_{\vec{n} + e_i}^{(\omega; \beta - \omega; \vec{a})} \tag{10}$$

where $L_{a_i}^{(\beta)} [\cdot]$ is defined by

$$L_{a_i}^{(\beta)} [y] = \frac{x}{1 - a_i^\omega} \nabla_\omega y - \left[\frac{a_i^\omega (\beta - \omega)}{1 - a_i^\omega} - x \right] y.$$

Proof The proof follows directly from the Rodrigues formula (4) for the ω -multiple Meixner polynomials of the first kind. □

Theorem 4.2 *Let ω be a positive real number. The lowering operator for ω -multiple Meixner polynomials of the first kind is*

$$\Delta_\omega M_{\vec{n}}^{(\omega; \beta; \vec{a})} = \sum_{i=1}^r \omega n_i M_{\vec{n} - e_i}^{(\omega; \beta + \omega; \vec{a})}. \tag{11}$$

In particular, for $r = 2$

$$\Delta_\omega M_{n_1, n_2}^{(\omega; \beta; a_1, a_2)} = \omega n_1 M_{n_1 - 1, n_2}^{(\omega; \beta + \omega; a_1, a_2)} + \omega n_2 M_{n_1, n_2 - 1}^{(\omega; \beta + \omega; a_1, a_2)}.$$

Proof Changing β for $\beta - \omega$ and applying the operator Δ_ω to both sides of the generating function (7) for the case $r = 2$ we obtain

$$\begin{aligned} \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \Delta_\omega M_{n_1, n_2}^{(\omega; \beta - \omega; a_1, a_2)} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} &= (\omega t_1 + \omega t_2) \left(1 - \frac{\omega t_1}{a_1^\omega - 1} - \frac{\omega t_2}{a_2^\omega - 1} \right)^{\frac{x}{\omega}} \\ &\times \left(1 - \frac{\omega t_1 a_1^\omega}{a_1^\omega - 1} - \frac{\omega t_2 a_2^\omega}{a_2^\omega - 1} \right)^{\frac{-(\beta+x)}{\omega}}. \end{aligned}$$

By use of the generating function (7) the above equation gets the following form:

$$\begin{aligned} \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \Delta_\omega M_{n_1, n_2}^{(\omega; \beta - \omega; a_1, a_2)} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} &= \omega \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty M_{n_1, n_2}^{(\beta, a_1, a_2)} \frac{t_1^{n_1+1} t_2^{n_2}}{n_1! n_2!} \\ &+ \omega \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty M_{n_1, n_2}^{(\beta, a_1, a_2)} \frac{t_1^{n_1} t_2^{n_2+1}}{n_1! n_2!}. \end{aligned}$$

Changing β for $\beta + \omega$ and then replacing n_1 with $n_1 - 1$ and n_2 with $n_2 - 1$ in the right side of equation we obtain

$$\begin{aligned} \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \Delta_\omega M_{n_1, n_2}^{(\omega; \beta; a_1, a_2)} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} &= \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty [\omega n_1 M_{n_1 - 1, n_2}^{(\omega; \beta + \omega; a_1, a_2)} + \omega n_2 M_{n_1, n_2 - 1}^{(\omega; \beta + \omega; a_1, a_2)}] \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!}. \end{aligned}$$

Finally, comparing the coefficients of $\frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!}$, appearing on both sides of the above equation the desired result is obtained for the case $r = 2$. □

Remark 4.1 In the case $\omega = 1$, the lowering operator for the ω -multiple Meixner polynomials of the first kind coincide with the lowering operator for the multiple Meixner polynomials of the first kind which is given in [7]. For the proof of the ω type we consider a different approach, where the generating function plays an important role and the proof becomes simpler than the corresponding proof when $\omega = 1$ [7, Theorem 2.4, p. 138].

Theorem 4.3 *Let ω be a positive real number. The difference equation for the ω -multiple Meixner polynomials $M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x)_{|\vec{n}|=0}^{\infty}$ is given by*

$$\begin{aligned}
 &L_{a_1}^{(\beta+2\omega-r\omega)} L_{a_2}^{(\beta+3\omega-r\omega)} \dots L_{a_r}^{(\beta+\omega)} \Delta_{\omega} M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) \\
 &+ \sum_{i=1}^r \omega n_i L_{a_1}^{(\beta+2\omega-r\omega)} \dots L_{a_{i-1}}^{(\beta+i\omega-r\omega)} L_{a_{i+1}}^{(\beta+(i+1)\omega-r\omega)} \dots \\
 &\times L_{a_r}^{(\beta)} L_{a_i}^{(\beta+\omega)} M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x) = 0.
 \end{aligned} \tag{12}$$

