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A global QP-free algorithm for mathematical programs with complementarity constraints

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Abstract

In this paper, a primal–dual interior point QP-free algorithm for mathematical programs with complementarity constraints is presented. Firstly, based on Fischer–Burmeister function and smoothing techniques, the investigated problem is approximated by a smooth nonlinear constrained optimization problem. Secondly, combining with an effective penalty function technique and working set, a QP-free algorithm is proposed to solve the smooth constrained optimization problem. At each iteration, only two reduced linear equations with the same coefficient matrix are solved to obtain the search direction. Under some mild conditions, the proposed algorithm possesses global convergence. Finally, some numerical results are reported.

Keywords: Complementarity constraints; Working set; QP-free algorithm; Global convergence

1 Introduction

In this paper, we discuss the following mathematical programming problem with complementarity constraints (MPCC for short):

$$\begin{aligned} \min \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0, \\ & 0 \le F(x,y) \perp y \ge 0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^{n+m} \to \mathbb{R}$, $g = (g_1,\dots,g_{m_g})^{\mathrm{T}} : \mathbb{R}^{n+m} \to \mathbb{R}^{m_g}$, $F = (F_1,\dots,F_m)^{\mathrm{T}} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ are continuously differentiable functions. "$F(x,y) \perp y$" means that the vectors $F(x,y)$ and y are perpendicular to each other.

MPCC (1) has a broad of applications in real world, such as engineering design, traffic transportation, game theory and so on. A detailed overview of MPCC applications can be found in [1] and the monographs [2–4].

Since MPCC (1) is a nonconvex optimization problem and the standard Mangasarian–Fromovitz constraint qualification (MFCQ) is violated at any feasible point, the well-developed algorithms for the standard nonlinear programs (for example, [5–18]) typically have severe difficulties if they are directly used to solve the MPCC (1). Hence, MPCC-tailed algorithms are desired.

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It is well known that the QP-free method is one of the efficient methods for nonlinear programming (see [14, 15, 19–25]). The nice properties of the QP-free method are as follows: (a) The search directions are determined only by solving systems of linear equations rather than solving QP-subproblems. As a consequence, the computational cost is decreased greatly. (b) Compared with the SQP method, the size of the investigated problem solved by QP-free method is larger. It is worth mentioning that the primal–dual interior point QP-free algorithm in [21] has an improvement, that is, the strict restriction that the Lagrangian Hessian estimate is uniformly positive is relaxed. It is noticed that the active set identifying technique is incorporated into the QP-free algorithms in [22, 23]. Consequently, the computational cost is further decreased and the numerical performance in [22] is encouraging.

Although there are many OP-free algorithms for nonlinear programming, to the best of our knowledge, there are few MPCC-tailed QP-free algorithms for the MPCC (1). Motivated by the ideas of the algorithms in [21, 24–27] and combining with smoothing techniques, we propose a primal–dual interior point QP-free algorithm for the MPCC (1). The proposed algorithm possesses the following nice properties:

(a) The technique of active set is incorporated into the algorithm. As a consequence, the size of SLEs becomes smaller and the computational cost is decreased.
(b) At each iteration, the search direction is obtained by solving two SLEs with the same coefficient matrix, which further decreases the computational cost.
(c) The uniformly positive definiteness on the Lagrangian Hessian estimate H_k is relaxed.
(d) The algorithm possesses global convergence without assuming that the stationary points are isolated.

2 Preliminaries and reformulation

In this section, for completeness, we first restate some definitions and results about the MPCC (1), then deduce an approximation problem of the MPCC (1) by Fischer–Burmeister function and smoothing techniques.

Denoted by X_0 the feasible set of the MPCC (1), and denoted by $I^{\mathcal{C}} = \{1, 2, \dots, m\}$ and $I^g = \{1, 2, \dots, m_g\}$ are the index sets of the complementarity constraints and the inequality constraints, respectively.

Definition 1 ([2]) Given $(x^*, y^*) \in X_0$, if

$$\bigl(y_i^*, F_i(x^*, y^*)\bigr) \neq (0, 0), \quad \forall i \in I^{\mathcal{C}}, \tag{2}$$

then we call that the lower-level strict complementarity (LLSC) is satisfied at (x^*, y^*).

Definition 2 ([2]) A point $(x^*, y^*) \in X_0$ is said to be a KKT stationary point of the MPCC (1) if there exists a KKT multiplier vector $(\lambda^*, \omega^*, \mu^*) \in \mathbb{R}^{m+m+m_g}$ such that

$$\begin{aligned}
&\nabla f(x^*, y^*) + \nabla g(x^*, y^*)\lambda^* - \nabla F(x^*, y^*)\omega^* - \begin{pmatrix} 0_{n\times m} \\ E_m \end{pmatrix} \mu^* = 0, \\
&0 \le -g(x^*, y^*) \perp \lambda^* \ge 0; \qquad F(x^*, y^*) \ge 0, \quad y^* \ge 0, \qquad F_i(x^*, y^*) y_i^* = 0; \\
&\omega_i^* = 0, \quad \text{if } F_i(x^*, y^*) > 0; \qquad \mu_i^* = 0, \quad \text{if } y_i^* > 0;
\end{aligned}$$

$$\omega_i^* \geq 0, \qquad \mu_i^* \geq 0, \quad \text{if } F_i(x^*, y^*) = y_i^* = 0,$$

where $E_m \in \mathbb{R}^{m\times m}$ is an mth-order identity matrix. The vector $((x^*,y^*),\lambda^*,\omega^*,\mu^*)$ is said to be a KKT stationary pair of the MPCC (1).

Proposition 1 ([2]) *Suppose that $(x^*,y^*)\in X_0$ satisfies LLSC, then $((x^*,y^*),\lambda^*,\omega^*,\mu^*)$ is a KKT stationary pair of the MPCC (1) if and only if there exists a vector $\iota^* = (\iota_i^*, i\in I^{\mathcal{C}})\in\mathbb{R}^m$, such that*

$$\begin{aligned}
&\begin{pmatrix} \nabla_x f(x^*,y^*) \\ \nabla_y f(x^*,y^*) \\ 0_{m\times 1} \end{pmatrix} + \begin{pmatrix} \nabla_x g(x^*,y^*) \\ \nabla_y g(x^*,y^*) \\ 0_{m\times l} \end{pmatrix}\lambda^* - \begin{pmatrix} \nabla_x F(x^*,y^*) \\ \nabla_y F(x^*,y^*) \\ -E_m \end{pmatrix}\omega^* - \begin{pmatrix} 0_{n\times m} \\ V^* \\ Y^* \end{pmatrix}\iota^* = 0, \\
&0 \leq -g(x^*,y^*) \perp \lambda^* \geq 0, \qquad 0 \leq F(x^*,y^*) \perp y^* \geq 0, \\
&\iota_i^* = \begin{cases} \mu_i^*/F_i(x^*,y^*), & \text{if } F_i(x^*,y^*) > 0, \\ \omega_i^*/y_i^*, & \text{if } y_i^* > 0, \end{cases}
\end{aligned}$$

where $V^* = \operatorname{diag}(F_i(x^*,y^*), i\in I^{\mathcal{C}})$, $Y^* = \operatorname{diag}(y_i^*, i\in I^{\mathcal{C}})$.

It is well known that the Fischer–Burmeister function is a complementarity function, which is defined by

$$\psi(a,b) = a + b - \sqrt{a^2+b^2}.$$

Obviously, ψ satisfies the following basic property:

$$\psi(a,b) = 0 \quad \Longleftrightarrow \quad a\geq 0, b\geq 0, ab = 0. \tag{3}$$

And ψ is continuous differentiable in $\{(a,b)\in\mathbb{R}^2 | (a,b)\neq(0,0)\}$, namely

$$\psi_a'(a,b) = 1 - \frac{a}{\sqrt{a^2+b^2}}; \qquad \psi_b'(a,b) = 1 - \frac{b}{\sqrt{a^2+b^2}}, \quad (a,b)\neq(0,0).$$

Let $v = F(x,y)$, according to the property (3), the MPCC (1) is equivalently transformed into the following nonsmooth nonlinear optimization problem:

$$\begin{aligned}
\min \quad & f(x,y) \\
\text{s.t.} \quad & v - F(x,y) = 0, \\
& \Psi(y,v) = 0, \\
& g(x,y) \leq 0,
\end{aligned} \tag{4}$$

where $\Psi(y,v) = (\psi(y_i,v_i), i\in I^{\mathcal{C}})$.

Obviously, the function ψ is not differentiable at the point $(0,0)$. Borrowing ideas from [28], we define the functions as follows:

$$\psi_\varepsilon(y_i,v_i) = \begin{cases} \psi(y_i,v_i), & i\in I^{\mathcal{C}}\setminus I^{\mathcal{C}}(y,v,\varepsilon), \\ \frac{2\varepsilon-y_i}{2\varepsilon}y_i + \frac{2\varepsilon-v_i}{2\varepsilon}v_i - \frac{\varepsilon}{2}, & i\in I^{\mathcal{C}}(y,v,\varepsilon), \end{cases}$$

$$\overline{\psi}_\varepsilon(y_i,v_i)=\begin{cases}0, & i\in I^{\mathcal{C}}\setminus I^{\mathcal{C}}(y,v,\varepsilon),\\ \frac{(\sqrt{y_i^2+v_i^2}-\varepsilon)^2}{2\varepsilon}, & i\in I^{\mathcal{C}}(y,v,\varepsilon),\end{cases}$$

where the index set $I^{\mathcal{C}}(y,v,\varepsilon)=\{i\in I^{\mathcal{C}}\mid \sqrt{y_i^2+v_i^2}<\varepsilon,\varepsilon>0\}$. Moreover, define

$$\Psi_\varepsilon(y,v)=\bigl(\psi_\varepsilon(y_i,v_i),i\in I^{\mathcal{C}}\bigr),\qquad \overline{\Psi}_\varepsilon(y,v)=\bigl(\overline{\psi}_\varepsilon(y_i,v_i),i\in I^{\mathcal{C}}\bigr),$$

so we have

$$\Psi(y,v)=\Psi_\varepsilon(y,v)+\overline{\Psi}_\varepsilon(y,v). \tag{5}$$

For any $\varepsilon>0$, the function $\psi_\varepsilon(y_i,v_i)$ is differentiable, and it follows that

$$\psi'_{\varepsilon,y_i}(y_i,v_i)=\frac{\partial\psi_\varepsilon(y_i,v_i)}{\partial y_i}=\begin{cases}1-\frac{y_i}{\sqrt{y_i^2+v_i^2}}, & i\in I^{\mathcal{C}}\setminus I^{\mathcal{C}}(y,v,\varepsilon),\\ 1-\frac{y_i}{\varepsilon}, & i\in I^{\mathcal{C}}(y,v,\varepsilon),\end{cases}$$

$$\psi'_{\varepsilon,v_i}(y_i,v_i)=\frac{\partial\psi_\varepsilon(y_i,v_i)}{\partial v_i}=\begin{cases}1-\frac{v_i}{\sqrt{y_i^2+v_i^2}}, & i\in I^{\mathcal{C}}\setminus I^{\mathcal{C}}(y,v,\varepsilon),\\ 1-\frac{v_i}{\varepsilon}, & i\in I^{\mathcal{C}}(y,v,\varepsilon).\end{cases}$$

So the smooth problem

$$\begin{aligned}\min\quad & f(x,y)\\ \text{s.t.}\quad & v-F(x,y)=0,\\ & \Psi_\varepsilon(y,v)=0,\\ & g(x,y)\le 0\end{aligned} \tag{6}$$

will be used as an approximation of the MPCC (1). Obviously, if $I^{\mathcal{C}}(y^*,v^*,\varepsilon)=\emptyset$, then the problem (6) is equivalent with the problem (4). Under some mild conditions, $I^{\mathcal{C}}(y^*,v^*,\varepsilon)=\emptyset$ is guaranteed, where (x^*,y^*,v^*) is an accumulation point of the iterative sequence $\{(x^k,y^k,v^k)\}$ (see the proof of Theorem 1 in the sequel).

