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# New generalized fractional versions of Hadamard and Fejér inequalities for harmonically convex functions

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## Abstract

The aim of this paper is to establish new generalized fractional versions of the Hadamard and the Fejér–Hadamard integral inequalities for harmonically convex functions. Fractional integral operators involving an extended generalized Mittag-Leffler function which are further generalized via a monotone increasing function are utilized to get these generalized fractional versions. The results of this paper give several consequent fractional inequalities for harmonically convex functions for known fractional integral operators deducible from utilized generalized fractional integral operators.

**Keywords:** Harmonically convex function; Hadamard inequality; Fejér–Hadamard inequality; Mittag-Leffler function; Fractional integral operators

## 1 Introduction

Fractional integral inequalities are generalizations of classical integral inequalities. Hadamard and Fejér–Hadamard inequalities are the inequalities which have been studied extensively for different fractional integral/derivative operators, see [1, 4–6, 8–10, 14, 16, 17, 23, 25, 27, 30, 33, 34, 36–38, 42, 44]. The main objective of this paper is to prove some new fractional generalizations of Hadamard and the Fejér–Hadamard inequalities for harmonically convex functions. We begin with fractional integral operators defined by Salim and Faraj in [35] containing generalized Mittag-Leffler function in their kernels as follows:

**Definition 1** ([35]) Let  $\sigma, \tau, k, r, \rho$  be positive real numbers and  $\omega \in \mathbb{R}$ . Then the generalized fractional integral operators containing Mittag-Leffler function for a real-valued continuous function  $f$  are defined by

$$(\epsilon_{\sigma, \tau, \delta, \omega, a^+}^{\rho, r, k} f)(x) = \int_a^x (x-t)^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k}(\omega(x-t)^\sigma) f(t) dt, \quad (1.1)$$

$$(\epsilon_{\sigma, \tau, \delta, \omega, b^-}^{\rho, r, k} f)(x) = \int_x^b (t-x)^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k}(\omega(t-x)^\sigma) f(t) dt, \quad (1.2)$$

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where  $E_{\sigma,\tau,\delta}^{\rho,r,k}(t)$  is the generalized Mittag-Leffler function defined as

$$E_{\sigma,\tau,\delta}^{\rho,r,k}(t) = \sum_{n=0}^{\infty} \frac{(\rho)_{kn} t^n}{\Gamma(\sigma n + \tau)(r)_{\delta n}}. \quad (1.3)$$

The connection of Mittag-Leffler function with fractional calculus is very useful and well-established. Its alliance with fractional integral operators as a kernel plays a vital role in the development of the theory and applications of fractional calculus in various subjects of science and engineering [12, 13, 18–22, 24, 28, 29, 31, 35, 39, 43]. In [2] Andrić et al. defined the following fractional integral operators containing an extended generalized Mittag-Leffler function in their kernels:

**Definition 2** ([2]) Let  $\omega, \tau, \delta, \rho, c \in \mathbb{C}$ ,  $\Re(\tau), \Re(\delta) > 0$ ,  $\Re(c) > \Re(\rho) > 0$ , with  $p \geq 0$ ,  $\sigma, r > 0$  and  $0 < k \leq r + \sigma$ . Let  $f \in L_1[a, b]$  and  $x \in [a, b]$ . Then the generalized fractional integral operators  $\epsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} f$  and  $\epsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} f$  are defined by

$$(\epsilon_{\sigma,\tau,\delta,\omega,a^+}^{\rho,r,k,c} f)(x; p) = \int_a^x (x-t)^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega(x-t)^\sigma; p) f(t) dt, \quad (1.4)$$

$$(\epsilon_{\sigma,\tau,\delta,\omega,b^-}^{\rho,r,k,c} f)(x; p) = \int_x^b (t-x)^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega(t-x)^\sigma; p) f(t) dt, \quad (1.5)$$

where

$$E_{\sigma,\tau,\delta}^{\rho,r,k,c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nk, c - \rho)(c)_{nk} t^n}{\beta(\rho, c - \rho) \Gamma(\sigma n + \tau)(\delta)_{nr}} \quad (1.6)$$

is the extended generalized Mittag-Leffler function.

