Konar et al. *Journal of Inequalities and Applications* (2020) 2020:250
https://doi.org/10.1186/s13660-020-02519-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Multivalued weak cyclic δ-contraction mappings

Pulak Konar[1], Samir Kumar Bhandari[2], Sumit Chandok[3*] and Aiman Mukheimer[4*]

*Correspondence:
sumit.chandok@thapar.edu;
chansok.s@gmail.com;
mukheimer@psu.edu.sa
[3]School of Mathematics, Thapar Institute of Engineering & Technology, Patiala 147-004, Punjab, India
[4]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we propose some new type of weak cyclic multivalued contraction mappings by generalizing the cyclic contraction using the δ-distance function. Several novel fixed point results are deduced for such class of weak cyclic multivalued mappings in the framework of metric spaces. Also, we construct some examples to validate the usability of the results. Various existing results of the literature are generalized.

MSC: 41A50; 47H10; 54H25

Keywords: Cyclic contraction; δ-distance; Fixed point; Geraghty contraction; Metric space

1 Introduction

In 2003, Kirk et al. [19] introduced the cyclic contraction and established some interesting results for such contractions in the setting of metric spaces. Thereafter many researchers worked in this arena and obtained astounding results, which have a lot of applications in various fields. Some well-known references consisting of similar type of work may be noted (see [7, 9–11, 22, 29]). Cyclic contractions are contractions useful to obtain fixed point and optimality results for non-self-mappings. Some coupling over the study of fixed points can be obtained through cyclic contractions; for details see [13]. The other utility of cyclic contractions is related to optimality problems; for details see [14].

Alber et al. [3] proposed weak contractions in Hilbert spaces and subsequently Rhoades [25] extended it. Several references to the literature are available with generalized weak contractions in metric and allied spaces with partially ordered metric spaces through [2–6, 8, 15, 16, 20, 21, 23, 26–28, 30]. An important contribution towards a generalized weak contraction was established by Choudhury et al. [12].

In this paper, we define multivalued $\mathscr{C}_S$-contractions and C_Γ-contractions mappings by generalizing cyclic contraction using δ-distance functions. Using the concept of Kirk et al. [19] with a blending of Geraghty contractions, we obtain some new fixed point results for such a class of weak cyclic mappings in the setting of metric spaces. Also, we provide some examples to show the usability of the results.

2 Main results

Throughout the paper, we suppose that $(\Delta, \wp)$ is a metric space and $\mathfrak{CB}(\Delta)$ denotes the family of nonempty closed and bounded subsets of Δ. Acar and Altun [1] define $\mathscr{D}(\sigma, \mathfrak{A})$ and $\delta(\mathfrak{A}, \mathfrak{B})$, for $\mathfrak{A}, \mathfrak{B} \in \mathfrak{CB}(\Delta)$, and $\sigma \in \Delta$, by

$$\mathscr{D}(\sigma, \mathfrak{A}) := \inf\bigl\{\wp(\sigma, \tilde{a})\bigr\}; \quad \text{for all } \tilde{a} \in \mathfrak{A}$$

and

$$\delta(\mathfrak{A}, \mathfrak{B}) := \sup\bigl\{\wp(\tilde{a}, \tilde{b}) : \tilde{a} \in \mathfrak{A}, \tilde{b} \in \mathfrak{B}\bigr\}.$$

Following Rakotch [24], Geraghty [17] introduced the following class of function:
Suppose that $\mathscr{S}$ is the class of functions $\varrho : R^+ \to [0, 1)$ with

(i) $R^+ = \{t \in R : t > 0\}$,
(ii) $\varrho(t_\beta) \to 1$ implies $t_\beta \to 0$.

Definition 1 ([18]) An element $\sigma \in \Delta$ is said to be a fixed point of a multi-valued mapping $\mathfrak{O} : \Delta \to \mathfrak{CB}(\Delta)$, such that $\sigma \in \mathfrak{O}(\sigma)$.

Now, we derive a fixed point theorem by applying Geraghty's contraction to $\mathfrak{O}$ to show that $\bigcap_{i=1}^k \mathfrak{CB}(A_i)$ is nonempty.
Simply put, if $j > k$ define $\mathfrak{A}_j = \mathfrak{A}_i$ where $i \equiv j(\mod k)$ and $1 \le i \le k$.

Definition 2 Suppose that $\{\mathfrak{A}_i\}_{i=1}^k$ are nonempty closed subsets of a metric space $(\Delta, \wp)$ and $\mathfrak{O} : \bigcup_{i=1}^k \mathfrak{A}_i \to \bigcup_{i=1}^k \mathfrak{CB}(\mathfrak{A}_i)$ such that $\mathfrak{O}(\mathfrak{A}_i) \subseteq \mathfrak{A}_{i+1}$ for $1 \le i \le k$ (where $\mathfrak{A}_{k+1} = \mathfrak{A}_1$). A mapping $\mathfrak{O}$ is called $\mathscr{C}_S$-contraction if for all $\sigma \in \mathfrak{A}_i$, $\varpi \in \mathfrak{A}_{i+1}$, $1 \le i \le k$, and a $\varrho \in \mathscr{S}$, we have

$$\delta(\mathfrak{O}\sigma, \mathfrak{O}\varpi) \le \varrho\bigl(\wp(\sigma, \varpi)\bigr)\mathscr{M}(\sigma, \varpi), \tag{2.1}$$

where

$$\mathscr{M}(\sigma, \varpi) = \max\left\{\wp(\sigma, \varpi), \frac{1}{2}\bigl[\mathscr{D}(\sigma, \mathfrak{O}\sigma) + \mathscr{D}(\varpi, \mathfrak{O}\varpi)\bigr], \frac{1}{2}\bigl[\mathscr{D}(\sigma, \mathfrak{O}\varpi) + \mathscr{D}(\varpi, \mathfrak{O}\sigma)\bigr]\right\}.$$

Theorem 1 *Every $\mathscr{C}_S$-contraction mapping on a complete metric space $(\Delta, \wp)$ has at least a fixed point in $\bigcap_{i=1}^k \mathfrak{CB}(A_i)$.*

Proof We present the proof of this theorem in the following steps.