Proof Since $L_{a_k}^{(n)} L_{a_m}^{(n+\omega)} = L_{a_m}^{(n)} L_{a_k}^{(n+\omega)}$ for $n, a_k, a_m \in \mathbb{R}$, we obtain

$$\begin{aligned}
 L_{a_1}^{(\beta+2\omega-r\omega)} L_{a_2}^{(\beta+3\omega-r\omega)} \dots L_{a_r}^{(\beta+\omega)} &= L_{a_1}^{(\beta+2\omega-r\omega)} \dots L_{a_{i-1}}^{(\beta+i\omega-r\omega)} \\
 &\times L_{a_{i+1}}^{(\beta+(i+1)\omega-r\omega)} \dots L_{a_r}^{(\beta)} L_{a_i}^{(\beta+\omega)},
 \end{aligned}$$

for $i = 1, \dots, r$.

Now applying $L_{a_1}^{(\beta+2\omega-r\omega)} L_{a_2}^{(\beta+3\omega-r\omega)} \dots L_{a_r}^{(\beta+\omega)}$ to the lowering operator (11) and using the raising operator (10), we get the result. \square

Theorem 4.4 *Let ω be a positive real number. The third order difference equation for the ω -multiple Meixner polynomials $M_{\vec{n}}^{(\omega; \beta; \vec{a})}(x)_{n_1+n_2=0}^{\infty}$ of the first kind is given as*

$$\begin{aligned}
 &x(x-\omega) \nabla_{\omega}^2 \Delta_{\omega} y + x[\beta(a_1^{\omega} + a_2^{\omega}) + (x-\omega)(a_1^{\omega} + a_2^{\omega} - 2)] \nabla_{\omega} \Delta_{\omega} y \\
 &+ [a_1^{\omega} \beta - x(1-a_1^{\omega})(a_2 \omega \beta - x\omega(1-a_1^{\omega})(1-a_2^{\omega}) - a_1^{\omega} a_2^{\omega} \beta \omega)] \Delta_{\omega} y \\
 &+ [n_1(1-a_1^{\omega}) + n_2(1-a_2^{\omega})] \omega x \nabla_{\omega} y \\
 &+ \omega(\beta-\omega)[[n_1 a_2^{\omega} + n_2 a_1^{\omega} - a_1^{\omega} a_2^{\omega} (n_1 + n_2)] \\
 &- (1-a_1^{\omega})(1-a_2^{\omega})(n_1 + n_2)x] y = 0.
 \end{aligned} \tag{13}$$

Proof Considering the case for $r = 2$ in Theorem 4.3, we have

$$L_{a_1}^{(\beta)} L_{a_2}^{(\beta+\omega)} \Delta_{\omega} y + \omega n_1 L_{a_2}^{(\beta)} y + \omega n_2 L_{a_1}^{(\beta)} y = 0,$$

where $y = M_{n_1, n_2}^{(\omega; \beta; a_1, a_2)}(x)$, which gives the proof. \square

Corollary 4.5 *The special cases of Theorem 4.1, Theorem 4.2 and Theorem 4.4, Theorem 4.4 when $\omega = 1$ can easily be reduced to the results obtained in [7]. For instance, when $\omega = 1$ in Theorem 4.1, we have the raising operator $L_{a_i}^{(\beta)}[\cdot]$ for any polynomial,*

$$L_{a_i}^{(\beta)}[y] = \frac{x}{1-a_i} \nabla y - \left[\frac{a_i(\beta-1)}{1-a_i} - x \right] y,$$

which is exactly the same formula as in [7, equation (2.4), p. 138].

When $\omega = 1$ in Theorem 4.2 the lowering operator for multiple Meixner polynomials of the first kind has the following form:

$$\Delta M_{\vec{n}}^{(\beta, \vec{a})} = \sum_{i=1}^r n_i M_{\vec{n}-e_i}^{(\beta+1, \vec{a})},$$

which coincides with the formula in [7, equation (2.5), p. 138].

When $\omega = 1$ in Theorem 4.4, the $(n + 1)$ th order difference equation for multiple Meixner polynomials of the first kind becomes

$$\begin{aligned} &L_{a_1}^{(\beta+2-r)} L_{a_2}^{(\beta+3-r)} \dots L_{a_r}^{(\beta+1)} \Delta M_{\vec{n}}^{(\beta, \vec{a})}(x) \\ &+ \sum_{i=1}^r n_i L_{a_1}^{(\beta+2-r)} \dots L_{a_{i-1}}^{(\beta+i-r)} L_{a_{i+1}}^{(\beta+(i+1)-r)} \dots L_{a_r}^{(\beta)} L_{a_i}^{(\beta+1)} M_{\vec{n}}^{(\beta, \vec{a})}(x) = 0, \end{aligned}$$

which reduces to the formula in [7, Theorem 2.5, p. 139].

