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The connectedness of the solutions set for set-valued vector equilibrium problems under improvement sets

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Abstract

In this paper, we provide the connectedness of the sets of weak efficient solutions, Henig efficient solutions and Benson proper efficient solutions for set-valued vector equilibrium problems under improvement sets.

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Keywords: Set-valued vector equilibrium problems; Improvement sets; Scalarization; Connectedness

1 Introduction

In recent years, many scholars paid attention to developing concepts to unify various kinds of solution notions of vector optimization problems, for instance, efficiency, weak efficiency, proper efficiency and ε -efficiency. Chicco et al. [1] putted forward a new concept of improvement set E and defined E -optimal solution in finite dimensional Euclidean space. E -optimal solution unifies some known concepts of exact and approximate solutions of vector optimization problems. Gutiérrez et al. [2] extended the notion of improvement set and E -optimal solution to a locally convex topological vector spaces. Much follow-up work about the improvement set E one finds in [3–12]. Chen et al. [13] introduced a new vector equilibrium problem based on improvement set E named the unified vector equilibrium problem (UVEP), linear scalarization characterizations of the efficient solutions, weak efficient solutions, Benson proper efficient solutions for (UVEP) were established, and some continuity results of parametric (UVEP) were obtained by applying scalarization method.

Vector equilibrium problems (shortly, VEP) provides a unified model of many significant problems (see [14–16]). An important topic about (VEP) is the connectedness of the solutions set. Lee et al. [17] and Cheng [18] discussed the path-connectedness and connectedness of weakly efficient solutions set for vector variational inequalities in finite dimensional Euclidean space, respectively. Applying the scalarization approaches, Gong [19] studied the connectedness of the sets of Henig efficient solutions and weakly efficient solutions for the vector Hartman–Stampacchia variational inequality in normed vector

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spaces (in short, n.v.s.). By employing scalarization results, Gong [20] investigated the connectedness and path-connectedness of sets of weak efficient solutions and various proper efficient solutions for (VEP) in locally convex spaces (in short, l.c.s.). By the density results, Gong and Yao [21] first showed the connectedness of efficient solutions set for (VEP) in l.c.s. Chen et al. [22] studied the connectedness and the compactness of weak efficient solutions set for set-valued vector equilibrium problems (shortly, SVEP) in n.v.s. Chen et al. [23] discussed the connectedness of the sets of ϵ -weak efficient solutions and ϵ -efficient solutions for (VEP) in l.c.s.

All the papers mentioned above, the hypotheses of compactness and monotonicity are essential in discussing the connectedness of the sets of various kinds of efficient solutions for (VEP). Han and Huang [24] studied the connectedness of the sets of (weakly) efficient solutions and various proper efficient solutions for the (GVEP) not using the conditions of compactness and monotonicity in n.v.s. Han and Huang [25] investigated the connectedness of the sets of weakly efficient solutions and ε -efficient solutions for the (SVEP) by using the scalarization results and the density results in n.v.s, respectively. The improvement set E is a tool to unify some exact and approximate solution notions, hence, it is very meaningful in establishing the connectedness of the solutions set for VEP based on improvement set.

Motivated by the work of [13, 24, 25], in this paper, by using the scalarization results, we study the connectedness of the sets of weakly efficient solutions, Henig efficient solutions and Benson proper efficient solutions for set-valued vector equilibrium problems under improvement sets. The main results unify and extend some exact and approximate cases.

2 Preliminaries

Throughout this paper, let X, Y be real locally convex Hausdorff topological vector spaces and let Z be a real vector topological space. Let Y^* be the topological dual space of Y and let C be a pointed closed convex cone in Y with its topological interior $\text{int } C \neq \emptyset$.

Let Q be a nonempty subset of Y , denote the closure of Q by $\text{cl } Q$ and the topological interior of Q by $\text{int } Q$. The cone hull of Q is defined by

$$\text{cone } Q := \{tq \mid t \geq 0, q \in Q\}.$$

We say Q is solid if $\text{int } Q \neq \emptyset$.