Define a penalty function $f_r(x,y)$ for the problem (6) by

$$f_r(x,y)=f(x,y)-r\sum_{i\in I^{\mathcal{C}}}\bigl(v_i-F_i(x,y)\bigr)-r\sum_{i\in I^{\mathcal{C}}}\psi_\varepsilon(y_i,v_i),$$

where $r>0$ is a penalty parameter. Similar to [29], we can convert the problem (6) to the following smoothing optimization only with inequality constraints:

$$\begin{aligned}\min\quad & f_r(x,y)\\ \text{s.t.}\quad & v_i-F_i(x,y)\le 0,\quad i\in I^{\mathcal{C}},\\ & \psi_\varepsilon(y_i,v_i)\le 0,\quad i\in I^{\mathcal{C}},\\ & g_i(x,y)\le 0,\quad i\in I^g.\end{aligned} \tag{7}$$

For simplicity, we use the following notations throughout this paper:

$$
\begin{aligned}
& z = (x,y,v), \qquad u = (x,y), \qquad w = (y,v),\\
& dz = (dx,dy,dv), \qquad du = (dx,dy), \qquad dw = (dy,dv),\\
& z^k = \left(x^k,y^k,v^k\right), \qquad u^k = \left(x^k,y^k\right), \qquad w^k = \left(y^k,v^k\right),\\
& dz^k = \left(dx^k,dy^k,dv^k\right), \qquad du^k = \left(dx^k,dy^k\right), \qquad dw^k = \left(dy^k,dv^k\right),\\
& e_I = (1,1,\dots,1)^T \in \mathbb{R}^{2m+m_g}, \qquad e_{I^{\mathcal{C}}} = (1,1,\dots,1)^T \in \mathbb{R}^m,\\
& X := \left\{z \in \mathbb{R}^{n+2m} \mid v_i - F_i(x,y) = 0, i \in I^{\mathcal{C}}; \psi_\varepsilon(y_i,v_i) = 0, i \in I^{\mathcal{C}}; g_i(x,y) \le 0, i \in I^g\right\},\\
& \widetilde{X} := \left\{z \in \mathbb{R}^{n+2m} \mid v_i - F_i(x,y) \le 0, i \in I^{\mathcal{C}}; \psi_\varepsilon(y_i,v_i) \le 0, i \in I^{\mathcal{C}}; g_i(x,y) \le 0, i \in I^g\right\},\\
& \widetilde{X}_0 := \left\{z \in \mathbb{R}^{n+2m} \mid v_i - F_i(x,y) < 0, i \in I^{\mathcal{C}}; \psi_\varepsilon(y_i,v_i) < 0, i \in I^{\mathcal{C}}; g_i(x,y) < 0, i \in I^g\right\},\\
& \varphi(x,y) = \max\left\{0; g_i(x,y), i \in I^g\right\}, \qquad I(x,y) = \left\{i \in I^g \mid g_i(x,y) = \varphi(x,y)\right\},\\
& I_0^F(z) = \left\{i \in I^{\mathcal{C}} \mid v_i - F_i(x,y) = 0\right\}, \qquad I_0^{\Psi_\varepsilon}(z) = \left\{i \in I^{\mathcal{C}} \mid \psi_\varepsilon(y_i,v_i) = 0\right\},\\
& I_0^g(z) = \left\{i \in I^g \mid g_i(x,y) = 0\right\}.
\end{aligned}
$$

The following proposition shows the equivalence between the problem (6) and the problem (7).

Proposition 2 *If (z,λ^r) is a KKT pair of the problem (7), and $v - F(x,y) = 0$, $\Psi_\varepsilon(y,v) = 0$, then (z,λ) with multiplier $\lambda = \lambda^r - r\tilde{e}$ is a KKT pair of the problem (6), where $\lambda^r := (\lambda^r_{F,I^{\mathcal{C}}}, \lambda^r_{\Psi_\varepsilon,I^{\mathcal{C}}}, \lambda^r_{g,I^g})$, $\lambda := (\lambda_{F,I^{\mathcal{C}}}, \lambda_{\Psi_\varepsilon,I^{\mathcal{C}}}, \lambda_{g,I^g})$, $\tilde{e} = (1,\dots,1_{mth},1,\dots,1_{(2m)th},0,\dots,0_{(2m+m_g)th})^{\mathrm{T}}$.*

Proof Since (z,λ^r) is a KKT pair of the problem (7), the vector pair (z,λ^r) satisfies the following relations:

$$
\begin{aligned}
& \nabla_z f_r(x,y) + \nabla_z\big(v - F(x,y)\big)\lambda^r_{F,I^{\mathcal{C}}} + \nabla_z \Psi_\varepsilon(y,v)\lambda^r_{\Psi_\varepsilon,I^{\mathcal{C}}} + \nabla_z g(x,y)\lambda^r_{g,I^g} = 0,\\
& \lambda^r_{F,i} \ge 0, v_i - F_i(x,y) \le 0, \quad \lambda^r_{F,i}\big(v_i - F_i(x,y)\big) = 0, \quad i \in I^{\mathcal{C}},\\
& \lambda^r_{\Psi_\varepsilon,i} \ge 0, \psi_\varepsilon(y_i,v_i) \le 0, \quad \lambda^r_{\Psi_\varepsilon,i}\psi_\varepsilon(y_i,v_i) = 0, \quad i \in I^{\mathcal{C}},\\
& \lambda^r_{g,i} \ge 0, g_i(x,y) \le 0, \quad \lambda^r_{g,i} g_i(x,y) = 0, \quad i \in I^g.
\end{aligned}
$$

Note that

$$
\nabla_z f_r(x,y) = \nabla_z f(x,y) - r\sum_{i\in I^{\mathcal{C}}} \nabla_z\big(v_i - F_i(x,y)\big) - r\sum_{i\in I^{\mathcal{C}}} \nabla_z \psi_\varepsilon(y_i,v_i),
$$

we get

$$
\nabla_z f(x,y) + \nabla_z\big(v - F(x,y)\big)\big(\lambda^r_{F,I^{\mathcal{C}}} - re_{I^{\mathcal{C}}}\big) + \nabla_z\Psi_\varepsilon(y,v)\big(\lambda^r_{\Psi_\varepsilon,I^{\mathcal{C}}} - re_{I^{\mathcal{C}}}\big) + \nabla_z g(x,y)\lambda^r_{g,I^g} = 0.
$$

Let $\lambda_{F,I^{\mathcal{C}}} = \lambda^r_{F,I^{\mathcal{C}}} - re_{I^{\mathcal{C}}}$, $\lambda_{\Psi_\varepsilon,I^{\mathcal{C}}} = \lambda^r_{\Psi_\varepsilon,I^{\mathcal{C}}} - re_{I^{\mathcal{C}}}$, $\lambda_{g,I^g} = \lambda^r_{g,I^g}$, we obtain

$$
\nabla_z f(x,y) + \nabla_z\big(v - F(x,y)\big)\lambda_{F,I^{\mathcal{C}}} + \nabla_z\Psi_\varepsilon(y,v)\lambda_{\Psi_\varepsilon,I^{\mathcal{C}}} + \nabla_z g(x,y)\lambda_{g,I^g} = 0.
$$

In view of $v - F(x,y) = 0$, $\Psi_\varepsilon(y,v) = 0$, $g(x,y) \le 0$, and

$$\lambda_{g,i} \ge 0, \quad \lambda_{g,i} g_i(x,y) = 0, \quad i \in I^g.$$

Let $\lambda := (\lambda_{F,I^C}, \lambda_{\Psi_\varepsilon,I^C}, \lambda_{g,I^g}) = \lambda^r - r\tilde{e}$, so (z,λ) is a KKT pair of the problem (6). □

3 Description of the algorithm

For the sake of theoretical analysis, we make a basic assumption throughout this paper.

Assumption 1

(1) For any $(x,y) \in \mathbb{R}^{n+m}$, the matrix $\nabla_y F(x,y)$ is a P_0 matrix, i.e., all the principal minors of $\nabla_y F(x,y)$ are nonnegative.
(2) For any $(x,y) \in \mathbb{R}^{n+m}$, the submatrix $(\nabla_y F(x,y))_{J^*J^*}$ of the matrix $\nabla_y F(x,y)$ is nonsingular, where the index set $J^* = \{i \in I^C \mid \psi'_{\varepsilon,y_i}(y_i,v_i) = 0\}$.

In order to construct the coefficient matrix of linear equations conveniently, denote

$$\Omega_J := \Omega_J(z,\varepsilon) = \begin{pmatrix} v - F(x,y) \\ \Psi_\varepsilon(y,v) \\ g_J(x,y) \end{pmatrix},$$

and denote by A_J the gradient matrix of Ω_J, that is,

$$A_J := A_J(z,\varepsilon) = \begin{pmatrix} -\nabla_x F(x,y) & 0 & \nabla_x g_J(x,y) \\ -\nabla_y F(x,y) & \nabla_y \Psi_\varepsilon(y,v) & \nabla_y g_J(x,y) \\ E_m & \nabla_v \Psi_\varepsilon(y,v) & 0 \end{pmatrix},$$

where $J \subseteq I^g$, the diagonal matrix

$$\nabla_y \Psi_\varepsilon(y,v) = \operatorname{diag}\bigl(\psi'_{\varepsilon,y_i}(y_i,v_i), i \in I^C\bigr), \qquad \nabla_v \Psi_\varepsilon(y,v) = \operatorname{diag}\bigl(\psi'_{\varepsilon,v_i}(y_i,v_i), i \in I^C\bigr).$$

Define the matrix

$$U := U(z,\varepsilon) = \begin{pmatrix} -\nabla_y F(x,y) & \nabla_y \Psi_\varepsilon(y,v) \\ E_m & \nabla_v \Psi_\varepsilon(y,v) \end{pmatrix}. \tag{8}$$

Similar to the proof in [30, 31], we can prove that the following proposition is true.

Proposition 3

(1) *Suppose that the matrix U is nonsingular. Then the matrix $A_{I(x,y)}$ is full of column rank, if and only if the matrix*

$$\nabla_x g_{I(x,y)}(x,y) - \nabla_x F(x,y)\bigl(U^{-1}\bigr)_m \nabla_y g_{I(x,y)}(x,y)$$

is full of column rank, where $(U^{-1})_m$ is the mth-order principal submatrix of U^{-1}, which consists of the first m rows and m columns of U^{-1}.
(2) *Suppose that Assumption 1 holds, then the matrix U is nonsingular.*

Assumption 2 For any $z \in \mathbb{R}^{n+2m}$, the matrix

$$\nabla_x g_{I(x,y)}(x,y) - \nabla_x F(x,y)\left(U^{-1}\right)_m \nabla_y g_{I(x,y)}(x,y)$$

is full of column rank.

Remark 1 According to Proposition 3, if Assumption 1 is true, then Assumption 2 is equivalent to the LICQ of the problem (7).

Based on Proposition 2, we know that if one can construct an efficient algorithm for the problem (7) and adjust penalty parameter r to force the iterate to asymptotically satisfy $v - F(x,y) = 0$, $\Psi_\varepsilon(y,v) = 0$, then the solution to the problem (6) can be yielded.

Define the following optimal identification function:

$$\rho(z,\lambda) = \left\|\Phi(z,\lambda)\right\|^{\vartheta}, \tag{9}$$

where

$$\Phi(z,\lambda) = \begin{pmatrix} \nabla_z L(z,\lambda) \\ v - F(x,y) \\ \Psi_\varepsilon(y,v) \\ \min\{-g(x,y), \lambda_{g,I^g}\} \end{pmatrix},$$

and $\lambda = (\lambda_{F,I^C}, \lambda_{\Psi_\varepsilon,I^C}, \lambda_{g,I^g})$, the parameter $\vartheta \in (0,1)$, $\|\cdot\|$ is the Euclidean norm, and we have the Lagrangian function

$$L(z,\lambda) = f(x,y) + \sum_{i\in I^C} \lambda_{F,i}\left(v_i - F_i(x,y)\right) + \sum_{i\in I^C} \lambda_{\Psi_\varepsilon,i}\psi_\varepsilon(y_i,v_i) + \sum_{i\in I^g} \lambda_{g,i} g_i(x,y).$$

Obviously, ρ is nonnegative and continuous. It follows from [32] that $\rho(z^*,\lambda^*) = 0$ if and only if (z^*,λ^*) is a KKT pair of the problem (6).