First Step: Assume $\sigma_0 \in \mathfrak{A}_1$ and $\sigma_\beta \in \mathfrak{O}^\beta\sigma_0$, $\beta = 1, 2, \ldots$, such that $\sigma_1 \in \mathfrak{O}\sigma_0, \sigma_2 \in \mathfrak{O}\sigma_1, \ldots$. If possible, for some $\beta \in \mathbb{N}$, let $\wp(\sigma_\beta, \sigma_{\beta+1}) > \wp(\sigma_{\beta-1}, \sigma_\beta)$. Consider

$$\begin{aligned}
\wp(\sigma_\beta, \sigma_{\beta+1}) &\le \delta\bigl(\mathfrak{O}^\beta\sigma_0, \mathfrak{O}^{\beta+1}\sigma_0\bigr) \\
&= \delta(\mathfrak{O}\sigma_{\beta-1}, \mathfrak{O}\sigma_\beta) \\
&\le \varrho\bigl(\wp(\sigma_{\beta-1}, \sigma_\beta)\bigr)\mathscr{M}(\sigma_{\beta-1}, \sigma_\beta) \\
&= \varrho\bigl(\wp(\sigma_{\beta-1}, \sigma_\beta)\bigr)\max\Bigl\{\wp(\sigma_{\beta-1}, \sigma_\beta), \frac{1}{2}\bigl[\mathscr{D}(\sigma_{\beta-1}, \mathfrak{O}\sigma_{\beta-1}) + \mathscr{D}(\sigma_\beta, \mathfrak{O}\sigma_\beta)\bigr],
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}\big[\mathscr{D}(\sigma_{\beta-1},\mathfrak{O}\sigma_\beta)+\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_{\beta-1})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\max\bigg\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big],\\
&\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_{\beta+1})+\wp(\sigma_\beta,\sigma_\beta)\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\max\bigg\{\wp(\sigma_{\beta-1},(\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_{\beta-1}\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big],\\
&\frac{1}{2}\big[\wp((\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\}\quad\text{[using the triangular inequality]}\\
&\le \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\max\bigg\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\max\bigg\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\max\big\{\wp(\sigma_{\beta-1},\sigma_\beta),\wp(\sigma_\beta,\sigma_{\beta+1})\big\}\\
&\le \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\wp(\sigma_\beta,\sigma_{\beta+1}).
\end{aligned}
$$

It implies that $\varrho(\wp(\sigma_{\beta-1},\sigma_\beta))\ge 1$, which is a contradiction since $\varrho\in\mathscr{S}$. Therefore, for all $\beta\ge 1$, $\wp(\sigma_\beta,\sigma_{\beta+1})\le\wp(\sigma_{\beta-1},\sigma_\beta)$. Hence $\{(\wp(\sigma_\beta,\sigma_{\beta+1}))\}$ is a decreasing sequence.

Furthermore, using (2.1), we have

$$
\begin{aligned}
\wp(\sigma_{\beta+1},\sigma_{\beta+2}) &\le \delta(\mathfrak{O}\sigma_\beta,\mathfrak{O}\sigma_{\beta+1})\\
&\le \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\mathscr{M}(\sigma_\beta,\sigma_{\beta+1})\\
&= \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_\beta)+\mathscr{D}(\sigma_{\beta+1},\mathfrak{O}\sigma_{\beta+1})\big],\\
&\quad\frac{1}{2}\big[\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_{\beta+1})+\mathscr{D}(\sigma_{\beta+1},\mathfrak{O}\sigma_\beta)\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_{\beta+1},\sigma_{\beta+2})\big],\\
&\quad\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+2})+\wp(\sigma_{\beta+1},\sigma_{\beta+1})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_{\beta+1},\sigma_{\beta+2})\big],\\
&\quad\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_{\beta+1},\sigma_{\beta+2})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_{\beta+1},\sigma_{\beta+2})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\}\\
&\le \varrho\big(\wp(\sigma_\beta,\sigma_{\beta+1})\big)\max\big\{\wp(\sigma_\beta,\sigma_{\beta+1}),\wp(\sigma_\beta,\sigma_{\beta+1})\big\}\\
&= \varrho\big(\wp(\sigma_{\beta-1},\sigma_\beta)\big)\wp(\sigma_\beta,\sigma_{\beta+1}).
\end{aligned}
$$

It implies that $\frac{\wp(\sigma_{\beta+1},\sigma_{\beta+2})}{\wp(\sigma_\beta,\sigma_{\beta+1})} \leq \varrho(\wp(\sigma_\beta,\sigma_{\beta+1})) < 1$, for $\beta = 1,2,3,\ldots$. Now, take $\beta \to +\infty$, and we get $\varrho(\wp(\sigma_\beta,\sigma_{\beta+1})) \to 1$, and since $\varrho \in \mathscr{S}$, we have $\wp(\sigma_\beta,\sigma_{\beta+1}) \to 0$.

Second Step: Suppose that there is $\rho > 0$ such that, for any $\beta_1 \in \mathbb{N}$, there exists $\beta > \alpha \geq \beta_1$ with $\beta - \alpha \equiv 1(\mod k)$ such that $\wp(\sigma_\beta,\sigma_\alpha) \geq \rho > 0$. Utilizing the triangle inequality, we get

$$\wp(\sigma_\beta,\sigma_\alpha) \leq \wp(\sigma_\beta,\sigma_{\beta+1}) + \wp(\sigma_{\beta+1},\sigma_{\alpha+1}) + \wp(\sigma_{\alpha+1},\sigma_\alpha)$$

and

$$\begin{aligned}
\mathscr{M}(\sigma_{\beta-1},\sigma_\beta) = {} & \max\Bigl\{\wp(\sigma_{\beta-1},\sigma_\beta), \frac{1}{2}\bigl[\mathscr{D}(\sigma_{\beta-1},\mathfrak{O}\sigma_{\beta-1}) + \mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_\beta)\bigr], \\
& \frac{1}{2}\bigl[\mathscr{D}(\sigma_{\beta-1},\mathfrak{O}\sigma_\beta) + \mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_{\beta-1})\bigr]\Bigr\} \\
= {} & \max\Bigl\{\wp(\sigma_{\beta-1},\sigma_\beta), \frac{1}{2}\bigl[\wp(\sigma_{\beta-1},\sigma_\beta) + \wp(\sigma_\beta,\sigma_{\beta+1})\bigr], \\
& \frac{1}{2}\bigl[\wp(\sigma_{\beta-1},\sigma_{\beta+1}) + \wp(\sigma_\beta,\sigma_\beta)\bigr]\Bigr\} \\
\leq {} & \max\Bigl\{\wp(\sigma_{\beta-1},\sigma_\beta), \frac{1}{2}\bigl[\wp(\sigma_{\beta-1},\sigma_\beta) + \wp(\sigma_\beta,\sigma_{\beta+1})\bigr], \\
& \frac{1}{2}\bigl[\wp(\sigma_{\beta-1},\mathfrak{O}\sigma_\beta) + \wp(\sigma_\beta,\sigma_{\beta+1})\bigr]\Bigr\} \\
= {} & \max\Bigl\{\wp(\sigma_{\beta-1},\sigma_\beta), \frac{1}{2}\bigl[\wp(\sigma_{\beta-1},\sigma_\beta) + \wp(\sigma_\beta,\sigma_{\beta+1})\bigr]\Bigr\} \\
= {} & \max\bigl\{\wp(\sigma_{\beta-1},\sigma_\beta), \wp(\sigma_{\beta-1},\sigma_\beta)\bigr\} \\
= {} & \wp(\sigma_{\beta-1},\sigma_\beta),
\end{aligned}$$

which implies $-\wp(\sigma_{\beta-1},\sigma_\beta) \leq -\mathscr{M}(\sigma_{\beta-1},\sigma_\beta)$.