The positive polar cone C^* and the strict positive polar cone $C^\#$ of C are defined as

$$C^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \forall y \in C\}$$

and

$$C^\# := \{y^* \in Y^* \mid y^*(y) > 0, \forall y \in C \setminus \{0_Y\}\},$$

respectively.

A nonempty convex set $B \subset C$ is said to be a base of C if

$$C = \text{cone } B \quad \text{and} \quad 0_Y \notin \text{cl } B.$$

It is clear that $C^\# \neq \emptyset$ if and only if C has a base. Let B be a base of C , because of $0_Y \notin \text{cl } B$, by the separation theorem of convex sets, there exists $0 \neq \varphi \in Y^*$ such that

$$\delta = \inf\{\varphi(b) \mid b \in B\} > \varphi(0_Y) = 0.$$

Define

$$V_B = \left\{ y \in Y \mid |\varphi(y)| < \frac{\delta}{2} \right\}.$$

Then V_B is an open convex balanced neighborhood of zero in Y . For each convex neighborhood U of zero with $U \subseteq V_B$, $B + U$ is a convex set and $0 \notin \text{cl}(B + U)$, let $C_U(B) := \text{cone}(U + B)$.

Remark 2.1 ([26])

- (i) $C_U(B)$ is a pointed convex cone;
- (ii) $C \setminus \{0_Y\} \subseteq \text{int } C_U(B)$.

For convenience, we denote by $N(0)$ the family of neighborhoods of zero in Y . Assume that B is a base of C and write

$$B^{st} := \{ \mu \in C^\# \mid \text{there exists } t > 0 \text{ such that } \mu(b) \geq t, \forall b \in B \}.$$

By the separation theorem of the convex sets (see [27]), we know $B^{st} \neq \emptyset$.

Lemma 2.2 ([28]) *Let B be a base of C and $y^* \in Y^* \setminus \{0_{Y^*}\}$. Then $y^* \in B^{st}$ if and only if there exists a convex neighborhood $N(0)$ such that $y^*(u - b) \leq 0, \forall u \in U, \forall b \in B$.*

Lemma 2.3 ([27]) *Let $\text{int } C \neq \emptyset$, then*

- (i) $\text{int } C \cup \{0_Y\}$ is a convex cone;
- (ii) $\text{int } C = C + \text{int } C$;
- (iii) $\text{int}(\text{int } C) = \text{int } C$.

Definition 2.4 ([2]) *Let E be a nonempty subset in Y . E is said to be an improvement set with respect to C if $0_Y \notin E$ and $E + C = E$. The class of the improvement sets with respect to C in Y is denoted by \mathfrak{S}_Y .*

Lemma 2.5 ([3]) *Let $E \in \mathfrak{S}_Y$ and be solid, then $\text{int } E = E + \text{int } C$.*

Lemma 2.6 ([3]) *Let $E \in \mathfrak{S}_Y$ and $\emptyset \neq N \subset Y$, then*

$$\text{cl}(\text{cone}(N + E)) = \text{cl}(\text{cone}(N + \text{int } E)).$$

Definition 2.7 ([29]) *Let $\emptyset \neq A \subset X$. The set-valued map $F : A \rightrightarrows Y$ is said to be nearly C -convexlike (closely C -convexlike) on A if $\text{cl}(F(A) + C)$ is a convex set in Y .*

Definition 2.8 ([5]) *Let $\emptyset \neq A \subset X$ and $E \in \mathfrak{S}_Y$. The set-valued map $F : A \rightrightarrows Y$ is said to be E -subconvexlike on A if $F(A) + \text{int } E$ is a convex set in Y .*

Definition 2.9 ([3]) Let $\emptyset \neq A \subset X$ and $E \in \mathfrak{S}_Y$. The set-valued map $F : A \rightrightarrows Y$ is said to be nearly E -subconvexlike on A if $\text{cl}(\text{cone}(F(A) + E))$ is a convex set in Y .