For the current iterate $z^k \in \widetilde{X}_0$, we define the corresponding multiplier vector $\lambda^k = (\lambda^k_{F,I^C}, \lambda^k_{\Psi_\varepsilon,I^C}, \lambda^k_{g,I^g})$ in (9) by

$$\lambda^0 = s^0, \qquad \lambda^k = \widehat{\lambda}^{k-1} - r_{k-1}\tilde{e}, \quad k \geq 1, \tag{10}$$

where $s^0 > 0$, and $(\widehat{\lambda}^{k-1}, r_{k-1})$ is computed in the previous $(k-1)$th iteration, we construct the working set I_k as follows:

$$I_k = \left\{i \in I^g \mid g_i\left(x^k,y^k\right) + \rho\left(z^k,\lambda^k\right) \geq 0\right\}. \tag{11}$$

When (z^k,λ^k) is sufficiently close to a KKT pair (z^*,λ^*) of the problem (6), and the second-order sufficient conditions and the MFCQ hold at (z^*,λ^*), the set I_k equals the precise identification set $I^g_0(z^*)$. Therefore, only inequality constraints associated with the working set I_k need to be considered.

For the current iterate $z^k \in \mathbb{R}^{n+2m}$, the first SLE in our algorithm is constructed as follows:

$$\text{SLE1:}\quad M_{I_k}\begin{pmatrix} dz \\ \lambda_{I_k}\end{pmatrix} = \begin{pmatrix} -\nabla_z f_{r_k}(x^k,y^k) \\ 0\end{pmatrix}, \tag{12}$$

where the coefficient matrix M_{I_k} is defined by

$$M_{I_k} := M_{I_k}(z^k, S_{I_k}) = \begin{pmatrix} H_k & A_{I_k} \\ S_{I_k}(A_{I_k})^{\mathrm{T}} & G_{I_k}\end{pmatrix} \tag{13}$$

with

$$G_{I_k} = \operatorname{diag}\bigl(v^k - F(x^k,y^k), \Psi_\varepsilon(y^k,v^k), g_{I_k}(x^k,y^k)\bigr),$$

the matrix $H_k \in \mathbb{R}^{n+2m}$ is an approximation of the Lagrangian Hessian, the matrix $S_{I_k} := \operatorname{diag}(s^k_{I_k})$, where the vector $s^k_{I_k} := (s^k_{F,I^{\mathcal{C}}}, s^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, s^k_{g,I_k}) \in \mathbb{R}^{2m+|I_k|}$ is an approximation of $\lambda^k_{I_k} := (\lambda^k_{F,I^{\mathcal{C}}}, \lambda^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \lambda^k_{g,I_k})$.

It is not difficult to prove that the following result is true.

Lemma 1 *Suppose that Assumptions 1–2 hold, and the symmetric matrix H_k satisfies the following relation*:

$$\begin{aligned} H_k \succ {}& \sum_{i\in I^{\mathcal{C}}} \frac{s^k_{F,i}}{v^k_i - F_i(x^k,y^k)} \nabla_z\bigl(v^k_i - F_i(x^k,y^k)\bigr)\nabla_z\bigl(v^k_i - F_i(x^k,y^k)\bigr)^{\mathrm{T}} \\ &+ \sum_{i\in I^{\mathcal{C}}} \frac{s^k_{\Psi_\varepsilon,i}}{\psi_\varepsilon(y^k_i,v^k_i)} \nabla_z\psi_\varepsilon\bigl(y^k_i,v^k_i\bigr)\nabla_z\psi_\varepsilon\bigl(y^k_i,v^k_i\bigr)^{\mathrm{T}} \\ &+ \sum_{i\in I_k} \frac{s^k_{g,i}}{g_i(x^k,y^k)} \nabla_z g_i\bigl(x^k,y^k\bigr)\nabla_z g_i\bigl(x^k,y^k\bigr)^{\mathrm{T}}, \end{aligned} \tag{14}$$

then the coefficient matrix M_{I_k} defined by (13) *is invertible, where the matrix order $A \succ B$ means that $A - B$ is positive definite.*

We now describe our algorithm as follows.

Algorithm A *Step 0.* (Initialization)

Choose an initial point $z^0 \in \widetilde{X}_0$; $\alpha \in (0,\frac{1}{2})$; $\vartheta,\beta,\theta \in (0,1)$; $r_0,\varepsilon_0,C,\nu,\gamma_1,\gamma_2,\gamma_3,\gamma_4 > 0$; $\tau > 1$; $q > 2$; termination accuracy $\epsilon > 0$; the vector $s^0 := (s^0_{F,I^{\mathcal{C}}}, s^0_{\Psi_\varepsilon,I^{\mathcal{C}}}, s^0_{g,I^g}) \in \mathbb{R}^{2m+m_g}$ with $s^0_i \in [s_{\min}, s_{\max}]$, where $s_{\max} > s_{\min} > 0$. Set $k := 0$.

Step 1. (Compute the working set)

Yield λ^k by (10), then compute $\rho(z^k,\lambda^k)$ by (9). If $\rho(z^k,\lambda^k) \le \epsilon$, and $I^{\mathcal{C}}(y^k,v^k,\varepsilon_k) = \emptyset$, then z^k is a KKT point of the MPCC (1), stop; otherwise, compute the working set I_k by (11).

Step 2. (Yield a matrix H_k)

Generate a symmetric matrix H_k such that it is an approximation of the Lagrangian Hessian of the problem (7) and satisfies the condition (14).

Step 3. (Generate a search direction)

(1) Solve the first SLE1 (12). Let its solution be $(\widehat{dz}^k, \widehat{\lambda}^k_{I_k})$, with $\widehat{\lambda}^k_{I_k} := (\widehat{\lambda}^k_{F,I^{\mathcal{C}}}, \widehat{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \widehat{\lambda}^k_{g,I_k}) \in \mathbb{R}^{2m+|I_k|}$. Set $\widehat{\lambda}^k := (\widehat{\lambda}^k_{F,I^{\mathcal{C}}}, \widehat{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \widehat{\lambda}^k_{g,I^g})$, where $\widehat{\lambda}^k_{g,I^g} := (\widehat{\lambda}^k_{g,I_k}, 0_{I^g\setminus I_k})$.

(2) If the following conditions hold:
   (i) $\|\widehat{dz}^k\| \le \gamma_1$,
   (ii) $\widehat{\lambda}^k \ge -\gamma_2 e_I$,
   (iii) $\widehat{\lambda}^k_{F,I^{\mathcal{C}}} \not> \gamma_3 e_{I^{\mathcal{C}}}$,
   (iv) $\widehat{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}} \not> \gamma_4 e_{I^{\mathcal{C}}}$,
   (v) $I^{\mathcal{C}}(y^k, v^k, \varepsilon_k) \ne \emptyset$, then update the parameter r_k by $r_{k+1} = \tau r_k$, set $z^{k+1} = z^k$, $s^{k+1} = s^k$, $H_{k+1} = H_k$, $I_{k+1} = I_k$, $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k$, and go back to Step 3(1); otherwise, set $r_{k+1} = r_k$, go to Step 3(3).

(3) Find $\phi^k_{F,I^{\mathcal{C}}} = (\phi^k_{F,i}) \in \mathbb{R}^m$, $D^k_{\Psi_\varepsilon,I^{\mathcal{C}}} = (D^k_{\Psi_\varepsilon,i}) \in \mathbb{R}^m$, $Q^k_{g,I_k} = (Q^k_{g,i}) \in \mathbb{R}^{|I_k|}$, respectively, as follows:

$$\begin{aligned}\phi^k_{F,i} &:= \phi_i\big(z^k, \widehat{\lambda}^k_{F,i}\big)\\ &= \min\big\{0, -\big(\max\big\{-\widehat{\lambda}^k_{F,i}, 0\big\}\big)^\nu - C\big(v^k_i - F_i\big(x^k, y^k\big)\big)\big\}, \quad i \in I^{\mathcal{C}},\end{aligned} \tag{15}$$

$$D^k_{\Psi_\varepsilon,i} := D_i\big(z^k, \widehat{\lambda}^k_{\Psi_\varepsilon,i}\big) = \min\big\{0, -\big(\max\big\{-\widehat{\lambda}^k_{\Psi_\varepsilon,i}, 0\big\}\big)^\nu - C\psi_\varepsilon\big(y^k_i, v^k_i\big)\big\}, \quad i \in I^{\mathcal{C}}, \tag{16}$$

$$Q^k_{g,i} := Q_i\big(z^k, \widehat{\lambda}^k_{g,i}\big) = \min\big\{0, -\big(\max\big\{-\widehat{\lambda}^k_{g,i}, 0\big\}\big)^\nu - Cg_i\big(x^k, y^k\big)\big\}, \quad i \in I_k. \tag{17}$$

Denote $\varpi^k = (\phi^k_{F,I^{\mathcal{C}}}, D^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, Q^k_{g,I_k})$, then compute

$$\zeta^k = \nabla_z f_{r_k}\big(x^k, y^k\big)^{\mathrm{T}} \widehat{dz}^k - \sum_{i\in I^{\mathcal{C}}} \frac{\widehat{\lambda}^k_{F,i}}{s^k_{F,i}}\phi^k_{F,i} - \sum_{i\in I^{\mathcal{C}}} \frac{\widehat{\lambda}^k_{\Psi_\varepsilon,i}}{s^k_{\Psi_\varepsilon,i}} D^k_{\Psi_\varepsilon,i} - \sum_{i\in I_k} \frac{\widehat{\lambda}^k_{g,i}}{s^k_{g,i}} Q^k_{g,i}, \tag{18}$$

$$\begin{aligned}b_k = {}& \big(\big\|\widehat{dz}^k\big\|^q + \big\|\varpi^k\big\|\big)\Big(\sum_{i\in I^{\mathcal{C}}} \widehat{\lambda}^k_{F,i} + \sum_{i\in I^{\mathcal{C}}} \widehat{\lambda}^k_{\Psi_\varepsilon i} + \sum_{i\in I_k} \widehat{\lambda}^k_{g,i}\Big)\\ &+ \sum_{i\in I^{\mathcal{C}}} \frac{\widehat{\lambda}^k_{F,i}}{s^k_{F,i}}\phi^k_{F,i} + \sum_{i\in I^{\mathcal{C}}} \frac{\widehat{\lambda}^k_{\Psi_\varepsilon,i}}{s^k_{\Psi_\varepsilon,i}} D^k_{\Psi_\varepsilon,i} + \sum_{i\in I_k} \frac{\widehat{\lambda}^k_{g,i}}{s^k_{g,i}} Q^k_{g,i},\end{aligned} \tag{19}$$

$$\varrho_k = \begin{cases} 1, & \text{if } b_k \le 0,\\ \min\{\frac{(1-\theta)|\zeta^k|}{b_k}, 1\}, & \text{if } b_k > 0.\end{cases} \tag{20}$$

(4) Solve the second SLE as follows:

$$\text{SLE2:}\quad M_{I_k}\begin{pmatrix} dz\\ \lambda_{I_k}\end{pmatrix} = \begin{pmatrix} -\nabla_z f_{r_k}(x^k, y^k)\\ \eta^k_{I_k}\end{pmatrix}, \tag{21}$$

where the perturbation vector $\eta^k_{I_k} = (\eta^k_{F,I^{\mathcal{C}}}, \eta^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \eta^k_{g,I_k})$ is determined by the following convex combination:

$$\eta^k_{I_k} = (1-\varrho_k)\varpi^k + \varrho_k\big(-\big\|\widehat{dz}^k\big\| - \big\|\varpi^k\big\|\big)s^k_{I_k} \tag{22}$$

with $s^k_{I_k} := (s^k_{F,I^{\mathcal{C}}}, s^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, s^k_{g,I_k}) \in \mathbb{R}^{2m+|I_k|}$.