In [7] (see, also [26]) Farid elegantly defined a unified integral operator as follows:

**Definition 3** Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$  be functions such that  $f$  is positive,  $f \in L_1[a, b]$ , and  $g$  is differentiable and strictly increasing. Also let  $\frac{\phi}{x}$  be an increasing function on  $[a, \infty)$  and  $\omega, \tau, \delta, \rho, c \in \mathbb{C}$ ,  $\Re(\tau), \Re(\delta) > 0$ ,  $\Re(c) > \Re(\rho) > 0$ , with  $p \geq 0$ ,  $\sigma, r > 0$  and  $0 < k \leq r + \sigma$ . Then for  $x \in [a, b]$  the integral operators  $({}_g F_{\sigma,\tau,\delta,\omega,a^+}^{\phi,\rho,r,k,c} f)$  and  $({}_g F_{\sigma,\tau,\delta,\omega,b^-}^{\phi,\rho,r,k,c} f)$  are defined by

$$({}_g F_{\sigma,\tau,\delta,\omega,a^+}^{\phi,\rho,r,k,c} f)(x; p) = \int_a^x \frac{\phi(g(x) - g(t))}{g(x) - g(t)} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega(g(x) - g(t))^\sigma; p) f(t) d(g(t)), \quad (1.7)$$

$$({}_g F_{\sigma,\tau,\delta,\omega,b^-}^{\phi,\rho,r,k,c} f)(x; p) = \int_x^b \frac{\phi(g(t) - g(x))}{g(t) - g(x)} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega(g(t) - g(x))^\sigma; p) f(t) d(g(t)). \quad (1.8)$$

A generalization of integral operators defined in (1.4), (1.5) can be deduced from the above definition by taking  $\phi(t) = t^\tau$  as follows:

**Definition 4** Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$  be functions such that  $f$  is positive,  $f \in L_1[a, b]$ , and  $g$  is differentiable and strictly increasing. Also let  $\omega, \tau, \delta, \rho, c \in \mathbb{C}$ ,  $\Re(\tau), \Re(\delta) > 0$ ,  $\Re(c) > \Re(\rho) > 0$ , with  $p \geq 0$ ,  $\sigma, r > 0$ , and  $0 < k \leq r + \sigma$ . Then for  $x \in [a, b]$  the integral operators

are defined by

$$({}_g \mathcal{I}_{\sigma, \tau, \delta, \omega, a^+}^{\rho, r, k, c} f)(x; p) = \int_a^x (g(x) - g(t))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(g(x) - g(t))^\sigma; p) f(t) d(g(t)), \quad (1.9)$$

$$({}_g \mathcal{I}_{\sigma, \tau, \delta, \omega, b^-}^{\rho, r, k, c} f)(x; p) = \int_x^b (g(t) - g(x))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(g(t) - g(x))^\sigma; p) f(t) d(g(t)). \quad (1.10)$$

Fractional integral operators (1.9), (1.10) produce some already known integral operators, see [33, Remark 1].

We are interested in utilizing fractional integral operators (1.9), (1.10) for the establishment of Hadamard and Fejér–Hadamard inequalities for harmonically convex functions. The classical Hadamard inequality is an elegant geometric interpretation of convex functions.

**Definition 5** ([41]) A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all  $x, y \in [a, b]$  and  $t \in [0, 1]$ .

Hadamard inequality is stated in the following theorem:

**Theorem 1.1** Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $a < b$ , be a convex function. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.11)$$

Fejér–Hadamard inequality is a weighted version of Hadamard inequality proved by Fejér in [11] which is stated in the following theorem:

**Theorem 1.2** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function and  $g : [a, b] \rightarrow \mathbb{R}$  be a nonnegative, integrable, and symmetric about  $\frac{a+b}{2}$ . Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x)g(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) dx. \quad (1.12)$$

Next we give the definition of harmonically convex functions [14].

**Definition 6** Let  $I$  be an interval of nonzero real numbers. Then a function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex if

$$f\left(\frac{ab}{ta + (1-t)b}\right) \leq tf(b) + (1-t)f(a) \quad (1.13)$$

holds for all  $a, b \in I$  and  $t \in [0, 1]$ . If the reversed inequality holds in (1.13), then  $f$  is called a harmonically concave function.

**Example 1.3** ([14]) Let  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = x$ , and  $g : (-\infty, 0) \rightarrow \mathbb{R}$ ,  $g(x) = x$ . Then  $f$  is a harmonically convex function and  $g$  is a harmonically concave function.

The above example gives following result.