Theorem 2.10 Let $E \in \mathfrak{S}_Y$ and $\emptyset \neq A \subset X$. If the set-valued map $F : A \rightrightarrows Y$ is E -subconvexlike on A , then F is nearly E -subconvexlike on A .

Proof It follows directly from Definition 2.8, Definition 2.9 and Lemma 2.6. □

Lemma 2.11 ([30]) Let $\emptyset \neq M \subset Y$ and $\text{int } C \neq \emptyset$. Then

$$\text{int}(\text{cl}(M + C)) = M + \text{int } C.$$

Theorem 2.12 Let $E \in \mathfrak{S}_Y$ be a convex set and $\text{int } C \neq \emptyset$. If the set-valued map $F : A \rightrightarrows Y$ is nearly C -convexlike on A , then F is nearly E -subconvexlike on A .

Proof As F is nearly C -convexlike on A , then $\text{cl}(F(A) + C)$ is convex, taking into account Lemma 2.3(i), one finds that $\text{int}(\text{cl}(F(A) + C))$ is convex, by Lemma 2.11, $F(A) + \text{int } C$ is convex too.

Moreover, observe that E is convex, then $F(A) + \text{int } C + E$ is convex. From Lemma 2.5, it follows that $F(A) + \text{int } E$ is convex. Therefore, in view of Lemma 2.6, we see that $\text{cl}(\text{cone}(F(A) + E))$ is convex. □

Lemma 2.13 ([31]) Let M and N be two nonempty convex subsets of a real topological linear space X with $\text{int } M \neq \emptyset$. Then $N \cap \text{int } M = \emptyset$ if and only if there are a linear functional $l \in X^* \setminus \{0_{X^*}\}$ and a real number α with $l(m) \leq \alpha \leq l(n)$ for all $m \in M$ and all $n \in N$, and

$$l(m) < \alpha \quad \text{for all } m \in \text{int } M.$$

In this paper, we let \mathbb{R} is the set of real numbers and $\mathbb{R}_+ = \{r \mid r \geq 0\}$, $\mathbb{R}_{++} = \{r \mid r > 0\}$.

From now on, we presume that A is a nonempty subset of X , $F : A \times A \rightrightarrows Y$ is a set-valued map.

We have the usual set-valued vector equilibrium problem (SVEP) of finding $\bar{x} \in A$ such that

$$(SVEP) \quad F(\bar{x}, x) \cap (-C \setminus \{0_Y\}) = \emptyset, \quad \forall x \in A.$$

For $x \in A$, we define

$$F(x, A) := \bigcup_{y \in A} \{F(x, y)\}.$$

Definition 2.14

- (i) An element $x \in A$ is called a weakly efficient solution of the (SVEP) (see [22]) if

$$F(x, A) \cap (-\text{int } C) = \emptyset.$$

(ii) An element $x \in A$ is called a Benson proper efficient solution of the (SVEP) (see [20]) if

$$\text{cl}(\text{cone}(F(x, A) + C)) \cap (-C) = \{0_Y\}.$$

(iii) An element $x \in A$ is called a C-Heing efficient solution of the (SVEP) (see [32]) if there exists some $U \in N(0)$ with $U \subset V_B$ such that

$$\text{cone}(F(x, A) + C) \cap (U - B) = \emptyset.$$

Remark 2.15 Let B be a base of C . It is easy to check that $x \in A$ is the C-Heing efficient solution of the (SVEP) with respect to B if and only if there exists some $U \in N(0)$ with $U \subset V_B$ such that

$$\text{cl}(\text{cone}(F(x, A) + C)) \cap (-C_U(B)) = \{0_Y\}.$$

Remark 2.16 Let B be a base of C . It is easy to check that $x \in A$ is the C-Heing efficient solution of the (SVEP) with respect to B if and only if there exists some $U \in N(0)$ with $U \subset V_B$ such that

$$\text{cl}(\text{cone}(F(x, A) + C)) \cap (-C_U(B)) = \{0_Y\}.$$

We consider the unified set-valued vector equilibrium problem (USVEP) through improvement set E of finding $\bar{x} \in A$ such that

$$\text{(USVEP)} \quad F(\bar{x}, x) \cap (-E) = \emptyset, \quad \forall x \in A.$$

Definition 2.17 An element $x \in A$ is said to be a weakly efficient solution of the (USVEP) if

$$F(x, A) \cap (-\text{int} E) = \emptyset.$$

Denote by $\text{We}(F, A; E)$ the set of weakly efficient solutions of the (USVEP).