Let its solution be $(dz^k, \lambda^k_{I_k})$ with $\lambda^k_{I_k} := (\lambda^k_{F,I^{\mathcal{C}}}, \lambda^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \lambda^k_{g,I_k}) \in \mathbb{R}^{2m+|I_k|}$. Set $\lambda^k := (\lambda^k_{F,I^{\mathcal{C}}}, \lambda^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \lambda^k_{g,I^g})$ with $\lambda^k_{g,I^g} = (\lambda^k_{g,I_k}, 0_{I^g\setminus I_k})$.

Step 4. (Perform line search) Compute the step size t_k, which is the first number t of the sequence $\{1,\beta,\beta^2,\ldots\}$ satisfying

$$f_{r_k}\left(z^k+t\,dz^k\right)\le f_{r_k}\left(z^k\right)+\alpha t\nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}}dz^k, \tag{23}$$

$$\left(v_i^k+t\,dv^k\right)-F_i\left(u^k+t\,du^k\right)<0,\quad \forall i\in I^{\mathcal{C}}, \tag{24}$$

$$\psi_\varepsilon\left(w_i^k+t\,dw^k\right)<0,\quad \forall i\in I^{\mathcal{C}}, \tag{25}$$

$$g_i\left(u^k+t\,du^k\right)<0,\quad \forall i\in I^g. \tag{26}$$

Step 5. (Update) Let $z^{k+1}=z^k+t_k\,dz^k$, and compute

$$\begin{aligned}
&s_i^{k+1}=\min\left\{\max\left\{\left\|dz^k\right\|^2+s_{\min},\lambda_i^k\right\},s_{\max}\right\},\\
&\xi_{k+1}=\min\left\{\sqrt{\left(y_i^{k+1}\right)^2+\left(v_i^{k+1}\right)^2},i\in I^{\mathcal{C}}\right\},\\
&\varepsilon_{k+1}=\begin{cases}\varepsilon_k, & \text{if } \varepsilon_k\le\xi_{k+1},\\ \frac{1}{2}\varepsilon_k, & \text{otherwise.}\end{cases}
\end{aligned} \tag{27}$$

Step 6. Let $k:=k+1$, and go back to Step 1.

Remark 2 The correction technique of penalty parameter r_k in Step 3(2) is from [33], when the conditions (i)–(ii) in Step 3(2) are satisfied, the current iteration point z^k is close to a KKT point of the problem (7). However, the conditions (iii)–(iv) in Step 3(2) indicate that z^k is far away from the feasible area of the problem (6), and the penalty parameter needs to be increased.

For convenience, denote

$$\begin{aligned}
\Lambda_k={}&H_k-\sum_{i\in I^{\mathcal{C}}}\frac{s_{F,i}^k}{v_i^k-F_i(x^k,y^k)}\nabla_z\left(v_i^k-F_i\left(x^k,y^k\right)\right)\nabla_z\left(v_i^k-F_i\left(x^k,y^k\right)\right)^{\mathrm{T}}\\
&-\sum_{i\in I^{\mathcal{C}}}\frac{s_{\Psi_\varepsilon,i}^k}{\psi_\varepsilon(y_i^k,v_i^k)}\nabla_z\psi_\varepsilon\left(y_i^k,v_i^k\right)\nabla_z\psi_\varepsilon\left(y_i^k,v_i^k\right)^{\mathrm{T}}\\
&-\sum_{i\in I_k}\frac{s_{g,i}^k}{g_i(x^k,y^k)}\nabla_z g_i\left(x^k,y^k\right)\nabla_z g_i\left(x^k,y^k\right)^{\mathrm{T}}.
\end{aligned}$$

Lemma 2 *For the directions $\widehat{dz}^k$ and dz^k found in Step* 3(1)(4), *the following conclusions hold*:

$$\nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}}\widehat{dz}^k=-\left(\widehat{dz}^k\right)^{\mathrm{T}}\Lambda_k\widehat{dz}^k\le 0,\quad \forall k\ge 0, \tag{28}$$

$$\nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}}dz^k\le\theta\zeta^k\le 0,\quad \forall k\ge 0. \tag{29}$$

Furthermore, when iteration goes into Step 3(3)(4), *we have $\widehat{dz}^k\neq 0$ and $\zeta^k<0$, so dz^k is a feasible descent direction of the problem* (7) *at z^k, hence, Algorithm* A *is well defined.*

Proof It follows from the SLE1 (12) that

$$H_k \widehat{dz}^k + A_{I_k} \widehat{\lambda}_{I_k}^k = -\nabla_z f_{r_k}\big(x^k, y^k\big), \tag{30}$$

$$S_{I_k} (A_{I_k})^{\mathrm{T}} \widehat{dz}^k + G_{I_k} \widehat{\lambda}_{I_k} = 0. \tag{31}$$

By (30)–(31), we have

$$\begin{aligned}
\nabla_z f_{r_k}\big(x^k, y^k\big) &= \sum_{i\in I^{\mathcal{C}}} \frac{s_{F,i}^k}{v_i^k - F_i(x^k, y^k)} \nabla_z\big(v_i^k - F_i\big(x^k, y^k\big)\big) \nabla_z\big(v_i^k - F_i\big(x^k, y^k\big)\big)^{\mathrm{T}} \widehat{dz}^k \\
&\quad + \sum_{i\in I^{\mathcal{C}}} \frac{s_{\Psi_\varepsilon,i}^k}{\psi_\varepsilon(y_i^k, v_i^k)} \nabla_z \psi_\varepsilon\big(y_i^k, v_i^k\big) \nabla_z \psi_\varepsilon\big(y_i^k, v_i^k\big)^{\mathrm{T}} \widehat{dz}^k \\
&\quad + \sum_{i\in I_k} \frac{s_{g,i}^k}{g_i(x^k, y^k)} \nabla_z g_i\big(x^k, y^k\big) \nabla_z g_i\big(x^k, y^k\big)^{\mathrm{T}} \widehat{dz}^k - H_k \widehat{dz}^k,
\end{aligned}$$

which indicates $\nabla_z f_{r_k}(x^k, y^k)^{\mathrm{T}} \widehat{dz}^k = -(\widehat{dz}^k)^{\mathrm{T}} \Lambda_k \widehat{dz}^k \le 0$.

By (15)–(18), we obtain

$$\phi_{F,i} \widehat{\lambda}_{F,i}^k \ge 0, \quad i \in I^{\mathcal{C}}; \qquad D_{\Psi_\varepsilon,i} \widehat{\lambda}_{\Psi_\varepsilon,i}^k \ge 0, \quad i \in I^{\mathcal{C}}; \qquad Q_{g,i} \widehat{\lambda}_{g,i}^k \ge 0, \quad i \in I_k. \tag{32}$$

It follows from (18) and (32) that

$$\zeta_k \le \nabla_z f_{r_k}\big(x^k, y^k\big)^{\mathrm{T}} \widehat{dz}^k \le 0. \tag{33}$$

Taking into account SLE1 (12) and SLE2 (21), we have

$$H_k \widehat{dz}^k + A_{I_k} \widehat{\lambda}_{I_k}^k = -\nabla_z f_{r_k}\big(x^k, y^k\big), \tag{34}$$

$$S_{I_k} (A_{I_k})^{\mathrm{T}} \widehat{dz}^k + G_{I_k} \widehat{\lambda}_{I_k}^k = 0, \tag{35}$$

$$H_k dz^k + A_{I_k} \lambda_{I_k}^k = -\nabla_z f_{r_k}\big(x^k, y^k\big), \tag{36}$$

$$S_{I_k} (A_{I_k})^{\mathrm{T}} dz^k + G_{I_k} \lambda_{I_k}^k = \eta_{I_k}^k. \tag{37}$$

It follows from (34) and (36) as well as the symmetry of the matrix H_k that

$$\begin{aligned}
&\nabla_z f_{r_k}\big(x^k, y^k\big)^{\mathrm{T}} dz^k - \nabla_z f_{r_k}\big(x^k, y^k\big)^{\mathrm{T}} \widehat{dz}^k \\
&\quad = \big(\widehat{dz}^k\big)^{\mathrm{T}} A_{I_k} \lambda_{I_k}^k - \big(dz^k\big)^{\mathrm{T}} A_{I_k} \widehat{\lambda}_{I_k}^k \\
&\quad = \big(\lambda_{I_k}^k\big)^{\mathrm{T}} (A_{I_k})^{\mathrm{T}} \widehat{dz}^k - \big(\widehat{\lambda}_{I_k}^k\big)^{\mathrm{T}} (A_{I_k})^{\mathrm{T}} dz^k.
\end{aligned}$$

From (35) and (37), we have

$$\begin{aligned}
&\big(\lambda_{I_k}^k\big)^{\mathrm{T}} (A_{I_k})^{\mathrm{T}} \widehat{dz}^k - \big(\widehat{\lambda}_{I_k}^k\big)^{\mathrm{T}} (A_{I_k})^{\mathrm{T}} dz^k \\
&\quad = -\sum_{i\in I^{\mathcal{C}}} \frac{\widehat{\lambda}_{F,i}^k \eta_{F,i}^k}{s_{F,i}^k} - \sum_{i\in I^{\mathcal{C}}} \frac{\widehat{\lambda}_{\Psi_\varepsilon,i}^k \eta_{\Psi_\varepsilon,i}^k}{s_{\Psi_\varepsilon,i}^k} - \sum_{i\in I_k} \frac{\widehat{\lambda}_{g,i}^k \eta_{g,i}^k}{s_{g,i}^k}.
\end{aligned}$$

By (18) and (19), one gets

$$\begin{aligned}
&\nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}} dz^k \\
&= \nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}} \widehat{dz}^k - \sum_{i\in I^C} \frac{\widehat{\lambda}_{F,i}^k \eta_{F,i}^k}{s_{F,i}^k} - \sum_{i\in I^C} \frac{\widehat{\lambda}_{\Psi_\varepsilon,i}^k \eta_{\Psi_\varepsilon,i}^k}{s_{\Psi_\varepsilon,i}^k} - \sum_{i\in I_k} \frac{\widehat{\lambda}_{g,i}^k \eta_{g,i}^k}{s_{g,i}^k} \\
&= \nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}} \widehat{dz}^k - \sum_{i\in I^C} \frac{\widehat{\lambda}_{F,i}^k}{s_{F,i}^k}\left[(1-\varrho_k)\phi_{F,i}^k + \varrho_k\left(-\|\widehat{dz}^k\|^q - \|\varpi^k\|\right)s_{F,i}^k\right] \\
&\quad - \sum_{i\in I^C} \frac{\widehat{\lambda}_{\Psi_\varepsilon,i}^k}{s_{\Psi_\varepsilon,i}^k}\left[(1-\varrho_k)D_{\Psi_\varepsilon,i}^k + \varrho_k\left(-\|\widehat{dz}^k\|^q - \|\varpi^k\|\right)s_{\Psi_\varepsilon,i}^k\right] \\
&\quad - \sum_{i\in I_k} \frac{\widehat{\lambda}_{g,i}^k}{s_{g,i}^k}\left[(1-\varrho_k)Q_{g,i}^k + \varrho_k\left(-\|\widehat{dz}^k\|^q - \|\varpi^k\|\right)s_{g,i}^k\right] \\
&= \left(\nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}} \widehat{dz}^k - \sum_{i\in I^C} \frac{\widehat{\lambda}_{F,i}^k}{s_{F,i}^k}\phi_{F,i}^k - \sum_{i\in I^C} \frac{\widehat{\lambda}_{\Psi_\varepsilon,i}^k}{s_{\Psi_\varepsilon,i}^k} D_{\Psi_\varepsilon,i}^k - \sum_{i\in I_k} \frac{\widehat{\lambda}_{g,i}^k}{s_{g,i}^k} Q_{g,i}^k\right) \\
&\quad + \varrho_k\left[\sum_{i\in I^C} \frac{\widehat{\lambda}_{F,i}^k}{s_{F,i}^k}\phi_{F,i}^k + \sum_{i\in I^C} \widehat{\lambda}_{F,i}^k\left(\|\widehat{dz}^k\|^q + \|\varpi^k\|\right)\right] \\
&\quad + \varrho_k\left[\sum_{i\in I^C} \frac{\widehat{\lambda}_{\Psi_\varepsilon,i}^k}{s_{\Psi_\varepsilon,i}^k} D_{\Psi_\varepsilon,i}^k + \sum_{i\in I^C} \widehat{\lambda}_{\Psi_\varepsilon,i}^k\left(\|\widehat{dz}^k\|^q + \|\varpi^k\|\right)\right] \\
&\quad + \varrho_k\left[\sum_{i\in I_k} \frac{\widehat{\lambda}_{g,i}^k}{s_{g,i}^k} Q_{g,i}^k + \sum_{i\in I_k} \widehat{\lambda}_{g,i}^k\left(\|\widehat{dz}^k\|^q + \|\varpi^k\|\right)\right] \qquad (38) \\
&= \zeta^k + \varrho_k b_k.
\end{aligned}$$