Since $\beta - \alpha \equiv 1(\mod k)$, σ_α and σ_β lie in different but consecutive sets $\mathfrak{A}_i$ and $\mathfrak{A}_{i+1}$ for some $1 \leq i \leq k$, by the contractive condition we get

$$\begin{aligned}
\bigl[1 - \varrho\bigl(\wp(\sigma_\beta,\sigma_\alpha)\bigr)\bigr]\rho &\leq \bigl[1 - \varrho\bigl(\wp(\sigma_\beta,\sigma_\alpha)\bigr)\bigr]\wp(\sigma_\beta,\sigma_\alpha) \\
&= \wp(\sigma_\beta,\sigma_\alpha) - \varrho\bigl(\wp(\sigma_\beta,\sigma_\alpha)\bigr)\wp(\sigma_\beta,\sigma_\alpha) \\
&\leq \wp(\sigma_\beta,\sigma_\alpha) - \varrho\bigl(\wp(\sigma_\beta,\sigma_\alpha)\bigr)\mathscr{M}(\sigma_\beta,\sigma_\alpha) \\
&\leq \wp(\sigma_\beta,\sigma_\alpha) - \delta(\mathfrak{O}\sigma_\beta,\mathfrak{O}\sigma_\alpha) \\
&\leq \wp(\sigma_\beta,\sigma_\alpha) - \wp(\sigma_{\beta+1},\sigma_{\alpha+1}) \\
&\leq \wp(\sigma_\beta,\sigma_{\beta+1}) + \wp(\sigma_{\beta+1},\sigma_{\alpha+1}) + \wp(\sigma_{\alpha+1},\sigma_\alpha) - \wp(\sigma_{\beta+1},\sigma_{\alpha+1}) \\
&= \wp(\sigma_\beta,\sigma_{\beta+1}) + \wp(\sigma_{\alpha+1},\sigma_\alpha).
\end{aligned}$$

Taking $\beta,\alpha \to +\infty$ with $\beta - \alpha \equiv 1(\mod k)$, we have $\varrho(\wp(\sigma_\beta,\sigma_\alpha)) \to 1$. But, since $\varrho \in \mathscr{S}$, we have $\wp(\sigma_\beta,\sigma_\alpha) \to 0$, which leads to a contradiction. Therefore, for given any $\epsilon > 0$ there exists $\beta_1 \in \mathbb{N}$ such that, for $\beta,\alpha \geq \beta_1$ and $\beta - \alpha \equiv 1(\mod k)$, we have $\wp(\sigma_\beta,\sigma_\alpha) < \epsilon/\rho$.

By the first step, we choose $\beta_2 \in \mathbb{N}$ so that $\wp(\sigma_\beta, \sigma_\alpha) < \epsilon/\rho$ if $\beta \geq \beta_2$. Considering $\beta, \alpha \geq \max\{\beta_1, \beta_2\}$ with $\beta > \alpha$. Then there exists $p \in \{1, 2, 3, \ldots, k\}$ such that $\beta - \alpha \equiv p(\mod k)$. Thus $\beta - \alpha + j \equiv 1(\mod k)$, where $j = k - p + 1$ and hence

$$\wp(\sigma_\beta, \sigma_\alpha) \leq \wp(\sigma_\alpha, \sigma_{\beta+j}) + \wp(\sigma_{\beta+j}, \sigma_{\beta+j-1}) + \cdots + \wp(\sigma_{\beta+1}, \sigma_\beta) < \rho \cdot \epsilon/\rho = \epsilon,$$

that is, $\wp(\sigma_\beta, \sigma_\alpha) < \epsilon$. This proves that $\{\sigma_\beta\}$ is a Cauchy sequence, and consequently that $\bigcap_{i=1}^{k} \mathfrak{CB}(A_i) \neq \emptyset$.

Third Step: Next we prove that there is a point $z \in \mathfrak{O}z$ which will be the fixed point of $\mathfrak{O}$. On the contrary assume that $z \notin \mathfrak{O}z$. Then there exist $n_0 \in \mathbb{N}$ and a subsequence $\{\sigma_{\beta_d}\}$ of $\{\sigma_\beta\}$ such that $\mathscr{D}(\sigma_{\beta_d+1}, \mathfrak{O}z) > 0$ for all $\beta_d \geq \beta_0$ else, there exists $\beta_1 \in \mathbb{N}$ such that $\sigma_\beta \in \mathfrak{O}z$ for all $\beta \geq \beta_1$, which implies that $z \in \mathfrak{O}z$, a contradiction to our assumption that $z \notin \mathfrak{O}z$. Since $\mathscr{D}(\sigma_{\beta_d+1}, \mathfrak{O}z) > 0$, for all $\beta_d \geq \beta_0$, we have

$$\begin{aligned}
\mathscr{D}(\sigma_{\beta_d+1}, \mathfrak{O}z) &\leq \delta(\mathfrak{O}\sigma_{\beta_d}, \mathfrak{O}z) \\
&\leq \varrho\big(\wp(\sigma_{\beta_d}, z)\big)\mathscr{M}(\sigma_{\beta_d}, z) \\
&\leq \mathscr{M}(\sigma_{\beta_d}, z) \\
&= \max\left\{\wp(\sigma_{\beta_d}, z), \frac{1}{2}\big[\mathscr{D}(\sigma_{\beta_d}, \mathfrak{O}\sigma_{\beta_d}) + \mathscr{D}(z, \mathfrak{O}z)\big],\right. \\
&\qquad \left.\frac{1}{2}\big[\mathscr{D}(\sigma_{\beta_d}, \mathfrak{O}z) + \mathscr{D}(z, \mathfrak{O}\sigma_{\beta_d})\big]\right\} \\
&\leq \max\left\{d(\sigma_{\beta_d}, z), \frac{1}{2}\big[\wp(\sigma_{\beta_d}, \mathfrak{O}\sigma_{\beta_{k+1}}) + \mathscr{D}(z, \mathfrak{O}z)\big],\right. \\
&\qquad \left.\frac{1}{2}\big[\mathscr{D}(\sigma_{\beta_d}, \mathfrak{O}z) + d(z, \mathfrak{O}\sigma_{\beta_{k+1}})\big]\right\}.
\end{aligned}$$

Taking the limit $d \to +\infty$, we get $\mathscr{D}(z, \mathfrak{O}z) \leq \frac{1}{2}\mathscr{D}(z, \mathfrak{O}z)$, which is a contradiction. Thus, we get $z \in \overline{\mathfrak{O}z} = \mathfrak{O}z$. Hence the result. □

By putting $\mathscr{M}(\sigma, \varpi) = \wp(\sigma, \varpi)$ in Theorem 1, we have the following result.