Definition 2.18 An element $x \in A$ is said to be a Benson proper efficient solution of the (USVEP) if

$$\text{cl}(\text{cone}(F(x, A) + E)) \cap (-C) = \{0_Y\}.$$

Denoted by $\text{Be}(F, A; E)$ the set of Benson proper efficient solutions of the (USVEP).

Definition 2.19 An element $x \in A$ is said to be a Heing efficient solution of the (USVEP) if there exists some $U \in N(0)$ with $U \subset V_B$ such that

$$\text{cl}(\text{cone}(F(x, A) + E)) \cap (-C_U(B)) = \{0_Y\}.$$

Denoted by $\text{He}(F, A, B; E)$ the set of Henig efficient solutions of the (USVEP).

Lemma 2.20 *Let $E \in \mathfrak{S}_Y$ and B be a base of C , then $\text{He}(F, A, B; E) \subseteq \text{Be}(F, A; E) \subseteq \text{We}(F, A; E)$.*

Proof Firstly, we prove $\text{He}(F, A, B; E) \subseteq \text{Be}(F, A; E)$.

Let $\bar{x} \in \text{He}(F, A, B; E)$, there exists $U \in \mathcal{N}(0)$ with $U \subset V_B$ such that

$$\text{cl}(\text{cone}(F(x, A) + E)) \cap (-C_U(B)) = \{0_Y\}.$$

Because of $C \subset \text{cone}B \subset \text{cone}(U + B)$,

$$\text{cl}(\text{cone}(F(\bar{x}, A) + E)) \cap (-C) = \{0_Y\}.$$

Then

$$\bar{x} \in \text{Be}(F, A; E).$$

In what follows, we prove

$$\text{Be}(F, A; E) \subseteq \text{We}(F, A; E).$$

Let $\hat{x} \in \text{Be}(F, A; E)$, then

$$\text{cl}(\text{cone}(F(\hat{x}, A) + E)) \cap (-C) = \{0_Y\}.$$

This implies

$$\text{cl}(\text{cone}(F(\hat{x}, A) + E)) \cap (-C \setminus \{0_Y\}) = \emptyset.$$

Thus

$$(F(\hat{x}, A) + E) \cap (-C \setminus \{0_Y\}) = \emptyset.$$

By $\text{int} C \subset C \setminus \{0\}$, then

$$(F(\hat{x}, A) + E) \cap (-\text{int} C) = \emptyset.$$

Therefore,

$$F(\hat{x}, A) \cap (-\text{int} E) = \emptyset.$$

Hence, $\hat{x} \in \text{We}(F, A; E)$. □

Remark 2.21

- (i) If $E = C \setminus \{0_Y\}$ or $E = \text{int} C$, then the weak efficiency of (USVEP) reduces to the weak efficiency of (SVEP).

- (ii) If $E = C \setminus \{0_Y\}$, then the Benson proper efficiency and Heing efficiency of (USVEP) reduce to the Benson proper efficiency and of C-Heing efficiency of (SVEP), respectively.

For $\forall \varphi \in Y^*$, we denote $\sigma_E(\varphi) := \inf_{e \in E} \varphi(e)$. Obviously, if $\varphi \in E^+$, then $\sigma_E(\varphi) \geq 0$. For all $\varphi \in E^+ \setminus \{0_{Y^*}\}$, an element $x \in A$ is said to a $\sigma_E(\varphi)$ -efficient solution of (USVEP) if

$$\varphi(F(x, y)) + \sigma_E(\varphi) \subseteq \mathbb{R}_+, \quad \forall y \in A.$$

Denote by $V_\varphi(F, A; E)$ the set of $\sigma_E(\varphi)$ -efficient solutions of (USVEP).