**Proposition 1.4** ([14]) *Let  $I \subset \mathbb{R} \setminus \{0\}$  be a real interval and  $f : I \rightarrow \mathbb{R}$  is a function.*

- (i) *If  $I \subset (0, \infty)$  and  $f$  is convex and nondecreasing function, then  $f$  is harmonically convex.*
- (ii) *If  $I \subset (0, \infty)$  and  $f$  is harmonically convex and nonincreasing function, then  $f$  is convex.*
- (iii) *If  $I \subset (-\infty, 0)$  and  $f$  is harmonically convex and nondecreasing function, then  $f$  is convex.*
- (iv) *If  $I \subset (-\infty, 0)$  and  $f$  is convex and nonincreasing function, then  $f$  is a harmonically convex.*

**Definition 7** ([25]) A function  $\varphi : [a, b] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  is said to be harmonically symmetric about  $\frac{a+b}{2ab}$  if

$$\varphi\left(\frac{1}{x}\right) = \varphi\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - x}\right), \quad x \in [a, b].$$

For some recent work on harmonically convex functions, we refer readers to [1, 3, 9, 14, 25, 30] and references therein. In this paper, we extend the work of Abbas et al. [1] and Farid et al. [9] for Hadamard and Fejér–Hadamard-type inequalities by using (1.9) and (1.10).

In Sect. 3, we prove two fractional versions of Hadamard and two fractional versions of Fejér–Hadamard-type inequalities for harmonically convex functions by using fractional integral operators (1.9) and (1.10). Furthermore, the associated published results are obtained which are identified in remarks, some corollaries are also given.

## 2 Main results

**Theorem 2.1** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$ ,  $\text{Range}(g) \subset [a, b]$ , be functions such that  $f$  is positive,  $f \in L_1[a, b]$ , and  $g$  is differentiable and strictly increasing. If  $f$  is a harmonically convex function on  $[a, b]$ , then for fractional integral operators (1.9) and (1.10) we have*

$$\begin{aligned} & f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right) \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-}^{\rho,r,k,c} 1\right) \left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \\ & \leq \frac{1}{2} \left( \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-}^{\rho,r,k,c} f \circ \psi\right) \left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \right. \\ & \quad \left. + \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}^{\rho,r,k,c} f \circ \psi\right) \left(g^{-1}\left(\frac{1}{g(a)}\right); p\right) \right) \\ & \leq \frac{f(g(a)) + f(g(b))}{2} \left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}^{\rho,r,k,c} 1\right) \left(g^{-1}\left(\frac{1}{g(a)}\right); p\right), \end{aligned} \quad (2.1)$$

where  $\psi(t) = \frac{1}{g(t)}$  for all  $t \in [\frac{1}{b}, \frac{1}{a}]$  and  $\omega' = \omega\left(\frac{g(a)(b)}{g(b)-g(a)}\right)^\sigma$ .

*Proof* Since  $f$  is harmonically convex on  $[a, b]$ , for  $x, y \in [a, b]$ , the following inequality holds:

$$f\left(\frac{2g(x)g(y)}{g(x)+g(y)}\right) \leq \frac{f(g(x)) + f(g(y))}{2}. \quad (2.2)$$

By taking  $g(x) = \frac{g(a)g(b)}{tg(b)+(1-t)g(a)}$  and  $g(y) = \frac{g(a)g(b)}{tg(a)+(1-t)g(b)}$  in (2.2), we have

$$2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right) \leq f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) + f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right). \quad (2.3)$$

Multiplying (2.3) by  $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)$  and integrating over  $[0, 1]$ , we get

$$\begin{aligned} & 2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right) \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) dt \\ & \leq \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) dt \\ & \quad + \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right) dt. \end{aligned} \quad (2.4)$$

By setting  $g(x) = \frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$  and  $g(y) = \frac{tg(a)+(1-t)g(b)}{g(a)g(b)}$  in (2.4) and using (1.9), (1.10), the first inequality of (2.1) can be obtained. On the other hand, using harmonic convexity of  $f$ , we have

$$f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) + f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right) \leq f(g(a)) + f(g(b)). \quad (2.5)$$

Multiplying (2.5) by  $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)$  and then integrating over  $[0, 1]$ , we get

$$\begin{aligned} & \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) dt \\ & \quad + \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right) dt \\ & \leq (f(g(a)) + f(g(b))) \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) dt. \end{aligned} \quad (2.6)$$

By setting  $g(x) = \frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$  and  $g(y) = \frac{tg(a)+(1-t)g(b)}{g(a)g(b)}$  in (2.6), and using (1.9), (1.10), the second inequality of (2.1) can be obtained.  $\square$

**Remark 2.2**

- (i) By setting  $p = 0$  and  $g = I$ , [1, Theorem 3.1] is obtained.
- (ii) By setting  $g = I$ , [9, Theorem 2.1] is obtained.
- (iii) By setting  $\omega = p = 0$ ,  $g = I$ , [15, Theorem 4] is obtained.