In view of (20), if $b_k \le 0$, then $\varrho_k b_k = b_k \le 0$. Furthermore, we have $\zeta^k + \varrho_k b_k \le \zeta^k \le \theta\zeta^k$. If $b_k > 0$, then we have $\zeta^k + \varrho_k b_k = \zeta^k + \min\{(1-\theta)|\zeta^k|, b_k\} \le \zeta^k + (\theta-1)\zeta^k = \theta\zeta^k$. In conclusion, we obtain $\zeta^k + \varrho_k b_k \le \theta\zeta^k$. From (33) and (38), it follows that

$$\nabla_z f_{r_k}\left(x^k,y^k\right)^{\mathrm{T}} dz^k = \zeta^k + \varrho_k b_k \le \theta\zeta^k \le 0,$$

which indicates that the inequality (29) is true.

If $\widehat{dz}^k = 0$, taking into account SLE1 (12), from $v^k - F(x^k,y^k) < 0$, $\Psi_\varepsilon(y^k,v^k) < 0$, $g(x^k,y^k) < 0$, we have $\widehat{\lambda}_{I_k}^k = 0$, so the iterate k does not go into Step 3(3)(4). Thus, $\widehat{dz}^k \neq 0$ when the iterative process goes into Step 3(3)(4).

By $\nabla_z f_{r_k}(x^k,y^k)^{\mathrm{T}} dz^k \le 0$, we know that dz^k is a feasible descent direction of the problem (7) at point z^k. Together with $v^k - F(x^k,y^k) < 0$, $\Psi_\varepsilon(y^k,v^k) < 0$, $g(x^k,y^k) < 0$ and $\zeta^k < 0$, Step 4 in Algorithm A can be finished by finite calculations, hence Algorithm A is well defined. □

4 Global convergence

In this paper, we assume that Algorithm A generates an infinite iteration sequence $\{z^k\}$. For the sake of convenience, let z^* be an accumulation point of the iteration sequence $\{z^k\}$. First, we show that the penalty parameter r_k can be augmented only in a finite number of steps. Then we prove that z^* is a KKT point of the problem (7). Finally, we show that z^*

is a KKT point of the MPCC (1). More precisely, we show that Algorithm A is globally convergent. For this purpose, the following assumptions are necessary.

Assumption 3 Suppose that the sequences $\{H_k\}$ and $\{z^k\}$ are all bounded, each accumulation point of $\{z^k\}$ satisfies LLSC (2). Assume that there exists a positive constant σ such that

$$d^{\mathrm{T}} \Lambda_k d \geq \sigma \|d\|^2, \quad \forall k, \forall d \in \mathbb{R}^{n+2m}. \tag{39}$$

Assumption 4 For any $z \in \widetilde{X}$, if $z \notin X$, then no scalars $\lambda_{F,i} \geq 0$, $i \in I_0^F(z)$; $\lambda_{\Psi_\varepsilon,i} \geq 0$, $i \in I_0^{\Psi_\varepsilon}(z)$ and $\lambda_{g,i} \geq 0$, $i \in I_0^g(z)$, exist such that

$$\begin{aligned} &\sum_{i\in I^C} \nabla_z\big(v_i - F_i(x,y)\big) + \sum_{i\in I^C} \nabla_z \psi_\varepsilon(y_i, v_i) \\ &\quad = \sum_{i\in I_0^F(z)} \lambda_{F,i}\nabla_z\big(v_i - F_i(x,y)\big) + \sum_{i\in I_0^{\Psi_\varepsilon}(z)} \lambda_{\Psi_\varepsilon,i}\nabla_z\psi_\varepsilon(y_i,v_i) + \sum_{i\in I_0^g(z)} \lambda_{g,i}\nabla_z g_i(x,y). \end{aligned}$$

Lemma 3 *Suppose that Assumptions 1–4 hold, then the penalty parameter r_k can be augmented only in a finite number of steps.*

Proof By contradiction, suppose that r_k is increased infinitely many times, i.e., there exists an infinite index set K such that $r_{k+1} > r_k$ for any $k \in K$. The increase of r_k must satisfy the following conditions for any $k \in K$:

(i) $\|\widehat{dz}^k\| \leq \gamma_1$; (ii) $\widehat{\lambda}^k \geq -\gamma_2 e_I$; (iii) $\widehat{\lambda}^k_{F,I^C} \ngtr \gamma_3 e_{I^C}$;
(iv) $\widehat{\lambda}^k_{\Psi_\varepsilon,I^C} \ngtr \gamma_4 e_{I^C}$; (v) $I^C\big(y^k, v^k, \varepsilon_k\big) \neq \emptyset$.

Denote

$$\begin{aligned} A_{Ig}^k &= \begin{pmatrix} -\nabla_x F(x^k,y^k) & 0 & \nabla_x g(x^k,y^k) \\ -\nabla_y F(x^k,y^k) & \nabla_y \Psi_\varepsilon(y^k,v^k) & \nabla_y g(x^k,y^k) \\ E_m & \nabla_v \Psi_\varepsilon(y^k,v^k) & 0 \end{pmatrix}, \\ G_{Ig}^k &= \operatorname{diag}\big(v^k - F\big(x^k,y^k\big), \Psi_\varepsilon\big(y^k,v^k\big), g\big(x^k,y^k\big)\big), \end{aligned}$$

and $S_{Ig}^k = \operatorname{diag}(s_{Ig}^k)$, where $s_{Ig}^k = (s_{F,I^C}^k, s_{\Psi_\varepsilon,I^C}^k, s_{g,Ig}^k)$. According to the SLE

$$\begin{pmatrix} H_k & A_{Ig}^k \\ S_{Ig}^k (A_{Ig}^k)^{\mathrm{T}} & G_{Ig}^k \end{pmatrix} \begin{pmatrix} \widehat{dz}^k \\ \widehat{\lambda}^k \end{pmatrix} = \begin{pmatrix} -\nabla_z f_{r_k}(x^k,y^k) \\ 0 \end{pmatrix},$$

we have

$$H_k \widehat{dz}^k + A_{Ig}^k \widehat{\lambda}^k + \nabla_z f_{r_k}\big(x^k,y^k\big) = 0, \tag{40}$$

$$S_{Ig}^k \big(A_{Ig}^k\big)^{\mathrm{T}} \widehat{dz}^k + G_{Ig}^k \widehat{\lambda}^k = 0. \tag{41}$$

Since $\{r_k\}$ tends to infinity, and $\lambda^k = \widehat{\lambda}^{k-1} - r_{k-1}\tilde{e}$, $k \geq 1$, such that $\{\|\widehat{\lambda}^k\|_\infty\}$ tends to infinity as well, we have

$$\beta_k = \max\{\|\widehat{\lambda}^k_{F,I^{\mathcal{C}}}\|_\infty, \|\widehat{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}}\|_\infty, \|\widehat{\lambda}^k_{g,I^g}\|_\infty, 1\} \xrightarrow{K} \infty.$$

Define

$$\bar{\lambda}^k_{F,I^{\mathcal{C}}} = \beta_k^{-1}\widehat{\lambda}^k_{F,I^{\mathcal{C}}}, \qquad \bar{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}} = \beta_k^{-1}\widehat{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \qquad \bar{\lambda}^k_{g,I^g} = \beta_k^{-1}\widehat{\lambda}^k_{g,I^g}, \tag{42}$$

for k ($\in K$) large enough, it follows that

$$\max\{\|\bar{\lambda}^k_{F,I^{\mathcal{C}}}\|_\infty, \|\bar{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}}\|_\infty, \|\bar{\lambda}^k_{g,I^g}\|_\infty\} = 1.$$

Since the sequence $\{z^k\}_{k\in K}$ is bounded by Assumption 3, there exists an infinite index set $K' \subseteq K$, $z^* \in \mathbb{R}^{n+2m}$, and $(\bar{\lambda}^*_{F,I^{\mathcal{C}}}, \bar{\lambda}^*_{\Psi_\varepsilon,I^{\mathcal{C}}}, \bar{\lambda}^*_{g,I^g}) \neq 0$, for all $k \in K'$, such that

$$\lim_{k\to\infty} z^k = z^*, \qquad \lim_{k\to\infty} \bar{\lambda}^k_{F,I^{\mathcal{C}}} = \bar{\lambda}^*_{F,I^{\mathcal{C}}}, \qquad \lim_{k\to\infty} \bar{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}} = \bar{\lambda}^*_{\Psi_\varepsilon,I^{\mathcal{C}}}, \qquad \lim_{k\to\infty} \bar{\lambda}^k_{g,I^g} = \bar{\lambda}^*_{g,I^g}.$$

In view of the boundedness of $\{z^k\}$ and the continuity of the functions, it follows that

$$\{\nabla_z(v^k - F(x^k,y^k))\}, \qquad \{\nabla_z\Psi_\varepsilon(y^k,v^k)\}, \qquad \{\nabla_z g(x^k,y^k)\},$$

are bounded. Further, $\{S^k_{I^g}\}$ are bounded by construction. Multiplying both sides of (41) by β_k^{-1}, we have

$$\beta_k^{-1}S^k_{I^g}(A^k_{I^g})^{\mathrm{T}}\widehat{dz}^k + \beta_k^{-1}G^k_{I^g}\widehat{\lambda}^k = 0.$$

Letting $k \xrightarrow{K'} \infty$, the condition (i) shows that $\beta_k^{-1}G^k_{I^g}\widehat{\lambda}^k \xrightarrow{K'} 0$, implying that $\beta_k^{-1}\widehat{\lambda}^k_{F,i} \xrightarrow{K'} 0$ for $i \in I^{\mathcal{C}} \setminus I^F_0(z^*)$. Similarly, $\beta_k^{-1}\widehat{\lambda}^k_{\Psi_\varepsilon,i} \xrightarrow{K'} 0$ for $i \in I^{\mathcal{C}} \setminus I^{\Psi_\varepsilon}_0(z^*)$.

From (10) and (42), we know that $\{\frac{r_k}{\beta_k}\}$ converges to $a \geq 0$ as $k \xrightarrow{K'} \infty$, with

$$\bar{\lambda}^*_{F,i} = a, \quad \forall i \in I^{\mathcal{C}} \setminus I^F_0(z^*); \qquad \bar{\lambda}^*_{\Psi_\varepsilon,i} = a, \quad \forall i \in I^{\mathcal{C}} \setminus I^{\Psi_\varepsilon}_0(z^*). \tag{43}$$

Then it follows from the condition (ii) and (42) that

$$\bar{\lambda}^*_{F,i} \leq a, \quad \forall i \in I^F_0(z^*); \qquad \bar{\lambda}^*_{\Psi_\varepsilon,i} \leq a, \quad \forall i \in I^{\Psi_\varepsilon}_0(z^*); \qquad \bar{\lambda}^*_{g,i} \leq 0, \quad \forall i \in I^g_0(z^*). \tag{44}$$

And from (41), we obtain

$$s^k_{g,i}\nabla_z g_i(x^k,y^k)^{\mathrm{T}}\widehat{dz}^k + g_i(x^k,y^k)\widehat{\lambda}^k_{g,i} = 0, \quad i \in I^g,$$

furthermore, multiplying both sides of this equation by β_k^{-1}, together with the condition (i) and taking the limit as $k \xrightarrow{K'} \infty$, we obtain

$$g_i(x^*,y^*)\widehat{\lambda}^*_{g,i} = 0, \quad i \in I^g,$$

so $\widehat{\lambda}^*_{g,i} = 0$ for all $i \in I^g \setminus I^g_0(z^*)$.