Corollary 1 *Let $\{\mathfrak{A}_i\}_{i=1}^{k}$ be nonempty closed subsets of a complete metric space $(\Delta, \wp)$. Suppose that $\mathfrak{O} : \bigcup_{i=1}^{k} \mathfrak{A}_i \to \bigcup_{i=1}^{k} \mathfrak{CB}(\mathfrak{A}_i)$ satisfies the following conditions*:

(i) $\mathfrak{O}(\mathfrak{A}_i) \subseteq \mathfrak{A}_{i+1}$ *for* $1 \leq i \leq k$, (*where* $\mathfrak{A}_{k+1} = \mathfrak{A}_1$);

(ii) $\delta(\mathfrak{O}\sigma, \mathfrak{O}\varpi) \leq \varrho(\wp(\sigma, \varpi))\wp(\sigma, \varpi)$ *for all* $\sigma \in \mathfrak{A}_i$, $\varpi \in \mathfrak{A}_{i+1}$ *for* $1 \leq i \leq k$, $\varrho \in \mathscr{S}$.

Then $\mathfrak{O}$ has at least a fixed point in $\cap_i \mathfrak{CB}(\mathfrak{A}_i)$.

The next corollary follows by imposing $\mathscr{M}(\sigma, \varpi) = \wp(\sigma, \varpi)$ and $\delta(\sigma, \varpi) = \wp(\sigma, \varpi)$ in Theorem 1.

Corollary 2 *Assume that $\{\mathfrak{A}_i\}_{i=1}^{k}$ is a nonempty closed subsets of a complete metric space $(\Delta, \wp)$. Suppose that $\mathfrak{O} : \bigcup_{i=1}^{k} \mathfrak{A}_i \to \bigcup_{i=1}^{k} \mathfrak{CB}(\mathfrak{A}_i)$ satisfies the conditions as follows*:

(i) $\mathfrak{O}(\mathfrak{A}_i) \subseteq \mathfrak{A}_{i+1}$ *for* $1 \leq i \leq k$, (*where* $\mathfrak{A}_{k+1} = \mathfrak{A}_1$);

(ii) $\wp(\mathfrak{O}\sigma, \mathfrak{O}\varpi) \leq \varrho(\wp(\sigma, \varpi))\wp(\sigma, \varpi)$ *for all* $\sigma \in \mathfrak{A}_i$, $\varpi \in \mathfrak{A}_{i+1}$ *for* $1 \leq i \leq k$, $\varrho \in \mathscr{S}$.

Then $\mathfrak{O}$ has at least a fixed point in $\cap_i \mathfrak{CB}(\mathfrak{A}_i)$.

By treating multivalued mapping $\mathfrak{O}$ as a singleton set, we have the following result.

Corollary 3 *Assume that* $\{\mathfrak{A}_i\}_{i=1}^k$ *is a nonempty closed subset of a complete metric space* $(\Delta,\wp)$. *Suppose that* $\mathfrak{O}:\bigcup_{i=1}^k\mathfrak{A}_i\to\bigcup_{i=1}^k\mathfrak{A}_i$ *satisfies the conditions as follows*:

(i) $\mathfrak{O}(\mathfrak{A}_i)\subseteq\mathfrak{A}_{i+1}$ *for* $1\le i\le k$, (*where* $\mathfrak{A}_{k+1}=\mathfrak{A}_1$);

(ii) $\wp(\mathfrak{O}\sigma,\mathfrak{O}\varpi)\le\varrho(\wp(\sigma,\varpi))\wp(\sigma,\varpi)$ *for all* $\sigma\in\mathfrak{A}_i$, $\varpi\in\mathfrak{A}_{i+1}$ *for* $1\le i\le k$, $\varrho\in\mathscr{S}$.

Then $\mathfrak{O}$ *has a fixed point in* $\cap_i\mathfrak{A}_i$.

Example 1 Let $\Delta=[0,1]$ with usual metric, $\mathfrak{A}_1=[0,1]$, $\mathfrak{A}_2=[0,1]$ such that $\Delta=\cup_{i=1}^2\mathfrak{A}_i$. Assume that $\mathfrak{O}x=\ln(1+\frac{x}{6})$. Here $\mathfrak{O}\mathfrak{A}_1\subseteq\mathfrak{A}_2$ and $\mathfrak{O}\mathfrak{A}_2\subseteq\mathfrak{A}_1$. Consider $\varrho(t)=\frac{1}{1+t}$, when $t\in(0,+\infty)$ and $\varrho(t)=1$, when $t=0$, so it satisfies the Geraghty condition. Here all the hypotheses of Corollary 3 are satisfied and 0 is a fixed point.

We denote by Γ the collection of all functions $\Psi:R^+\to[0,+\infty)$ satisfying the following conditions:

(a) Ψ is upper semi-continuous from the right;

(b) $0\le\Psi(t)<t$ for $t>0$.

Definition 3 Suppose that $\{\mathfrak{A}_i\}_{i=1}^k$ are nonempty closed subsets of a metric space $(\Delta,\wp)$ and $\mathfrak{O}:\bigcup_{i=1}^k\mathfrak{A}_i\to\bigcup_{i=1}^k\mathfrak{CB}(\mathfrak{A}_i)$ such that $\mathfrak{O}(\mathfrak{A}_i)\subseteq\mathfrak{A}_{i+1}$ for $1\le i\le k$ (where $\mathfrak{A}_{p+1}=\mathfrak{A}_1$). A mapping $\mathfrak{O}$ is called a C_Γ-contraction if there exists $\Psi\in\Gamma$ and, for all $\sigma\in A_i$, $\varpi\in A_{i+1}$, $1\le i\le k$, we have

$$\delta(\mathfrak{O}\sigma,\mathfrak{O}\varpi)\le\Psi\bigl(\mathscr{M}(\sigma,\varpi)\bigr),\tag{2.2}$$

where $M(\sigma,\varpi)=\max\{\wp(\sigma,\varpi),\frac{1}{2}[\mathscr{D}(\sigma,\mathfrak{O}\sigma)+\mathscr{D}(\varpi,\mathfrak{O}\varpi)],\frac{1}{2}[\mathscr{D}(\sigma,\mathfrak{O}\varpi)+\mathscr{D}(\varpi,\mathfrak{O}x)]\}$.