**Corollary 2.3** *If we take  $\psi(x) = x$  in Theorem 2.1, then we get the following inequalities:*

$$\begin{aligned} & f\left(\frac{2}{a+b}\right) \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(\frac{1}{a})^-}^{\rho,r,k,c} - 1\right) \left(\frac{1}{b}; p\right) \\ & \leq \frac{1}{2} \left( \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(\frac{1}{a})^-}^{\rho,r,k,c} - f\right) \left(\frac{1}{b}; p\right) + \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(\frac{1}{b})^+}^{\rho,r,k,c} - f\right) \left(\frac{1}{a}; p\right) \right) \\ & \leq \frac{f(\frac{1}{a}) + f(\frac{1}{b})}{2} \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(\frac{1}{b})^+}^{\rho,r,k,c} + 1\right) \left(\frac{1}{a}; p\right), \end{aligned}$$

where  $g$  is the reciprocal function.

The following lemma is useful to give the next result.

**Lemma 2.4** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$ ,  $\text{Range}(g) \subset [a, b]$ , be functions such that  $f$  is positive,  $f \in L_1[a, b]$ , and  $g$  is differentiable and strictly increasing. If  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , then for operators (1.9) and (1.10) we have:*

$$\begin{aligned} & \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &= \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(a)}))^-}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right) \\ &= \frac{1}{2} \left( \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \right. \\ &\quad \left. + \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(a)}))^-}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right) \right), \end{aligned} \quad (2.7)$$

where  $\psi(t) = \frac{1}{g(t)}$  for all  $t \in [\frac{1}{b}, \frac{1}{a}]$ .

*Proof* For operators (1.9) and (1.10), we can write

$$\begin{aligned} & \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &= \int_{g^{-1}(\frac{1}{g(b)})}^{g^{-1}(\frac{1}{g(a)})} \left( \frac{1}{g(a)} - g(t) \right)^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c} \left( \omega \left( \frac{1}{g(a)} - g(t) \right)^{\sigma}; p \right) f \left( \frac{1}{g(t)} \right) d(g(t)). \end{aligned} \quad (2.8)$$

Replacing  $g(t)$  by  $\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)$  in equation (2.8) and then using  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , we have

$$\begin{aligned} & \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &= \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(a)}))^-}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right). \end{aligned} \quad (2.9)$$

By adding  $\left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right)$  on both sides of (2.9), we have

$$\begin{aligned} & 2 \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &= \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(b)}))^+}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \end{aligned} \quad (2.10)$$

$$+ \left( {}_g\mathcal{Y}_{\sigma, \tau, \delta, \omega, (g^{-1}(\frac{1}{g(a)}))^-}^{\rho, r, k, c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right). \quad (2.11)$$

Equations (2.9) and (2.11) give required result.  $\square$

**Theorem 2.5** *Let  $f, g, h : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$ ,  $\text{Range}(g), \text{Range}(h) \subset [a, b]$ , be functions such that  $f$  is positive,  $f \in L_1[a, b]$ ,  $g$  is differentiable, strictly increasing, and  $h$  is non-negative and integrable. If  $f$  is harmonically convex and  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , then for*

fractional integral operators (1.9) and (1.10) we have:

$$\begin{aligned}
 & f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}^{\rho,r,k,c}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \\
 & \quad +\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-}^{\rho,r,k,c}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
 & \leq\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}^{\rho,r,k,c}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \\
 & \quad +\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-}^{\rho,r,k,c}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
 & \leq\frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}^{\rho,r,k,c}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \\
 & \quad +\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^-}^{\rho,r,k,c}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right), \tag{2.12}
 \end{aligned}$$

where  $\psi(t) = \frac{1}{g(t)}$  for all  $t \in [\frac{1}{b}, \frac{1}{a}]$ ,  $fh \circ \psi = (f \circ \psi)(h \circ \psi)$  and  $\omega' = \omega(\frac{g(a)g(b)}{g(b)-g(a)})^\sigma$ .