In view of the condition (i) and the continuity of the functions, multiplying (40) by β_k^{-1} and taking the limit as $k \xrightarrow{K'} \infty$, we get $\beta_k^{-1} A_{I^g}^k \widehat{\lambda}^k \xrightarrow{K'} 0$, i.e.,

$$\sum_{i\in I^{\mathcal{C}}} \bar{\lambda}_{F,i}^* \nabla_z\big(v_i^* - F_i(x^*,y^*)\big) + \sum_{i\in I^{\mathcal{C}}} \bar{\lambda}_{\Psi_\varepsilon i}^* \nabla_z \psi_\varepsilon\big(y_i^*,v_i^*\big) + \sum_{i\in I_0^g(z^*)} \bar{\lambda}_{g,i}^* \nabla_z g_i\big(x^*,y^*\big) = 0. \tag{45}$$

Since $\bar{\lambda}_{F,I^{\mathcal{C}}}^*$, $\bar{\lambda}_{\Psi_\varepsilon,I^{\mathcal{C}}}^*$, and $\bar{\lambda}_{g,I^g}^*$ are not zero, combining with (45) and Assumption 2, it follows that

$$I_0^F\big(z^*\big) \neq \{1,\dots,m\}, \qquad I_0^{\Psi_\varepsilon}\big(z^*\big) \neq \{1,\dots,m\},$$

i.e., $z^* \notin X$ and $a > 0$.

Based on (43), dividing both sides of (45) by a, we obtain

$$\begin{aligned}
&\sum_{i\in I^{\mathcal{C}}\setminus I_0^F(z^*)} \nabla_z\big(v_i^* - F_i(x^*,y^*)\big) + \sum_{i\in I^{\mathcal{C}}\setminus I_0^{\Psi_\varepsilon}(z^*)} \nabla_z \psi_\varepsilon\big(y_i^*,v_i^*\big)\\
&\quad = -\sum_{i\in I_0^F(z^*)} \frac{\bar{\lambda}_{F,i}^*}{a} \nabla_z\big(v_i^* - F_i(x^*,y^*)\big) - \sum_{i\in I_0^{\Psi_\varepsilon}(z^*)} \frac{\bar{\lambda}_{\Psi_\varepsilon,i}^*}{a} \nabla_z \psi_\varepsilon\big(y_i^*,v_i^*\big)\\
&\qquad - \sum_{i\in I_0^g(z^*)} \frac{\bar{\lambda}_{g,i}^*}{a} \nabla_z g_i\big(x^*,y^*\big).
\end{aligned} \tag{46}$$

Adding both sides of the above equality to

$$\sum_{i\in I_0^F(z^*)} \nabla_z\big(v_i^* - F_i(x^*,y^*)\big) + \sum_{i\in I_0^{\Psi_\varepsilon}(z^*)} \nabla_z \psi_\varepsilon\big(y_i^*,v_i^*\big),$$

we obtain

$$\begin{aligned}
&\sum_{i\in I^{\mathcal{C}}} \nabla_z\big(v_i^* - F_i(x^*,y^*)\big) + \sum_{i\in I^{\mathcal{C}}} \nabla_z \psi_\varepsilon\big(y_i^*,v_i^*\big)\\
&\quad = \sum_{i\in I_0^F(z^*)} \lambda_{F,i} \nabla_z\big(v_i^* - F_i(x^*,y^*)\big) + \sum_{i\in I_0^{\Psi_\varepsilon}(z^*)} \lambda_{\Psi_\varepsilon,i} \nabla_z \psi_\varepsilon\big(y_i^*,v_i^*\big)\\
&\qquad + \sum_{i\in I_0^g(z^*)} \lambda_{g,i} \nabla_z g_i\big(x^*,y^*\big),
\end{aligned}$$

where $\lambda_{F,i} = 1 - \frac{\bar{\lambda}_{F,i}^*}{a}$, $i \in I_0^F(z^*)$; $\lambda_{\Psi_\varepsilon,i} = 1 - \frac{\bar{\lambda}_{\Psi_\varepsilon,i}^*}{a}$, $i \in I_0^{\Psi_\varepsilon}(z^*)$; $\lambda_{g,i} = -\frac{\bar{\lambda}_{g,i}^*}{a}$, $i \in I_0^g(z^*)$. In view of (44) and $z^* \notin X$. This contradicts Assumption 4, so the penalty parameter r_k is updated at most a finite number of times. □

Lemma 4 *Suppose that Assumptions 1–4 hold, then there exists an integer k_0, such that $\varepsilon_k \equiv \varepsilon_{k_0}$ for $k \geq k_0$.*

Proof By Step 3(2) and Step 5 of Algorithm A, the sequence $\{\varepsilon_k\}$ is monotonically decreasing and bounded from below, so the sequence $\{\varepsilon_k\}$ is convergent. Let

$$K_1 = \{k \in K \mid \varepsilon_k > \xi_{k+1}\},$$
$$K_2 = \{k \in K \mid \widehat{dz}^k = 0, \widehat{\lambda}^k_{g,I_k} \geq 0, I^{\mathcal{C}}(y^k, v^k, \varepsilon_k) \neq \emptyset\},$$

where $K = \{0, 1, 2, \ldots\}$. Now we turn to prove that K_1 and K_2 are finite sets.

By contradiction, if K_1 is infinite set, then $\lim_{k\to\infty} \varepsilon_k = 0$. There exist $i \in I^{\mathcal{C}}$ and $\overline{K}_1 \subseteq K_1$, such that

$$\xi_{k+1} = \sqrt{(y_i^{k+1})^2 + (v_i^{k+1})^2} < \varepsilon_k, \quad k \in \overline{K}_1 \subseteq K_1. \tag{47}$$

Together with the boundedness of $\{z^{k+1}\}_{\overline{K}_1}$, we suppose that $z^{k+1} \xrightarrow{\overline{K}_1} z^*$. From (47), we have $(y_i^*)^2 + (v_i^*)^2 = 0$, which contradicts Assumption 3. So K_1 is a finite set. Similarly, we show that K_2 is finite. Based on the finiteness of K_1 and K_2, from the update rule (27) of ε_k and Step 3(2), we can conclude that Lemma 4 holds. □

Based on Lemmas 3–4, in the following analysis, we suppose, without loss of generality, that $r_k \equiv \widehat{r}$, $\varepsilon_k \equiv \varepsilon$.

Lemma 5 *Suppose that Assumptions 1–4 hold. Then*

(1) *The coefficient matrix sequence $\{M_{I_k}\}$ is unified invertible, and there exists a positive constant L such that $\|(M_{I_k})^{-1}\| \leq L$, $\forall k \geq 0$.*
(2) *The sequences $\{(\widehat{dz}^k, \widehat{\lambda}^k)\}$ and $\{(dz^k, \lambda^k)\}$ are bounded.*

Proof (1) By contradiction, suppose that there exists an infinite subset K such that $\|(M_{I_k})^{-1}\| \xrightarrow{K} \infty$. Since $\{z^k\}$ and $\{H_k\}$ are bounded, and the finite choice of I_k, without loss of generality, for all $k \in K$, let

$$I_k \equiv I', \qquad z^k \xrightarrow{K} z^*, \qquad H_k \xrightarrow{K} H_*, \qquad s^k \xrightarrow{K} s^* \geq s_{\min} e_I > 0.$$

Denote

$$A^*_{I'} = \begin{pmatrix} -\nabla_x F(x^*, y^*) & 0_{n\times m} & \nabla_x g_{I'}(x^*, y^*) \\ -\nabla_y F(x^*, y^*) & \nabla_y \Psi_\varepsilon(y^*, v^*) & \nabla_y g_{I'}(x^*, y^*) \\ E_m & \nabla_v \Psi_\varepsilon(y^*, v^*) & 0_{m\times m_g} \end{pmatrix},$$
$$G^*_{I'} = \operatorname{diag}\big(v^* - F(x^*, y^*), \Psi_\varepsilon(y^*, v^*), g_{I'}(x^*, y^*)\big),$$

then

$$M_{I_k} \xrightarrow{K} M^*_{I'} = \begin{pmatrix} H_* & A^*_{I'} \\ S^*_{I'}(A^*_{I'})^{\mathrm{T}} & G^*_{I'} \end{pmatrix}, \tag{48}$$

where the matrix $S^*_{I'} = \operatorname{diag}(s^*_{I'})$ with $s^*_{I'} = (s^*_{F,I^{\mathcal{C}}}, s^*_{\Psi_\varepsilon,I^{\mathcal{C}}}, s^*_{g,I'})$.

Under the Assumptions, it can be shown easily that $M^*_{I'}$ is nonsingular. So we obtain

$$\|(M_{I_k})^{-1}\| \xrightarrow{K} \|(M^*_{I'})^{-1}\| < \infty,$$

which contradicts $\|(M_{I_k})^{-1}\| \xrightarrow{K} \infty$.

(2) In view of SLE1 (12), and Lemma 3 as well as $r_k \equiv \widehat{r}$, we see that $\{(\widehat{dz}^k, \widehat{\lambda}^k)\}$ is bounded. By (15)–(22) and the boundedness of $\{(\widehat{dz}^k, \widehat{\lambda}^k)\}$ as well as the boundedness of $\{s^k\}$, we know that $\{\eta^k_{I_k}\}$ is bounded. Hence, the boundedness of $\{(dz^k, \lambda^k)\}$ follows by SLE2 (21). □

Lemma 6 *Suppose that Assumptions 1–4 hold, z^* is an accumulation point of the infinite sequence $\{z^k\}$ generated by Algorithm A, and $\{z^k\}_K$ converges to z^*. If $\{\zeta^k\}_K \to 0$, then z^* is a KKT point of the problem (7), and both $\{\widehat{\lambda}^k\}_K$ and $\{\lambda^k\}_K$ converge to the unique multiplier vector λ^* corresponding to z^*.*

Proof Let $(\widehat{\lambda}^*; \lambda^*)$ be any accumulation point of $\{(\widehat{\lambda}^k; \lambda^k)\}_K$. First, we verify that $(z^*, \widehat{\lambda}^*)$ is a KKT pair of the problem (7). Taking into account Assumption 3, Lemma 5, and the finite choice of I_k, there exists an infinite index $K' \subseteq K$ such that

$$\begin{aligned} &I_k \equiv I', \qquad (\widehat{\lambda}^k; \lambda^k) \xrightarrow{K'} (\widehat{\lambda}^*; \lambda^*), \qquad H_k \xrightarrow{K'} H_*, \\ &\widehat{dz}^k \xrightarrow{K'} \widehat{dz}^*, \qquad s^k \xrightarrow{K'} s^* \ge s_{\min} e_I. \end{aligned} \tag{49}$$

It follows from (33), (28), Assumption 3 and $\{\zeta^k\}_K \to 0$ that $\widehat{dz}^* = 0$. Further, taking the limit in SLE1 (12) for $k \in K'$, one obtains

$$\nabla_z f_{\widehat{r}}(x^*, y^*) + A^*_{I'} \widehat{\lambda}^*_{I'} = 0, \qquad G^*_{I'} \widehat{\lambda}^*_{I'} = 0. \tag{50}$$