When $\omega = 1$ in Theorem 4.4, the third order difference equation for multiple Meixner polynomials of the first kind is obtained as

$$\begin{aligned} &x(x - 1)\nabla^2 \Delta y + x[\beta(a_1 + a_2) + (x - 1)(a_1 + a_2 - 2)]\nabla \Delta y \\ &+ [(a_1\beta - x(1 - a_1))(a_2\beta - x(1 - a_1)(1 - a_2) - a_1a_2\beta)]\Delta y \\ &+ [n_1(1 - a_1) + n_2(1 - a_2)]x\nabla y \\ &+ (\beta - 1)[[n_1a_2 + n_2a_1 - a_1a_2(n_1 + n_2)] - (1 - a_1)(1 - a_2)(n_1 + n_2)x]y = 0, \end{aligned}$$

which coincides with the formula in [7, Corollary 2.6, p. 139].

5 1/2-Multiple Meixner polynomials of the first kind

As we mentioned before for the case when $\omega = 1$, ω -multiple Meixner polynomials of the first kind reduce to the known multiple Meixner polynomials of the first kind. For the other values of ω we have new classes for multiple Meixner polynomials of the first kind where ω is positive real number. In this section we exhibit the case $\omega = 1/2$ and state some relations for 1/2-multiple Meixner polynomials of the first kind such as weight functions, orthogonality conditions, the explicit form, the generating function and a third order difference equation.

1/2-Multiple Meixner polynomials of the first kind have the following weight functions:

$$w_i^{(1/2;\beta)}(x) = \frac{\Gamma_{1/2}(\beta + x)a_i^x}{\Gamma_{1/2}(\beta)\Gamma_{1/2}(x + 1/2)} = \frac{\Gamma(2(\beta + x))a_i^x}{\Gamma(2\beta)\Gamma(2(x + 1/2))} \quad i = 1, 2, \dots, r.$$

By using these weight functions in (2), the orthogonality conditions for 1/2-multiple Meixner polynomials of the first kind can be written as

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{(1/2;\beta;\vec{a})}\left(\frac{x}{2}\right) (-x)_j \frac{\Gamma(2\beta + x)a_i^{x/2}}{\Gamma(2\beta)\Gamma(x + 1)} = 0, \quad j = 0, 1, \dots, n_i - 1.$$

The explicit form for 1/2-multiple Meixner polynomials can easily be obtained from (5) as follows:

$$M_{\vec{n}}^{(1/2;\beta;\vec{a})}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \times \prod_{j=1}^r \left[\frac{(a_j^{1/2})^{n_j-k_j}}{(2a_j^{1/2}-2)^{n_j}} (-2x)_{|\vec{k}|} (2\beta+2x)_{|\vec{n}|-|\vec{k}|} \right].$$

1/2-Multiple Meixner polynomials of the first kind have the following generating function:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} M_{\vec{n}}^{(1/2;\beta;\vec{a})}(x) \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1! \cdots n_r!} = \left(1 - \frac{t_1}{2(a_1^{1/2}-1)} - \cdots - \frac{t_r}{2(a_r^{1/2}-1)} \right)^{2x} \times \left(1 - \frac{t_1 a_1^{1/2}}{2(a_1^{1/2}-1)} - \cdots - \frac{t_r a_r^{1/2}}{2(a_r^{1/2}-1)} \right)^{-2(\beta+x)}.$$

1/2-Multiple Meixner polynomials of the first kind satisfy the following third order difference equation:

$$\begin{aligned} &x(x-1/2)\nabla_{1/2}^2 \Delta_{1/2} y + x[\beta(a_1^{1/2}+a_2^{1/2})+(x-1/2)(a_1^{1/2}+a_2^{1/2}-2)]\nabla_{1/2} \Delta_{1/2} y \\ &+ [a_1^{1/2}\beta-x(1-a_1^{1/2})((a_2\beta)/2-[x(1-a_1^{1/2})(1-a_2^{1/2})]/2-[a_1^{1/2}a_1^{1/2}\beta]/2)]\Delta_{1/2} y \\ &+ [n_1(1-a_1^{1/2})+n_2(1-a_2^{1/2})]x/2\nabla_{1/2} y \\ &+ (\beta-1/2)/2[[n_1a_2^{1/2}+n_2a_1^{1/2}-a_1^{1/2}a_2^{1/2}(n_1+n_2)] \\ &- (1-a_1^{1/2})(1-a_2^{1/2})(n_1+n_2)x]y = 0, \end{aligned}$$

where $\Delta_{1/2}f(x) = f(x+1/2) - f(x)$ and $\nabla_{1/2}f(x) = f(x) - f(x-1/2)$.

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