*Proof* Multiplying (2.3) by  $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)h(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)})$ , then integrating over  $[0, 1]$  we get

$$\begin{aligned}
 & 2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \\
 & \leq \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \\
 & \quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt. \tag{2.13}
 \end{aligned}$$

By setting  $g(x) = \frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$  in (2.13) and using (1.9), (1.10), as well as the condition  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , we have

$$\begin{aligned}
 & 2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-}^{\rho,r,k,c}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
 & \leq\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}^{\rho,r,k,c}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \\
 & \quad +\left({}_g\Upsilon_{\sigma,\tau,\delta,\omega',\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-}^{\rho,r,k,c}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right). \tag{2.14}
 \end{aligned}$$

By using Lemma 2.4 in the above inequality, one can get the first inequality of (2.12). For the second inequality of (2.12), multiplying (2.5) by  $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)h(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)})$ , then integrating over  $[0, 1]$ , we get

$$\begin{aligned}
 & \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \\
 & \quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt
 \end{aligned}$$

$$\leq (f(g(a)) + f(g(b))) \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) h\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt. \quad (2.15)$$

Setting  $g(x) = \frac{tg(b) + (1-t)g(a)}{g(a)g(b)}$  in (2.15) and using (1.9), (1.10), as well as the condition  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , we have

$$\begin{aligned} & \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{1}{g(b)}))}^{\rho,r,k,c} f h \circ \psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right) \\ & + \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{1}{g(a)}))}^{\rho,r,k,c} f h \circ \psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \\ & \leq (f(g(a)) + f(g(b))) \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{1}{g(b)}))}^{\rho,r,k,c} h \circ \psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right). \end{aligned} \quad (2.16)$$

Again using Lemma 2.4 in (2.16), one can get the second inequality of (2.12).  $\square$

**Remark 2.6**

- (i) By setting  $p = 0$ ,  $h(x) = 1$  and  $g = I$ , [1, Theorem 3.1] is obtained.
- (ii) By setting  $g = I$  and  $h(x) = 1$ , [9, Theorem 2.1] is obtained.
- (iii) By setting  $\omega = p = 0$ ,  $h(x) = 1$  and  $g = I$ , [15, Theorem 4] is obtained.
- (iv) By setting  $\omega = p = 0$ ,  $\tau = 1$  and  $g = I$ , [3, Theorem 8] is obtained.
- (v) By setting  $\omega = p = 0$ ,  $\tau = 1$ ,  $h(x) = 1$  and  $g = I$ , [25, Theorem 2.4] is obtained.

**Theorem 2.7** Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$ ,  $\text{Range}(g) \subset [a, b]$ , be functions such that  $f$  is positive,  $f \in L_1[a, b]$ , and  $g$  is differentiable and strictly increasing. If  $f$  is harmonically convex on  $[a, b]$  and  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , then for operators (1.9) and (1.10) we have:

$$\begin{aligned} & f\left(\frac{2g(a)g(b)}{g(a) + g(b)}\right) \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} 1\right)\left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \\ & \leq \frac{1}{2} \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right) \\ & + \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \\ & \leq \frac{f(g(a)) + f(g(b))}{2} \left({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} 1\right)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right), \end{aligned} \quad (2.17)$$

where  $\psi(t) = \frac{1}{g(t)}$  for  $t \in [\frac{1}{b}, \frac{1}{a}]$  and  $\omega' = \omega(\frac{g(a)g(b)}{g(b)-g(a)})^\sigma$ .

**Proof** Multiplying (2.3) by  $2t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)$  then integrating over  $[0, \frac{1}{2}]$ , we have

$$\begin{aligned} & 2f\left(\frac{2g(a)g(b)}{g(a) + g(b)}\right) \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) dt \\ & \leq \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(a) + (1-t)g(b)}\right) dt \\ & + \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt. \end{aligned} \quad (2.18)$$



Setting  $g(x) = \frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$  in (2.18) and using  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , as well as (1.9) and (1.10), the first inequality of (2.17) can be obtained.