Next we will prove $\widehat{\lambda}^*_{I'} = (\widehat{\lambda}^*_{F,I^{\mathcal{C}}}, \widehat{\lambda}^*_{\Psi_\varepsilon,I^{\mathcal{C}}}, \widehat{\lambda}^*_{g,I'}) \ge 0$, and $\widehat{\lambda}^* = (\widehat{\lambda}^*_{I'}, 0_{I^g \setminus I'}) \ge 0$. From $\widehat{\lambda}^*_{F,i}(v^*_i - F_i(x^*, y^*)) = 0$, for $i \in I^{\mathcal{C}} \setminus I^F_0(z^*)$, we have $\widehat{\lambda}^*_{F,I^{\mathcal{C}} \setminus I^F_0(z^*)} = 0$. Similarly,

$$\widehat{\lambda}^*_{\Psi_\varepsilon, I^{\mathcal{C}} \setminus I^{\Psi_\varepsilon}_0(z^*)} = 0, \qquad \widehat{\lambda}^*_{g, I' \setminus I^g_0(z^*)} = 0.$$

By (18) and $(\zeta^k, \widehat{dz}^k) \xrightarrow{K'} (0, 0)$, one obtains

$$\sum_{i \in I^{\mathcal{C}}} \frac{\widehat{\lambda}^k_{F,i}}{s^k_{F,i}} \phi^k_{F,i} + \sum_{i \in I^{\mathcal{C}}} \frac{\widehat{\lambda}^k_{\Psi_\varepsilon,i}}{s^k_{\Psi_\varepsilon,i}} D^k_{\Psi_\varepsilon,i} + \sum_{i \in I'} \frac{\widehat{\lambda}^k_{g,i}}{s^k_{g,i}} Q^k_{g,i} \xrightarrow{K'} 0.$$

Furthermore, by (32), we have $\frac{\widehat{\lambda}^k_{F,i} \phi^k_{F,i}}{s^k_{F,i}} \ge 0$ for every $i \in I^{\mathcal{C}}$. Combining with (49), we have $\widehat{\lambda}^k_{F,i} \phi^k_{F,i} \xrightarrow{K'} 0$, $i \in I^{\mathcal{C}}$. So, in view of (15), one has

$$\widehat{\lambda}^*_{F,i} \min\{0, -(\max\{-\widehat{\lambda}^*_{F,i}, 0\})^{\nu} - C(v^*_i - F_i(x^*, y^*))\} = 0, \quad i \in I^{\mathcal{C}},$$

which implies that $\widehat{\lambda}^*_{F,i} \ge 0$ for $i \in I^{\mathcal{C}} \cap I^F_0(z^*)$. Similarly, one obtains

$$\widehat{\lambda}^*_{\Psi_\varepsilon,i} \ge 0 \quad \text{for } i \in I^{\mathcal{C}} \cap I^{\Psi_\varepsilon}_0(z^*); \qquad \widehat{\lambda}^*_{g,i} \ge 0 \quad \text{for } i \in I' \cap I^g_0(z^*).$$

So we can claim that $\widehat{\lambda}^*_{I'} \ge 0$. Obviously, $\widehat{\lambda}^* = (\widehat{\lambda}^*_{I'}, 0_{I^g \setminus I'}) \ge 0$ is immediate.

Therefore, in view of $z^* \in \widetilde{X}$ and (50), we see that $(z^*, \widehat{\lambda}^*)$ is a KKT pair of the problem (7). Moreover, the above analysis indicates that the sequence $\{\widehat{\lambda}^k\}_K$ possesses a unique accumulation point $\widehat{\lambda}^*$, so $\lim_{k\in K} \widehat{\lambda}^k = \widehat{\lambda}^*$.

Last, noticing that $(\widehat{dz}^k, \widehat{\lambda}^k) \xrightarrow{K'} (0, \widehat{\lambda}^*) \geq 0$, (15)–(17) and (22), we have

$$\phi^k_{F,I^{\mathcal{C}}} \xrightarrow{K'} 0, \qquad D^k_{\Psi_\varepsilon,I^{\mathcal{C}}} \xrightarrow{K'} 0, \qquad Q^k_{g,I'} \xrightarrow{K'} 0, \qquad \eta^k_{I'} = \left(\eta^k_{F,I^{\mathcal{C}}}, \eta^k_{\Psi_\varepsilon,I^{\mathcal{C}}}, \eta^k_{g,I'}\right) \xrightarrow{K'} 0.$$

Denote

$$M_{I_k}\begin{pmatrix} dz^k \\ \lambda^k_{I'} \end{pmatrix} = \begin{pmatrix} -\nabla_z f_{\tilde{r}}(x^k, y^k) \\ \eta^k_{I'} \end{pmatrix}, \tag{51}$$

so it follows from the above SLE and SLE1 (12) that

$$M_{I_k}\begin{pmatrix} dz^k - \widehat{dz}^k \\ \lambda^k_{I'} - \widehat{\lambda}^k_{I'} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^k_{I'} \end{pmatrix} \xrightarrow{K'} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{52}$$

which together with Lemma 5(1) gives

$$\lambda^* = \lim_{k\in K'} \lambda^k = \lim_{k\in K'} \widehat{\lambda}^k = \widehat{\lambda}^*. \qquad \square$$

Based on the lemmas above, we now present the global convergence of Algorithm A.

Theorem 1 *Suppose that Assumptions 1–4 hold. z^* is an accumulation point of $\{z^k\}$ generated by Algorithm A, then z^* is a KKT point of the MPCC (1).*

Proof A. We first show that z^* is a KKT point of the problem (6) (called Conclusion A).

Let K' be an infinite index set such that

$$I_k \equiv I', \qquad (\widehat{\lambda}^k; \lambda^k) \xrightarrow{K'} (\widehat{\lambda}^*; \lambda^*), \qquad H_k \xrightarrow{K'} H_*,$$
$$\widehat{dz}^k \xrightarrow{K'} \widehat{dz}^*, \qquad s^k \xrightarrow{K'} s^* \geq s_{\min} e_I.$$

Denote

$$A^*_{I'} = \begin{pmatrix} -\nabla_x F(x^*, y^*) & 0 & \nabla_x g_{I'}(x^*, y^*) \\ -\nabla_y F(x^*, y^*) & \nabla_y \Psi_\varepsilon(y^*, v^*) & \nabla_y g_{I'}(x^*, y^*) \\ E_m & \nabla_v \Psi_\varepsilon(y^*, v^*) & 0 \end{pmatrix},$$
$$G^*_{I'} = \operatorname{diag}\left(v^* - F\left(x^*, y^*\right), \Psi_\varepsilon\left(y^*, v^*\right), g_{I'}\left(x^*, y^*\right)\right),$$
$$M^*_{I'} = \begin{pmatrix} H_* & A^*_{I'} \\ S^*_{I'}(A^*_{I'})^{\mathrm{T}} & G^*_{I'} \end{pmatrix},$$

where the matrix $S^*_{I'} = \operatorname{diag}(s^*_{I'})$ with $s^*_{I'} = (s^*_{F,I^{\mathcal{C}}}, s^*_{\Psi_\varepsilon,I^{\mathcal{C}}}, s^*_{g,I'})$.

By contradiction, suppose that z^* is not a KKT point of the problem (6). From Lemma 5, without loss of generality, suppose that $\lambda^k = \widehat{\lambda}^{k-1} - \widetilde{r}\widetilde{e} \xrightarrow{K'} \widehat{\lambda}'$. Therefore, it follows that

$(z^*,\widehat{\lambda}')$ is not a KKT pair of the problem (6), which implies $\rho(z^*,\widehat{\lambda}') > 0$ and $I_0^g(z^*) \subseteq I_k$, $k \in K'$ large enough. The following two cases are discussed.

Case I. z^* is a KKT point of the problem (7). Then there exists a multiplier $\widehat{\lambda}'' = (\widehat{\lambda}''_{I_*}, 0_{I^g\setminus I'}) \geq 0$, such that the KKT condition of the problem (7) is satisfied at $(z^*,\widehat{\lambda}''_{I_*})$. For $k \in K'$ large enough, we have $I_0^g(z^*) \subseteq I_k \equiv I'$. It is easy to see that $(0,\widehat{\lambda}''_{I_*})$ is a solution to the following SLE:

$$M_{I'}^* \begin{pmatrix} dz \\ \lambda_{I_k} \end{pmatrix} = -\begin{pmatrix} \nabla_z f_{\widehat{r}}(x^*,y^*) \\ 0 \end{pmatrix}. \tag{53}$$

Passing to the limit in SLE1 (12), we know that $(\widehat{dz}^*,\widehat{\lambda}^*_{I_*})$ is the solution to (53) above. Let $\widehat{\lambda}^* = (\widehat{\lambda}^*_{I_*}, 0_{I^g\setminus I'})$. In view of Lemma 5 (1), taking into account the nonsingularity of the matrix $M_{I'}^*$, we see that the solution of (53) is unique. Hence, we obtain $\widehat{dz}^* = 0$, $\widehat{\lambda}^*_{I_*} = \widehat{\lambda}''_{I_*} \geq 0$, which implies $\widehat{\lambda}^* = \widehat{\lambda}'' \geq 0$. The conditions (i) and (ii) in Step 3 (2) are satisfied for $k \in K'$ large enough. Moreover, in view of $r_k \equiv \widehat{r} < \infty$ for k large enough, Step 3 (2) implies $\widehat{\lambda}^k_{F,I^{\mathcal{C}}} > \gamma_3 e_{I^{\mathcal{C}}}$, $\widehat{\lambda}^k_{\Psi_\varepsilon,I^{\mathcal{C}}} > \gamma_4 e_{I^{\mathcal{C}}}$ for $k \in K'$ large enough, which further implies that

$$\widehat{\lambda}^*_{F,I^{\mathcal{C}}} \geq \gamma_3 e_{I^{\mathcal{C}}} > 0, \qquad \widehat{\lambda}^*_{\Psi_\varepsilon,I^{\mathcal{C}}} \geq \gamma_4 e_{I^{\mathcal{C}}} > 0.$$

Since the pair $(z^*,\widehat{\lambda}'')$ satisfies the complementary slackness, one gets

$$\begin{aligned} &\widehat{\lambda}''_{F,i}\bigl(v_i^* - F_i\bigl(x^*,y^*\bigr)\bigr) = \widehat{\lambda}^*_{F,i}\bigl(v_i^* - F_i\bigl(x^*,y^*\bigr)\bigr) = 0, \quad i \in I^{\mathcal{C}}, \\ &\widehat{\lambda}''_{\Psi_\varepsilon,i}\psi_\varepsilon\bigl(y_i^*,v_i^*\bigr) = \widehat{\lambda}^*_{\Psi_\varepsilon,i}\psi_\varepsilon\bigl(y_i^*,v_i^*\bigr) = 0, \quad i \in I^{\mathcal{C}}. \end{aligned}$$

So $v^* - F(x^*,y^*) = 0$, $\Psi_\varepsilon(y^*,v^*) = 0$, which together with Proposition 2 shows that z^* is a KKT point of the problem (6), which contradicts the assumption that z^* is not a KKT point of the problem (6).

Case II. z^* is not a KKT point of the problem (7). It follows from Lemma 6 and $\zeta^k \leq 0$ that $\zeta^k \xrightarrow{K'} \bar{\zeta} < 0$. By (18), (28) and Assumption 3, we have

$$\lim_{k\in K'}\bigl(\|\widehat{dz}^k\|^q + \|\varpi^k\|\bigr) > 0,$$

so there exist a subset $K'' \subseteq K'$ and a positive constant $\bar{c}$ such that

$$\zeta^k \leq \frac{\bar{\zeta}}{2} < 0; \qquad \bigl(\|\widehat{dz}^k\|^q + \|\varpi^k\|\bigr) \geq \bar{c} > 0, \quad k \in K''.$$

In the following we first show that there exists a positive constant $\bar{t}$, such that $t_k \geq \bar{t}$ for all $k \in K''$.