Theorem 2 *Every* C_Γ*-contraction mapping on a complete metric space* $(\Delta,\wp)$ *has at least a fixed point in* $\bigcap_{i=1}^k\mathfrak{CB}(A_i)$.

Proof Let $\sigma_0\in A_1$ and $\sigma_\beta\in\mathfrak{O}^\beta\sigma_0$, $\beta=1,2,\dots$, such that $\sigma_1\in\mathfrak{O}\sigma_0,\dots$.

First Step:

If possible, for some β, let $\wp(\sigma_\beta,\sigma_{\beta+1})>\wp(\sigma_{\beta-1},\sigma_\beta)$. Now, utilizing the triangular property, we have

$$\begin{aligned}
\wp(\sigma_\beta,\sigma_{\beta+1}) &\le \delta\bigl(\mathfrak{O}^\beta\sigma_0,T^{\beta+1}\sigma_0\bigr)\\
&=\delta(\mathfrak{O}\sigma_{\beta-1},\mathfrak{O}\sigma_\beta)\\
&\le\Psi\bigl(\mathscr{M}(\sigma_{\beta-1},\sigma_\beta)\bigr)\\
&<\mathscr{M}(\sigma_{\beta-1},\sigma_\beta)\\
&=\max\biggl\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\bigl[\mathscr{D}(\sigma_{\beta-1},\mathfrak{O}\sigma_{\beta-1})+\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_\beta)\bigr],\\
&\quad\frac{1}{2}\bigl[\mathscr{D}(\sigma_{\beta-1},\mathfrak{O}\sigma_\beta)+\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_{\beta-1})\bigr]\biggr\}\\
&\le\max\biggl\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\bigl[\wp(\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\bigr],
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_{\beta+1})+\wp(\sigma_\beta,\sigma_\beta)\big]\bigg\} \\
&\le \max\bigg\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big], \\
&\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\} \\
&\le \max\bigg\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_{\beta-1},\sigma_\beta)+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\} \\
&\le \max\bigg\{\wp(\sigma_{\beta-1},\sigma_\beta),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_\beta,\sigma_{\beta+1})\big]\bigg\} \\
&\le \max\big\{\wp(\sigma_{\beta-1},\sigma_\beta),\wp(\sigma_\beta,\sigma_{\beta+1})\big\} \\
&\le \wp(\sigma_\beta,\sigma_{\beta+1}),
\end{aligned}$$

which implies $\wp(\sigma_\beta,\sigma_{\beta+1}) < \wp(\sigma_\beta,\sigma_{\beta+1})$, which leads to a contradiction. Therefore, for all $\beta \ge 1$, $\wp(\sigma_\beta,\sigma_{\beta+1}) \le \wp(\sigma_{\beta-1},\sigma_\beta)$. Hence $\{(\wp(\sigma_\beta,\sigma_{\beta+1}))\}$ is a decreasing sequence.

Again assume that $\lim_{\beta\to+\infty}\wp(\sigma_\beta,\sigma_{\beta+1}) = \gamma \ge 0$. Say $\gamma > 0$. Using (2.2), we have

$$\begin{aligned}
\wp(\sigma_{\beta+1},\sigma_{\beta+2}) &\le \delta(\mathfrak{O}\sigma_\beta,\mathfrak{O}\sigma_{\beta+1}) \\
&\le \Psi\big(\mathscr{M}(\sigma_\beta,\sigma_{\beta+1})\big) \\
&= \Psi\bigg(\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_\beta)+\mathscr{D}(\sigma_{\beta+1},\mathfrak{O}\sigma_{\beta+1})\big], \\
&\quad \frac{1}{2}\big[\mathscr{D}(\sigma_\beta,\mathfrak{O}\sigma_{\beta+1})+\mathscr{D}(\sigma_{\beta+1},\mathfrak{O}\sigma_\beta)\big]\bigg\}\bigg) \\
&\le \Psi\bigg(\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_{\beta+1},\sigma_{\beta+2})\big], \\
&\quad \frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+2})+\wp(\sigma_{\beta+1},\sigma_{\beta+1})\big]\bigg\}\bigg) \\
&\le \Psi\bigg(\max\bigg\{\wp(\sigma_\beta,\sigma_{\beta+1}),\frac{1}{2}\big[\wp(\sigma_\beta,\sigma_{\beta+1})+\wp(\sigma_{\beta+1},\sigma_{\beta+2})\big]\bigg\}\bigg).
\end{aligned}$$

Taking $\beta \to +\infty$, we see that $\gamma \le \Psi(\gamma)$ which is possible only when $\gamma = 0$.

Therefore, $\lim_{\beta\to+\infty}\wp(\sigma_\beta,\sigma_{\beta+1}) = 0$.

Second Step: In this step we prove that the sequence $\{\sigma_\beta\}$ is a Cauchy sequence. If possible let there exists $\epsilon > 0$ such that, for any $d \in \mathbb{N}$, there exist $\alpha_d > \beta_d \ge d$ such that $\wp(\sigma_{\alpha_d},\sigma_{\beta_d}) \ge \epsilon$. Again, we say that, for each d, α_d is chosen to be the smallest number greater that β_d then the above is true. So,

$$\lim_{d\to+\infty}\wp(\sigma_{\alpha_d},\sigma_{\alpha_d-1}) = 0.$$

Furthermore, we have

$$\epsilon \le \wp(\sigma_{\alpha_d},\sigma_{\beta_d}) \le \wp(\sigma_{\alpha_d},\sigma_{\alpha_{d-1}}) + \wp(\sigma_{\alpha_{d-1}},\sigma_{\beta_d}) \le \wp(\sigma_{\alpha_d},\sigma_{\alpha_{d-1}}) + \epsilon.$$

Therefore,

$$\lim_{d\to+\infty} \wp(\sigma_{\alpha_d}, \sigma_{\beta_d}) = \epsilon.$$

Also

$$\wp(\sigma_{\alpha_d}, \sigma_{\beta_d}) - \wp(\sigma_{\alpha_d+1}, \sigma_{\alpha_d}) \leq \wp(\sigma_{\alpha_d+1}, \sigma_{\beta_d}) \leq \wp(\sigma_{\alpha_d+1}, \sigma_{\alpha_d}) + \wp(\sigma_{\alpha_d}, \sigma_{\beta_d}).$$

Therefore, we get

$$\lim_{d\to+\infty} \wp(\sigma_{\alpha_{d+1}}, \sigma_{\beta_d}) = \epsilon.$$

So, there is j, with $0 \leq j \leq k-1$, such that $\alpha_d - \beta_d + j \equiv 1(\mod k)$ for infinitely many d.