For the second inequality, multiplying (2.5) by  $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p)$  then integrating over  $[0, \frac{1}{2}]$ , we get

$$\begin{aligned} & \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) dt \\ & + \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right) dt \\ & \leq (f(g(a)) + f(g(b))) \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) dt. \end{aligned} \quad (2.19)$$

Setting  $g(x) = \frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$  in (2.19) and using  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , as well as (1.9) and (1.10), the second inequality of (2.17) can be obtained.  $\square$

**Remark 2.8**

- (i) By setting  $p = 0$  and  $g = I$ , [1, Theorem 3.3] is obtained.
- (ii) By setting  $g = I$ , [9, Theorem 2.3] is obtained.
- (iii) By setting  $\omega = p = 0$  and  $g = I$ , [25, Theorem 4] is obtained.

To prove the next result, we will use the following lemma:

**Lemma 2.9** Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$ ,  $\text{Range}(g) \subset [a, b]$ , be functions such that  $f$  is positive,  $f \in L_1[a, b]$ , and  $g$  is differentiable and strictly increasing. If  $f$  is harmonically convex and  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , then for fractional integral operators (1.9) and (1.10) we have:

$$\begin{aligned} & ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi) \left( g^{-1}\left(\frac{1}{g(a)}\right); p \right) \\ & = ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi) \left( g^{-1}\left(\frac{1}{g(b)}\right); p \right) \\ & = \frac{1}{2} \left( ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi) \left( g^{-1}\left(\frac{1}{g(a)}\right); p \right) \right. \\ & \quad \left. + ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi) \left( g^{-1}\left(\frac{1}{g(b)}\right); p \right) \right), \\ & \psi(t) = \frac{1}{g(t)}, t \in \left[ \frac{1}{b}, \frac{1}{a} \right]. \end{aligned} \quad (2.20)$$

*Proof* By using Definition 4, we can write

$$\begin{aligned} & ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))}^{\rho,r,k,c} f \circ \psi) \left( g^{-1}\left(\frac{1}{g(a)}\right); p \right) \\ & = \int_{g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}}^{g^{-1}(\frac{1}{g(a)})} \left( \frac{1}{g(a)} - g(t) \right)^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c} \left( \omega \left( \frac{1}{g(a)} - g(t) \right)^\sigma; p \right) f\left(\frac{1}{g(t)}\right) d(g(t)). \end{aligned} \quad (2.21)$$

By replacing  $g(t)$  with  $\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)$  in equation (2.21) and using the condition  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , we have

$$\begin{aligned} & \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &= \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right). \end{aligned} \quad (2.22)$$

By adding  $({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} f \circ \psi)(g^{-1}(\frac{1}{g(a)}); p)$  on both sides of (2.22), we have

$$\begin{aligned} & 2 \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &= \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &+ \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega,(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} f \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right) \end{aligned} \quad (2.23)$$

Equations (2.22) and (2.23) give the required result.  $\square$

**Theorem 2.10** Let  $f, g, h : [a, b] \rightarrow \mathbb{R}$ ,  $0 < a < b$ ,  $\text{Range}(g), \text{Range}(h) \subset [a, b]$ , be functions such that  $f$  is positive,  $f \in L_1[a, b]$ ,  $g$  is differentiable, strictly increasing, and  $h$  is non-negative and integrable. If  $f$  is harmonically convex and  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , then for fractional integral operators (1.9) and (1.10) we have

$$\begin{aligned} & f \left( \frac{2g(a)g(b)}{g(a) + g(b)} \right) \left( \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} h \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \right. \\ &+ \left. \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} h \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right) \right) \\ &\leq \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} fh \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \\ &+ \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} fh \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right) \\ &\leq \frac{f(g(a)) + f(g(b))}{2} \left( \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^+}^{\rho,r,k,c} h \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(a)} \right); p \right) \right. \\ &+ \left. \left( {}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega',(g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} h \circ \psi \right) \left( g^{-1} \left( \frac{1}{g(b)} \right); p \right) \right), \end{aligned} \quad (2.24)$$

where  $\psi(t) = \frac{1}{g(t)}$  for  $t \in [\frac{1}{b}, \frac{1}{a}]$ ,  $fh \circ \psi = (f \circ \psi)(h \circ \psi)$  and  $\omega' = \omega(\frac{g(a)g(b)}{g(b)-g(a)})^\sigma$ .