Define a set

$$M := \{x \in A \mid F(x, y) \subset E, \forall y \in A\}.$$

3 Scalarization

Theorem 3.1 *Let $E \in \mathfrak{S}_Y$ and $\text{int} E \neq \emptyset$. Suppose that, for all $x \in A$, $F(x, \cdot)$ is nearly E -subconvexlike on A , $0_Y \in F(x, x)$. Then*

$$\text{We}(F, A; E) = \bigcup_{\varphi \in E^+ \setminus \{0_{Y^*}\}} V_\varphi(F, A; E).$$

Proof Necessity. Let $x_0 \in \text{We}(F, A; E)$, then $F(x_0, x) \cap (-\text{int} E) = \emptyset, \forall x \in A$. Thus, $F(x_0, A) \cap (-\text{int} E) = \emptyset$. It follows from Lemma 2.5 that

$$(F(x_0, A) + E) \cap (-\text{int} C) = \emptyset. \tag{1}$$

Because $\text{int} C \cup \{0\}$ is convex, we have

$$\text{cone}(F(x_0, A) + E) \cap (-\text{int} C) = \emptyset. \tag{2}$$

We assert that

$$\text{cl}(\text{cone}(F(x_0, A) + E)) \cap (-\text{int} C) = \emptyset. \tag{3}$$

Otherwise, there exists $\bar{y} \in \text{cl}(\text{cone}(F(x_0, A) + E)) \cap (-\text{int} C)$. Therefore, there exists $U_1 \in N(0)$ such that

$$\bar{y} + U_1 \in \text{cone}(F(x_0, A) + E).$$

Since $y \in -\text{int} C = \text{int}(-\text{int} C)$, there exists $U_2 \in N(0)$ such that

$$\bar{y} + U_2 \in -\text{int} C.$$

As a result

$$\bar{y} + U_0 \in \text{cone}(F(x_0, A) + E) \cap (-\text{int} C), \quad U_0 \in U_1 \cap U_2,$$

which contradicts (2). Consequently, (3) holds. By Lemma 2.13, there exists $\bar{\varphi} \in Y^* \setminus \{0_{Y^*}\}$ such that

$$\bar{\varphi}(z + e) \geq 0, \quad \forall e \in E, \forall x \in A, \forall z \in F(x_0, x). \tag{4}$$

Taking $x = x_0$ in (4), we get $\bar{\varphi}(e) \geq 0, \forall e \in E$, and so $\varphi \in E^+ \setminus \{0_{Y^*}\}$.

On the other hand, by (4),

$$\bar{\varphi}(e) \geq -\bar{\varphi}(z), \quad \forall e \in E, \forall x \in A, \forall z \in F(x_0, x).$$

Hence

$$\sigma_E(\bar{\varphi}) = \inf_{e \in E} \bar{\varphi}(e) \geq -\bar{\varphi}(z), \quad \forall e \in E, \forall x \in A, \forall z \in F(x_0, x).$$

Consequently,

$$\bar{\varphi}(z) + \sigma_E(\bar{\varphi}) \geq 0, \quad \forall x \in A, \forall z \in F(x_0, x),$$

i.e.,

$$\bar{\varphi}(F(x_0, x)) + \sigma_E(\bar{\varphi}) \subseteq \mathbb{R}_+, \quad \forall x \in A.$$

Hence, $x_0 \in V_{\bar{\varphi}}(F, A; E) \subset \bigcup_{\varphi \in E^+ \setminus \{0_{Y^*}\}} V_{\varphi}(F, A; E)$.