If $j = 0$, then, for some d, we have

$$\begin{aligned}
\wp(\sigma_{\alpha_d}, \sigma_{\beta_d}) &\leq \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \wp(\sigma_{\alpha_{d+1}}, \sigma_{\beta_{d+1}}) + \wp(\sigma_{\beta_{d+1}}, \sigma_{\beta_d}) \\
&\leq \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \Psi\big(\mathscr{M}(\sigma_{\alpha_d}, \sigma_{\beta_d})\big) + \wp(\sigma_{\beta_{d+1}}, \sigma_{\beta_d}) \\
&< \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \mathscr{M}(\sigma_{\alpha_d}, \sigma_{\beta_d}) + \wp(\sigma_{\beta_{d+1}}, \sigma_{\beta_d}) \\
&= \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \max\bigg\{\wp(\sigma_{\alpha_d}, \sigma_{\beta_d}), \frac{1}{2}\big[\mathscr{D}(\sigma_{\alpha_d}, \mathfrak{O}\sigma_{\alpha_d}) + \mathscr{D}(\sigma_{\beta_d}, \mathfrak{O}\sigma_{\beta_d})\big], \\
&\quad \frac{1}{2}\big[\mathscr{D}(\sigma_{\alpha_d}, \mathfrak{O}\sigma_{\beta_d}) + \mathscr{D}(\sigma_{\beta_d}, \mathfrak{O}\sigma_{\alpha_d})\big]\bigg\} + \wp(\sigma_{\beta_{d+1}}, \sigma_{\beta_d}) \\
&\leq \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \max\bigg\{\wp(\sigma_{\alpha_d}, \sigma_{\beta_d}), \frac{1}{2}\big[\wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \wp(\sigma_{\beta_d}, \sigma_{\beta_{d+1}})\big], \\
&\quad \frac{1}{2}\big[\wp(\sigma_{\alpha_d}, \sigma_{\beta_{d+1}}) + \wp(\sigma_{\beta_d}, \sigma_{\alpha_{d+1}})\big]\bigg\} + \wp(\sigma_{\beta_{d+1}}, \sigma_{\beta_d}) \\
&\leq \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \max\bigg\{\wp(\sigma_{\alpha_d}, \sigma_{\beta_d}), \frac{1}{2}\big[\wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}}) + \wp(\sigma_{\beta_d}, \sigma_{\beta_{d+1}})\big], \\
&\quad \frac{1}{2}\big[\wp(\sigma_{\alpha_d}, \sigma_{\beta_d}) + \wp(\sigma_{\beta_d}, \sigma_{\beta_{d+1}}) + \wp(\sigma_{\beta_d}, \sigma_{\alpha_d}) + \wp(\sigma_{\alpha_d}, \sigma_{\alpha_{d+1}})\big]\bigg\} \\
&\quad + \wp(\sigma_{\beta_{d+1}}, \sigma_{\beta_d}).
\end{aligned}$$

Taking $d \to +\infty$, we have $\epsilon \leq \Psi(\epsilon)$, which is again a contradiction to our assumption $\Psi(t) < t$ for $t > 0$. Hence,

$$\wp(\sigma_\alpha, \sigma_\beta) < \epsilon.$$

Similarly, we can prove for $j \neq 0$. This proves that $\{\sigma_\beta\}$ is a Cauchy sequence, and consequently $\bigcap_{i=1}^{k} \mathfrak{CB}(A_i) \neq \emptyset$.

Now, it is easy to prove the existence of fixed points along similar lines to Theorem 1. □

Assuming $\mathscr{M}(\sigma, \varpi) = \wp(\sigma, \varpi)$ and $\delta(\sigma, \varpi) = \wp(\sigma, \varpi)$ in Theorem 2, we have the following result.

Corollary 4 *Let $\{\mathfrak{A}_i\}_{i=1}^k$ be nonempty closed subsets of a complete metric space $(\Delta, \wp)$. Suppose that $\mathfrak{O} : \bigcup_{i=1}^k \mathfrak{A}_i \to \bigcup_{i=1}^k \mathfrak{CB}(\mathfrak{A}_i)$ satisfies the conditions as follows*:

(i) $\mathfrak{O}(\mathfrak{A}_i) \subseteq \mathfrak{A}_{i+1}$ *for* $1 \le i \le k$; (*where* $\mathfrak{A}_{k+1} = \mathfrak{A}_1$);

(ii) $\wp(\mathfrak{O}\sigma, \mathfrak{O}\varpi) \le \Psi(\wp(\sigma, \varpi))$ *for all* $\sigma \in \mathfrak{A}_i$, $\varpi \in \mathfrak{A}_{i+1}$ *for* $1 \le i \le k$, *where* $\Psi : R^+ \to [0, +\infty)$ *is upper semi-continuous from the right and satisfies* $0 \le \Psi(t) < t$ *for* $t > 0$.

Then $\mathfrak{O}$ *has at least a fixed point in* $\cap_i \mathfrak{CB}(\mathfrak{A}_i)$.

Example 2 Let $\Delta = \{-1, 0, 1\}$, $\mathfrak{A}_1 = \{-1, 0\}$, $\mathfrak{A}_2 = \{0, 1\}$ such that $\Delta = \cup_{i=1}^2 \mathfrak{A}_i$ with usual metric $\wp$. Assume that

$$\mathfrak{O}(x) = \begin{cases} \{0\}, & x = 0, \\ \{-x\}, & x \in \Delta \setminus \{0\}. \end{cases}$$

Here $\mathfrak{O}\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\mathfrak{O}\mathfrak{A}_2 \subseteq \mathfrak{A}_1$. Consider $\psi(t) = \begin{cases} 0, & t = 0, \\ t, & t > 0. \end{cases}$

Here all the hypotheses of Theorem 2 are satisfied and 0 is a fixed point.