(i) By (24)–(26), we have

$$\begin{aligned} &v_i^* - F_i\bigl(x^*,y^*\bigr) < 0, \quad i \notin I_0^F\bigl(z^*\bigr); \qquad \psi_\varepsilon\bigl(y_i^*,v_i^*\bigr) < 0, \quad i \notin I_0^{\Psi_\varepsilon}\bigl(z^*\bigr); \\ &g_i\bigl(x^*,y^*\bigr) < 0, \quad i \notin I_0^g\bigl(z^*\bigr). \end{aligned}$$

Based on the boundedness of $\{dz^k\}_{K''}$ and the continuity of functions, for all $k \in K''$ large enough and $t > 0$ sufficiently small, one gets

$$\left(v_i^k + t\,dv^k\right) - F_i\left(u^k + t\,du^k\right) < 0, \qquad \psi_\varepsilon\left(w_i^k + t\,dw^k\right) < 0, \qquad g_i\left(u^k + t\,du^k\right) < 0.$$

In view of $i \in I_0^F(z^*)$, we have $v_i^* - F_i(u^*) = 0$. Similarly, $\psi_\varepsilon(w_i^*) = 0$ for $i \in I_0^{\Psi_\varepsilon}(z^*)$, and $g_i(u^*) = 0$ for $i \in I_0^g(z^*)$. From (22) and SLE2 (21), we have

$$S_{I_k}(A_{I_k})^{\mathrm{T}}\,dz^k + G_{I_k}\lambda_{I_k}^k = \eta_{I_k}^k,$$

that is,

$$\begin{aligned}
&\nabla_z\left(v_i^k - F_i\left(u^k\right)\right)^{\mathrm{T}} dz^k = \frac{\eta_{F,i}^k - \lambda_{F,i}^k(v_i^k - F_i(u^k))}{s_{F,i}^k}, \quad i \in I^{\mathcal{C}},\\
&\nabla_z\psi_\varepsilon\left(w_i^k\right)^{\mathrm{T}} dz^k = \frac{\eta_{\Psi_\varepsilon,i}^k - \lambda_{\Psi_\varepsilon,i}^k\psi_\varepsilon(w_i^k)}{s_{\Psi_\varepsilon,i}^k}, \quad i \in I^{\mathcal{C}},\\
&\nabla_z g_i\left(u^k\right)^{\mathrm{T}} dz^k = \frac{\eta_{g,i}^k - \lambda_{g,i}^k g(w_i^k)}{s_{g,i}^k}, \quad i \in I_k.
\end{aligned}$$

By Taylor's expansion and $\|\widehat{dz}^k\| \geq 0$, for $t > 0$ sufficient small, we get

$$\begin{aligned}
&\left(v_i^k + t\,dv^k\right) - F_i\left(u^k + t\,du^k\right)\\
&\quad = \left(v_i^k - F_i\left(u^k\right)\right) + t\nabla_z\left(v_i^k - F_i\left(u^k\right)\right)^{\mathrm{T}} dz^k + o(t)\\
&\quad = \left(v_i^k - F_i\left(u^k\right)\right) + t\frac{\eta_{F,i}^k - \lambda_{F,i}^k(v_i^k - F_i(u^k))}{s_{F,i}^k} + o(t)\\
&\quad = \left(1 - t\frac{\lambda_{F,i}^k}{s_{F,i}^k}\right)\left(v_i^k - F_i\left(u^k\right)\right) + t\frac{1-\varrho_k}{s_{F,i}^k}\phi_{F,i}^k - t\varrho_k\left(\left\|\widehat{dz}^k\right\|^q + \left\|\varpi^k\right\|\right)\\
&\quad \leq -t\varrho_k\left(\left\|\widehat{dz}^k\right\|^q + \left\|\varpi^k\right\|\right) + o(t).
\end{aligned} \tag{54}$$

Furthermore, we consider $\zeta^k \leq \frac{\bar{\zeta}}{2} < 0$, by (19), (20), and the boundedness of b_k by Lemma 5, we see that there exists a constant $\bar{a} > 0$ such that $\varrho_k \geq \bar{a} > 0$, $k \in K''$. For $k \in K''$ large enough and $t > 0$ small enough, we have

$$\left(v_i^k + t\,dv^k\right) - F_i\left(u^k + t\,du^k\right) \leq -\bar{a}\bar{c}t + o(t) < 0, \quad i \in I_0^F\left(z^*\right).$$

Similarly, for $k \in K''$ large enough and $t > 0$ small enough, we have

$$\begin{aligned}
&\psi_\varepsilon\left(w_i^k + t\,dw^k\right) \leq -\bar{a}\bar{c}t + o(t) < 0, \quad i \in I_0^{\Psi_\varepsilon}\left(z^*\right),\\
&g\left(u_i^k + t\,du^k\right) \leq -\bar{a}\bar{c}t + o(t) < 0, \quad i \in I_0^g\left(z^*\right).
\end{aligned}$$

(ii) Consider the inequality (23). By (29) and Taylor expansion, for $k \in K''$ large enough and $t > 0$ small enough, it follows that

$$\begin{aligned} & f_{\hat{r}}(z^k + t\,dz^k) - f_{\hat{r}}(z^k) - \alpha t \nabla_z f_{\hat{r}}(x^k, y^k)^{\mathrm{T}} dz^k \\ & \quad = (1-\alpha) t \nabla_z f_{\hat{r}}(x^k, y^k)^{\mathrm{T}} dz^k + o(t) \\ & \quad \le (1-\alpha) t\theta\zeta^k + o(t) \le (1-\alpha) t\theta \frac{\bar{\zeta}}{2} + o(t) \le 0. \end{aligned}$$

By summarizing the above discussion, we can conclude that the inequalities (23)–(26) hold for $k \in K''$ large enough and $t > 0$ small enough, that is, $\bar{t} = \inf\{t_k : k \in K''\} > 0$.

In what follows, based on $t_k \ge \bar{t} > 0$ $(k \in K'')$ we will deduce a contradiction. In view of the monotone decreasing property of $\{f_{\hat{r}}(z^k)\}$ and $\lim_{k \in K''} f_{\hat{r}}(z^k) = f_{\hat{r}}(z^*)$, we have $\lim_{k \to \infty} f_{\hat{r}}(z^k) = f_{\hat{r}}(z^*)$. Moreover, from (23) and (29), for $k \in K''$ large enough, we have

$$f_{\hat{r}}(z^{k+1}) - f_{\hat{r}}(z^k) \le \alpha t_k \nabla_z f_{\hat{r}}(x^k, y^k)^{\mathrm{T}} dz^k \le \alpha t_k \theta \zeta^k \le \alpha \bar{t} \theta \frac{\bar{\zeta}}{2}.$$

Taking the limit $k \xrightarrow{K''} \infty$ in the inequality above, we obtain a contradiction.

From the above analysis, we can conclude that z^* is a KKT point of the problem (6).

B. We now show that $I^{\mathcal{C}}(y^*, v^*, \varepsilon) = \emptyset$ (called Conclusion B).

From Lemma 4 and the updating rule of ε_k in Step 5, we see that

$$\sqrt{(y_i^*)^2 + (v_i^*)^2} \ge \varepsilon, \quad \forall i \in I^{\mathcal{C}},$$

which implies that $I^{\mathcal{C}}(y^*, v^*, \varepsilon) = \emptyset$.

Based on Conclusions A and B, and combining with Proposition 1, we can conclude that z^* is a KKT point of the MPCC (1). □

5 Numerical experiments

In this section, a preliminary implementation of Algorithm A is given on a Intel(R) Core(TM) i5-7200U CPU (@ 2.50 GHz, 2.71 GHz) and RAM (4 GB). A Matlab code (Version R2014a) is written corresponding to this implementation.

In our tests, referring to [21], we initialize $H_0 = E_n$, and for $k = 1, 2, \ldots$, the approximate Hessian matrix H_k is determined by the following equation:

$$H_k := \nabla^2_{zz} L_{r_k}(z^k, s^k) + h_k E_{n+2m},$$

where E_{n+2m} is an $(n+2m)$th-order identity matrix, h_k is chosen as follows:

$$h_k := \begin{cases} 0, & \text{if } \lambda_{\min} > s_{\max}; \\ -\lambda_{\min} + s_{\min}, & \text{if } |\lambda_{\min}| \le s_{\max}; \\ 2|\lambda_{\min}|, & \text{otherwise,} \end{cases}$$

Table 1 The computational results

Problem	*dim*	f_{opt}	Algorithm A		filtermpec	snopt	loqo	knitro
			f^*	*itr*	*itr*	*itr*	*itr*	*itr*
bard3m	6	−12.6787	−12.6787	38	4	10	23	66
flp2	4	0	6.2848e−11	9	–	–	–	–
gauvin	3	20	20.0000	34	7	10	20	29
kth2	2	0	−5.2411e−06	12	2	4	11	7
kth3	2	0.5	0.5000	5	4	6	10	22
scholtes2	3	15	14.4898	15	2	3	11	16
scholtes3	2	0.5	0.5000	6	4	0	29	10
scholtes5	3	1.0	1.0000	8	1	3	10	8

where $s_{\min} > 0$ and $s_{\max} > 0$ are sufficiently small and sufficiently large, respectively, $\lambda_{\min}$ is the smallest eigenvalue of the following matrix:

$$\begin{aligned}\mathcal{B}_k = {} & \nabla^2_{zz} L_{r_k}\big(z^k, s^k\big) - \sum_{i\in I^C} \frac{s^k_{F,i}}{v_i - F_i(x^k, y^k)} \nabla_z\big(v_i - F_i\big(x^k, y^k\big)\big) \nabla_z\big(v_i - F_i\big(x^k, y^k\big)\big)^{\mathrm{T}} \\ & - \sum_{i\in I^C} \frac{s^k_{\Psi_\varepsilon,i}}{\psi_\varepsilon(y^k_i, v^k_i)} \nabla_z\big(\psi_\varepsilon\big(y^k_i, v^k_i\big)\big) \nabla_z\big(\psi_\varepsilon\big(y^k_i, v^k_i\big)\big)^{\mathrm{T}} \\ & - \sum_{i\in I_g} \frac{s^k_{g,i}}{g_i(x^k, y^k)} \nabla_z g_i\big(x^k, y^k\big) \nabla_z g_i\big(x^k, y^k\big)^{\mathrm{T}}.\end{aligned}$$

In the numerical experiments, the parameters are chosen as follows:

$$\begin{aligned} & \alpha = 0.49, \qquad \varepsilon = 10^{-5}, \qquad \beta = 0.5, \qquad \theta = 0.99, \qquad q = 3, \qquad \tau = 2, \\ & r = 1, \qquad C = 100, \qquad \vartheta = 0.5, \qquad \nu = 1, \qquad \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1, \\ & s_{\max} = 10^{20}, \qquad s_{\min} = 10^{-5}. \end{aligned}$$

Algorithm A stops if one of the following termination criteria is satisfied:
(1) $\rho(z^k, \lambda^k) \le 10^{-5}$ and $I^C(y^k, v^k, \varepsilon_k) = \emptyset$.
(2) $\|\widehat{dz}^k\| \le 10^{-5}$, $\max\{-\widehat{\lambda}^k_{g,i}, i \in I^g\} < 10^{-5}$ and $I^C(y^k, v^k, \varepsilon_k) = \emptyset$.

The following test problems are selected from [34]. Algorithm A can find their solutions within a small number of iterations. We compared Algorithm A with four algorithms, i.e., filtermpec, snopt, loqo, and knitro given in [35]. The computational results are given in Table 1. The meaning of some notations in Table 1 are as follows:

Problem: The problem in [34].
dim: the dimensions of the variables (x, y);
itr: the number of iterations;
f_{opt}: the optimal value given in [34];
f^*: the optimal value obtained by Algorithm A.

6 Conclusions

In this paper, based on Fischer–Burmeister function and working set techniques, a primal–dual interior point QP-free algorithm for mathematical programs with complementarity constraints is presented. At each iteration, only two reduced linear equations with the same coefficient matrix are solved to yield the search direction. The use of the working set decreases the computational cost. Moreover, the uniformly positive definiteness on the Lagrangian Hessian estimate H_k is relaxed. Under some mild conditions, the proposed algorithm is globally convergent. The preliminary numerical results show that the proposed algorithm is effective.

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