Sufficiency. Let $x_0 \in \bigcup_{\varphi \in E^+ \setminus \{0_{Y^*}\}} V_{\varphi}(F, A; E)$, then there exists $\hat{\varphi} \in E^+ \setminus \{0_{Y^*}\}$ such that $x_0 \in V_{\hat{\varphi}}(F, A; E)$. Thus,

$$\hat{\varphi}(F(x_0, x)) + \sigma_E(\hat{\varphi}) \subseteq \mathbb{R}_+, \quad \forall x \in A. \tag{5}$$

Suppose that $x_0 \notin \text{We}(F, A; E)$, then there exist $\hat{x} \in A$ and $\hat{e} \in \text{int } E$ such that $-\hat{e} \in F(x_0, \hat{x})$. Since $\hat{\varphi}(\hat{e}) > \sigma_E(\hat{\varphi})$,

$$\hat{\varphi}(-\hat{e}) + \sigma_E(\hat{\varphi}) < 0,$$

which contradicts (5). Hence, $x_0 \in \text{We}(F, A; E)$. □

Remark 3.2 By Theorem 2.12, the condition nearly E -subconvexlike in Theorem 3.1 is weaker than nearly C -convexlike and the convexity of E in Theorem 3.1 of [13]. So, compared with Theorem 3.1 in [13], Theorem 3.1 extends the model from vector-valued maps to set-valued maps under the weaker condition.

Lemma 3.3 *Let D and C be two closed convex cones in a locally convex vector space Y , and let C be pointed and have a compact base. If $D \cap (-C) = \{0_Y\}$, then there exists $\varphi \in C^\#$ such that $\varphi \in D^*$.*

Theorem 3.4 *Let $E \in \mathfrak{S}_Y$. Suppose that, for each $x \in A$, $F(x, \cdot)$ is nearly E -subconvexlike on A .*

- (i) If C has a base B , then $\text{He}(F, A, B; E) = \bigcup_{\varphi \in B^{st}} V_{\varphi}(F, A; E)$.
- (ii) If C has a compact base B , then $\text{Be}(F, A; E) = \bigcup_{\varphi \in C^{\#}} V_{\varphi}(F, A; E)$.

Proof (i) Necessity. Let $\bar{x} \in S_{pH}(F, A, B; E)$, by Definition 2.19, there exists $U \in N(0)$ such that

$$\text{cl}(\text{cone}(F(\bar{x}, A)) + E) \cap (-C_U(B)) = \{0_Y\}.$$

By Remark 2.1,

$$\text{cl}(\text{cone}(F(\bar{x}, A) + E)) \cap (-\text{int}(C_U(B))) = \emptyset$$

and

$$\text{int}(C_U(B)) = \emptyset.$$

On the other hand, from the near E -subconvexlikeness of $F(x, \cdot)$, we have $\text{cl}(\text{cone}(F(\bar{x}, A) + E))$ is convex. By Lemma 2.13, there exists $\hat{\varphi} \in Y^* \setminus \{0_{Y^*}\}$ such that

$$\hat{\varphi}(y_1) \geq \hat{\varphi}(y_2), \quad \forall y_1 \in \text{cl}(\text{cone}(F(\bar{x}, A) + E)), \forall y_2 \in -C_U(B). \tag{6}$$

Taking $y_1 = 0_Y$ in (6) that

$$\hat{\varphi}(y_2) \geq 0, \quad \forall y_2 \in C_U(B).$$

Then

$$\hat{\varphi}(u + b) \geq 0, \quad \forall b \in B, \forall u \in U,$$

that is,

$$\hat{\varphi}(b - u) \geq 0, \quad \forall b \in B, \forall u \in U.$$

By Lemma 2.2, we have $\hat{\varphi} \in B^{st}$.