Example 3 Let $\Delta = \{-\frac{1}{2}, -\frac{1}{2^2}, \dots, -\frac{1}{2^n}, \dots\} \cup \{0\} \cup \{\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^n}, \dots\}$, $\mathfrak{A}_1 = \{\frac{-1}{2}, \frac{-1}{2^2}, \dots\} \cup \{0\}$, $\mathfrak{A}_2 = \{\frac{1}{2}, \frac{1}{2^2}, \dots\} \cup \{0\}$ such that $\Delta = \cup_{i=1}^2 \mathfrak{A}_i$ with usual metric $\wp$. Assume that

$$\mathfrak{O}(x) = \begin{cases} \{0\}, & x = 0, \\ \{\frac{1}{2^{2n+1}}\}, & x = -\frac{1}{2^n}, n \ge 1, \\ \{-\frac{1}{2^{2n+1}}\}, & x = \frac{1}{2^n}, n \ge 1. \end{cases}$$

Here $\mathfrak{O}\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\mathfrak{O}\mathfrak{A}_2 \subseteq \mathfrak{A}_1$. Consider $\psi(t) = \begin{cases} \frac{t}{2}, & t > 0, \\ 0, & t = 0. \end{cases}$

Here all the hypotheses of Theorem 2 are satisfied and $\mathfrak{O}$ has a fixed point.

Theorem 3 *Let $\{\mathfrak{A}_i\}_{i=1}^k$ be nonempty closed subsets of a complete metric space $(\Delta, \wp)$. Suppose that $\Psi_i : \mathfrak{A}_i \to R$ is lower semi-continuous and bounded below for $i = 1, 2, \dots, k$ and $\mathfrak{O} : \bigcup_{i=1}^k A_i \to \bigcup_{i=1}^k \mathfrak{CB}(A_i)$ satisfies the following conditions*:

(i) $\mathfrak{O}(\mathfrak{A}_i) \subseteq \mathfrak{A}_{i+1}$ *for* $1 \le i \le k$, (*where* $\mathfrak{A}_{k+1} = \mathfrak{A}_1$);

(ii) $\delta(\sigma, \mathfrak{O}\sigma) \le \Psi_i(\sigma) - \Psi_{i+1}(\mathfrak{O}(\sigma))$ *for all* $\sigma \in \mathfrak{A}_i$, $1 \le i \le k$.

Then $\mathfrak{O}$ *has at least a fixed point in* $\bigcap_{i=1}^k \mathfrak{CB}(A_i)$.

Proof Let $\sigma_1 \in \mathfrak{A}_1$ and $\sigma_\beta \in \mathfrak{O}^{\beta-1}(\sigma_1)$. From condition (*ii*), we get

$$\Psi_1(\sigma_1) \ge \delta(\sigma_1, \mathfrak{O}\sigma_1) + \Psi_2(\mathfrak{O}\sigma_1) \ge \wp(\sigma_1, \sigma_2) + \Psi_2(\sigma_2) \ge \Psi_2(\sigma_2),$$

that is, $\Psi_1(\sigma_1) \ge \Psi_2(\sigma_2)$. Iterating in the same way, we get

$$\Psi_1(\sigma_1) \ge \Psi_2(\sigma_2) \ge \cdots \ge \Psi_\beta(\sigma_\beta) \ge \cdots, \quad \beta = 1, 2, \dots,$$

where $\Psi_i = \Psi_j$ if $i \equiv j(\mod k)$.

Therefore $\lim_{i \to +\infty} \Psi_i(\sigma_i) = \gamma$.

Now we fix $\sigma_\beta \in \mathfrak{A}_\beta$, and $\alpha > \beta$. Consider

$$\begin{aligned}
\wp(\sigma_\beta,\sigma_\alpha) &\le \wp(\sigma_\beta,\sigma_{\beta+1}) + \wp(\sigma_{\beta+1},\sigma_{\beta+2}) + \cdots + \wp(\sigma_{\alpha-1},\sigma_\alpha) \\
&\le \delta(\sigma_\beta,\mathfrak{D}\sigma_\beta) + \delta(\mathfrak{D}\sigma_\beta,\mathfrak{D}\sigma_{\beta+1}) + \cdots + \delta(\mathfrak{D}\sigma_{\alpha-2},\mathfrak{D}\sigma_{\alpha-1}) \\
&= \delta(\sigma_\beta,\mathfrak{D}\sigma_\beta) + \delta(\mathfrak{D}\sigma_\beta,\mathfrak{D}\mathfrak{D}\sigma_\beta) + \cdots + \delta(\mathfrak{D}\sigma_{\alpha-2},\mathfrak{D}\mathfrak{D}\sigma_{\alpha-2}) \\
&\le \bigl[\Psi_\beta(\sigma_\beta) - \Psi_{\beta+1}(\mathfrak{D}\sigma_\beta)\bigr] + \bigl[\Psi_{\beta+1}(\mathfrak{D}\sigma_\beta) - \Psi_{\beta+2}(\mathfrak{D}\sigma_{\beta+1})\bigr] \\
&\quad + \cdots + \bigl[\Psi_{\alpha-1}(\mathfrak{D}\sigma_{\alpha-2}) - \Psi_\alpha(\mathfrak{D}\sigma_{\alpha-1})\bigr] \\
&= \Psi_\beta(\sigma_\beta) - \Psi_\alpha(\mathfrak{D}\sigma_{\alpha-1}) \\
&= \Psi_\beta(\sigma_\beta) - \Psi_\alpha(\sigma_\alpha).
\end{aligned}$$

Therefore, $\{\sigma_\beta\}$ is a Cauchy sequence, and in turn $\bigcap_{i=1}^k \mathfrak{CB}(\mathfrak{A}_i) \neq \emptyset$.

Now, we have a particular situation when $\mathfrak{D} : \mathfrak{A}_\mathrm{i} \to \mathfrak{A}_\mathrm{i}$ and

$$\delta(\sigma,\mathfrak{D}\sigma) \le \min_{1\le i\le k}\bigl[\Psi_i(\sigma) - \Psi_{i+1}(\mathfrak{D}\sigma)\bigr],$$

for all $\sigma \in \mathfrak{A}_i$. Thus,

$$\begin{aligned}
k\delta(\sigma,\mathfrak{D}\sigma) &\le \Psi_1(\sigma) - \Psi_2(\mathfrak{D}\sigma) + \Psi_2(\sigma) - \Psi_3(\mathfrak{D}\sigma) + \cdots + \Psi_k(\sigma) - \Psi_1(\mathfrak{D}\sigma) \\
&= \sum_{i=1}^k \bigl[\Psi_i(\sigma) - \Psi_i(\mathfrak{D}\sigma)\bigr].
\end{aligned}$$

Now define $\Xi : \mathfrak{A} \to R$ by $\Xi(\sigma) = k^{-1}\sum_{i=1}^k \Psi_i(\sigma)$, $\sigma \in \mathfrak{A}$, where Φ is lower semi-continuous and bounded below and, moreover,

$$\delta(\sigma,\mathfrak{D}\sigma) \le \Xi(\sigma) - \Xi(\mathfrak{D}\sigma),$$

for each $\sigma \in \mathfrak{A}_i$.