*Proof* Multiplying (2.3) by  $t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) h(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)})$  then integrating over  $[0, \frac{1}{2}]$ , we have

$$2f \left( \frac{2g(a)g(b)}{g(a) + g(b)} \right) \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) h \left( \frac{g(a)g(b)}{tg(b) + (1-t)g(a)} \right) dt$$

$$\begin{aligned} &\leq \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(a) + (1-t)g(b)}\right) h\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt \\ &\quad + \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) h\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt. \end{aligned} \quad (2.25)$$

By choosing  $g(x) = \frac{tg(b) + (1-t)g(a)}{g(a)g(b)}$  and using the condition  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$  in (2.25), we have

$$\begin{aligned} &2f\left(\frac{2g(a)g(b)}{g(a) + g(b)}\right) ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} h \circ \psi)\left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \\ &\leq ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^{+}}^{\rho,r,k,c} fh \circ \psi)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right) \\ &\quad + ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} fh \circ \psi)\left(g^{-1}\left(\frac{1}{g(b)}\right); p\right). \end{aligned} \quad (2.26)$$

Using Lemma 2.9 in the above inequality, one can get the first inequality of (2.24). For the second part of inequality of (2.24), multiplying (2.5) by  $t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) h(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)})$  then integrating over  $[0, \frac{1}{2}]$ , we have

$$\begin{aligned} &\int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) h\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt \\ &\quad + \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) f\left(\frac{g(a)g(b)}{tg(a) + (1-t)g(b)}\right) h\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt \\ &\leq (f(g(a)) + f(g(b))) \int_0^{\frac{1}{2}} t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; p) h\left(\frac{g(a)g(b)}{tg(b) + (1-t)g(a)}\right) dt. \end{aligned} \quad (2.27)$$

Setting  $g(x) = \frac{tg(b) + (1-t)g(a)}{g(a)g(b)}$  in (2.27) and using (1.9), (1.10), as well as the condition  $f(\frac{1}{g(x)}) = f(\frac{1}{\frac{1}{g(a)} + \frac{1}{g(b)} - g(x)})$ , we have

$$\begin{aligned} &({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^{+}}^{\rho,r,k,c} fh \circ \psi)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right) \\ &\quad + ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^-}^{\rho,r,k,c} fh \circ \psi)\left(g^{-1}\left(\frac{1}{g(b)}\right); p\right) \\ &\leq (f(g(a)) + f(g(b))) ({}_g\mathcal{Y}_{\sigma,\tau,\delta,\omega', (g^{-1}(\frac{g(a)+g(b)}{2g(a)g(b)}))^{+}}^{\rho,r,k,c} h \circ \psi)\left(g^{-1}\left(\frac{1}{g(a)}\right); p\right). \end{aligned} \quad (2.28)$$

Again using Lemma 2.9 in (2.28), the second inequality of (2.24) can be obtained.  $\square$

**Remark 2.11**

- (i) By setting  $p = 0$  and  $g = I$ , [1, Theorem 3.6] is obtained.
- (ii) By setting  $g = I$ , [9, Theorem 2.6] is obtained.
- (iii) By setting  $\omega = p = 0$ ,  $g = I$  and  $\tau = 1$ , [3, Theorem 8], is obtained.

**Corollary 2.12** *Setting  $\omega = p = 0$  and  $g = I$  in Theorem 2.10, we get the following inequalities via Riemann–Liouville fractional integrals:*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) \left( (I_{\frac{a+b}{2ab}}^{\tau} h \circ \psi) \left( \frac{1}{a} \right) + (I_{\frac{a+b}{2ab}}^{\tau} h \circ \psi) \left( \frac{1}{b} \right) \right) \\ & \leq (I_{\frac{a+b}{2ab}}^{\tau} fh \circ \psi) \left( \frac{1}{a} \right) + (I_{\frac{a+b}{2ab}}^{\tau} fh \circ \psi) \left( \frac{1}{b} \right) \\ & \leq \frac{f(a) + f(b)}{2} \left( (I_{\frac{a+b}{2ab}}^{\tau} h \circ \psi) \left( \frac{1}{a} \right) + (I_{\frac{a+b}{2ab}}^{\tau} h \circ \psi) \left( \frac{1}{b} \right) \right). \end{aligned}$$

### 3 Concluding remarks

This paper investigates generalized fractional integral inequalities of Hadamard and Fejér–Hadamard-type for harmonically convex functions. Presented results are generalizations of several inequalities given in [1, 3, 9, 15, 25]. The results of this paper also hold for fractional integral operators defined in [2, 31, 32, 35, 40] and are deducible from the generalized fractional integral operators given in (1.9) and (1.10), see [33, Remark 1].

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