On the other hand, by (6),

$$\hat{\varphi}(z + e) \geq -\hat{\varphi}(y_2), \quad \forall y_2 \in -C_U(B), \forall e \in E, \forall x \in A, \forall z \in F(\bar{x}, x). \tag{7}$$

Taking $y_2 = 0_Y$ in (7), as a result $\hat{\varphi}(e) \geq -\hat{\varphi}(z), \forall e \in E, \forall x \in A, \forall z \in F(\bar{x}, x)$. Then

$$\sigma_E(\hat{\varphi}) \geq -\hat{\varphi}(z), \quad \forall x \in A, \forall z \in F(\bar{x}, x).$$

In consequence, $\hat{\varphi}(z) + \sigma_E(\hat{\varphi}) \geq 0, \forall x \in A, \forall z \in F(\bar{x}, x)$. Thus,

$$\hat{\varphi}(F(x_0, x)) + \sigma_E(\hat{\varphi}) \subseteq \mathbb{R}_+, \quad \forall x \in A.$$

So, $\bar{x} \in V_{\hat{\varphi}}(F, A; E) \subset \bigcup_{\varphi \in B^{st}} V_{\varphi}(F, A; E)$.

Sufficiency. Let $\bar{x} \in \bigcup_{\varphi \in B^{st}} V_{\varphi}(F, A; E)$, then there exists $\varphi' \in B^{st}$ such that $\bar{x} \in V_{\varphi'}(F, A; E)$. Hence, we have

$$\varphi'(F(x_0, x)) + \sigma_E(\varphi') \subseteq \mathbb{R}_+, \quad \forall x \in A. \tag{8}$$

Suppose that $\bar{x} \notin S_{pH}(F, A, B; E)$, then there exists $U \in N(0)$ such that

$$\text{cl}(\text{cone}(F(\bar{x}, A) + E)) \cap (-C_U(B)) \neq \{0_Y\}.$$

Hence,

$$\text{cone}(F(\bar{x}, A) + E) \cap (-C_U(B)) \neq \{0_Y\}.$$

Let $0_Y \neq z \in \text{cone}(F(\bar{x}, A) + E) \cap (-C_U(B))$.

Since $z \in -C_U(B)$, there exist $\alpha > 0, u \in U, b \in B$ such that

$$z = -\alpha(-u + b),$$

i.e.,

$$z = \alpha(u - b).$$

By $\varphi' \in B^{st}$, then

$$\varphi'(z) < 0. \tag{9}$$

Since $0_Y \neq z \in \text{cone}(F(\bar{x}, A) + E)$, there exists $t > 0, \bar{y} \in A, \bar{z} \in F(\bar{x}, \bar{y}), e \in E$ such that $z = t(\bar{z} + e)$. From (9), it follows that

$$t(\varphi'(\bar{z} + e)) < 0,$$

then

$$\varphi'(e) < -\varphi'(\bar{z}).$$

Whence

$$\sigma_E(\varphi') < -\varphi'(\bar{z}),$$

i.e.,

$$\varphi'(\bar{z}) + \sigma_E(\varphi') < 0. \tag{10}$$

This contradicts (8). Therefore, $\bar{x} \in S_{pH}(F, A, B; E)$.

(ii) Necessity. Let $\bar{x} \in \text{Be}(F, A; E)$, then $\text{cl}(\text{cone}(F(x, A) + E)) \cap (-C) = \{0_Y\}$. From the near E -subconvexlike of $F(x, \cdot)$, one has $\text{cl}(\text{cone}(F(\bar{x}, A) + E))$ is a closed convex cone. By Lemma 3.3, there exists $\tilde{\varphi} \in C^{\#}$ such that $\tilde{\varphi} \in D^*$, where $D := \text{cl}(\text{cone}(F(\bar{x}, A) + E))$.

Thus, we have

$$\tilde{\varphi}(z) \geq 0, \quad \forall z \in \text{cl}(\text{cone}(F(x, A) + E)), \tag{11}$$

Furthermore,

$$\tilde{\varphi}(z + e) \geq 0, \quad \forall e \in E, \forall x \in A, \forall z \in F(\bar{x}, x). \tag{12}$$

We conclude $\tilde{\varphi}(e) \geq -\tilde{\varphi}(z), \forall e \in E, \forall x \in A, \forall z \in F(\bar{x}, x)$. Then

$$\sigma_E(\tilde{\varphi}) \geq -\tilde{\varphi}(z), \quad \forall x \in A, \forall z \in F(\bar{x}, x).$$