Following the similar methodology as in the Caristi type result [9], the proof of the remaining part of the theorem is obvious. □

Remark 1 (i) In this paper, we have not assumed the continuity of ϱ in any sense.

(ii) The concept of δ-distance is different from other distances in metric spaces. Many generalized contractions and cyclic contractions are used to obtain fixed point results with the help of multivalued mappings.

(iii) Existence and uniqueness of fixed point with this kind of multivalued cyclic δ-Meir–Keeler type contractions may be one of the challenging issues.

Acknowledgements

The authors are thankful to anonymous referees for valuable suggestions. The third author is thankful to NBHM-DAE grant 02011/11/2020/NBHM(RP)/R&D-II/7830. The fourth author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Funding

The fourth author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Availability of data and materials
All materials and data are available.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Amity University, Kadampukur, 24PGS(S), Kolkata, West Bengal 700135, India. [2]Department of Mathematics, Bajkul Milani Mahavidyalaya, P.O- Kismat Bajkul, Dist - Purba Medinipur, Bajkul, West Bengal 721655, India. [3]School of Mathematics, Thapar Institute of Engineering & Technology, Patiala 147-004, Punjab, India. [4]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 21 July 2020 Accepted: 16 November 2020 Published online: 23 November 2020

References
1. Acar, Ö., Altun, I.: A fixed point theorem for multivalued mappings with δ-distance. Abstr. Appl. Anal. **2014**, Article ID 497092 (2014)
2. Alamgir, N., Kiran, Q., Isık, H., Aydi, H.: Fixed point results via a Hausdorff controlled type metric. Adv. Differ. Equ. **2020**, 24 (2020)
3. Alber, Ya.I., Guerre-Delabriere, S.: Principle of weakly contractive maps in Hilbert spaces. New Res. Oper. Theory Appl. **98**, 7–22 (1997)
4. Ali, A., Isik, H., Aydi, H., Ameer, E., Lee, J.R., Arshad, M.: On multivalued Suzuki-type θ-contractions and related applications. Open Math. **18**, 386–399 (2020)
5. Ameer, E., Aydi, H., Arshad, M., De la Sen, M.: Hybrid Ćirić type graphic (Υ, Λ)-contraction mappings with applications to electric circuit and fractional differential equations. Symmetry **12**(3), 467 (2020)
6. Aydi, H., Lakzian, H., Mitrović, Z.D., Radenović, S.: Best proximity points of MT-cyclic contractions with property UC. Numer. Funct. Anal. Optim. **41**(7), 871–882 (2020)
7. Boyd, D.W., Wong, J.S.W.: On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458–464 (1969)
8. Budhia, L., Aydi, H., Ansari, A.H., Gopal, D.: Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations. Nonlinear Anal., Model. Control **25**(4), 580–597 (2020)
9. Caristi, J.: Fixed point theorems for mappings satisfying inwardness conditions. Transl. Am. Math. Soc. **215**, 241–251 (1976)
10. Chandok, S., Ansari, A.H., Narang, T.D.: A fixed point result for c-Kannan type cyclic weakly contractions. J. Nonlinear Funct. Anal. **2016**, 39 (2016)
11. Chandok, S., Postolache, M.: Fixed point theorem for weakly Chatterjea-type cyclic contractions. Fixed Point Theory Appl. **2013**, 28 (2013)
12. Choudhury, B.S., Konar, P., Rhoades, B.E., Metiya, N.: Fixed point theorems for generalized weakly contractive mappings. Nonlinear Anal. **74**(6), 2116–2126 (2011)
13. Choudhury, B.S., Maity, P., Konar, P.: A global optimality result using Geraghty type contraction. Int. J. Optim. Control Theor. Appl. **4**, 99–104 (2014)
14. Choudhury, B.S., Maity, P., Konar, P.: Fixed point results for couplings on metric spaces. Bull. Fac. Sci. Ibaraki Univ., Ser. A., Math. **79**(1), 80–88 (2017)
15. Ćirić, L.: Some Recent Results in Metrical Fixed Point Theory. University of Belgrade, Beograd (2003)
16. Doric, D.: Common fixed point for generalized ψ, ϕ-weak contractions. Appl. Math. Lett. **22**, 1896–1900 (2009)
17. Geraghty, M.A.: On contractive mappings. Proc. Am. Math. Soc. **40**, 604–608 (1973)
18. Gordji, M.E., Baghani, H., Khodaei, H., Ramezani, M.: A generalization of Nadler's fixed point theorem. J. Nonlinear Sci. Appl. **3**, 148–151 (2010)
19. Kirk, W.A., Srinivasan, P.S., Veeramani, P.: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory **4**, 79–89 (2003)
20. Popescu, O.: Fixed points for (ψ, ϕ)-weak contractions. Appl. Math. Lett. **24**, 1–4 (2011)
21. Radenović, S.: Some remarks on mappings satisfying cyclical contractive conditions. Afr. Math. **27**(1–2), 291–295 (2016)
22. Radenovic, S., Chandok, S., Shatanawi, W.: Some cyclic fixed point results for contractive mappings. Univ. Thought, Publ. Nat. Sci. **6**, 38–40 (2016)
23. Radenović, S., Došenović, T., Aleksić-Lampert, T., Golubović, Z.: A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations. Appl. Math. Comput. **273**, 155–164 (2016)
24. Rakotch, E.: A note on contractive mappings. Proc. Am. Math. Soc. **13**, 459–465 (1962)
25. Rhoades, B.E.: Some theorems on weakly contractive maps. Nonlinear Anal. **47**(4), 2683–2693 (2001)
26. Shatanawi, W., Karapinar, E., Aydi, H., Fulga, A.: Wardowski type contractions with applications on Caputo type nonlinear fractional differential equations. University Politechnica of Bucharest Scientific Bulletin-Series A-Applied **82**(2), 157–170 (2020)
27. Shatanawi, W., Samet, B.: On (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces. Comput. Math. Appl. **62**(8), 3204–3214 (2011)
28. Todorcević, V.: Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics. Springer, Switzerland (2019)
29. Wong, J.S.W.: Mappings of contractive type on abstract spaces. J. Math. Anal. Appl. **37**, 331–340 (1972)
30. Zhang, Q., Song, Y.: Fixed point theory for generalized ψ-weak contractions. Appl. Math. Lett. **22**, 75–78 (2009)