In consequence, $\tilde{\varphi}(z) + \sigma_E(\tilde{\varphi}) \geq 0, \forall x \in A, \forall z \in F(\bar{x}, x)$. Thus,

$$\tilde{\varphi}(F(\bar{x}, x)) + \sigma_E(\tilde{\varphi}) \subseteq \mathbb{R}_+, \quad \forall x \in A.$$

So, $\bar{x} \in V_{\tilde{\varphi}}(F, A; E) \subset \bigcup_{\varphi \in C^\sharp} V_\varphi(F, A; E)$.

Sufficiency. Let $\bar{x} \in \bigcup_{\varphi \in C^\sharp} V_\varphi(F, A; E)$, then there exists $\varphi' \in C^\sharp$ such that $\bar{x} \in V_{\varphi'}(F, A; E)$. Hence, we have

$$\varphi'(F(\bar{x}, x)) + \sigma_E(\varphi') \subseteq \mathbb{R}_+, \quad \forall x \in A. \tag{13}$$

In the following, we show that $\bar{x} \in \text{Be}(F, A; E)$.

Let $d \in \text{cl}(\text{cone}(F(\bar{x}, A) + E)) \cap (-C)$, from $E \in \mathfrak{S}_Y$, one has

$$d \in \text{cl}(\text{cone}(F(\bar{x}, A) + E + C)) \cap (-C).$$

Hence, there exist sequences $\{\mu_n\} \subset \mathbb{R}_+, \{c_n\} \subset C, \{e_n\} \subset E$ and $\{y_n\} \subset A$ such that

$$\mu_n(z_n + e_n + c_n) \rightarrow d, \quad \forall z_n \in F(\bar{x}, y_n).$$

Since $\varphi' \in C^\sharp$,

$$\mu_n(\varphi'(z_n + e_n + c_n)) \rightarrow \varphi'(d), \quad \forall z_n \in F(\bar{x}, y_n). \tag{14}$$

By (13), $\varphi'(z_n + e_n) \geq 0, \forall z_n \in F(\bar{x}, y_n)$. Furthermore, from $\varphi'(c_n) \geq 0$ and (14), it follows that $\varphi'(d) \geq 0$. Moreover, since $\varphi' \in C^\sharp$ and $d \in -C, \varphi'(d) \leq 0$. As a result $\varphi'(d) = 0$. As $\varphi' \in C^\sharp$, we have $d = \{0_Y\}$. Hence, $\text{cl}(\text{cone}(F(\bar{x}, A) + E)) \cap (-C) = \{0_Y\}$, and so $\bar{x} \in \text{Be}(F, A; E)$. \square

Remark 3.5 By Theorem 2.10, the condition nearly E -subconvexlike in Theorem 3.4 is weaker than E -subconvexlike in Theorem 3.4 of [13]. So, to be compared with Theorem 3.4 in [13], Theorem 3.4 extends the model from vector-valued map to set-valued map under weaker condition.

4 Connectedness of the solutions set

In this section, we discuss the connectedness of $We(F, A; E)$, $He(F, A, B; E)$ and $Be(F, A; E)$.

Lemma 4.1 ([33]) *Suppose that $\{A_\gamma : \gamma \in \Gamma\}$ is a family of connected sets in topological space Φ . If $\bigcap_{\gamma \in \Gamma} A_\gamma \neq \emptyset$, then $\bigcup_{\gamma \in \Gamma} A_\gamma$ is a connected set in topological space Φ .*

Definition 4.2 ([27]) Let A be a nonempty convex subset of X . A set-valued map $F : A \rightrightarrows Y$ is called C -concave if and only if, for all $x_1, x_2 \in A$ and $t \in [0, 1]$, we have

$$F(tx_1 + (1 - t)x_2) \subseteq tF(x_1) + (1 - t)F(x_2) + C.$$

Theorem 4.3 Let $E \in \mathfrak{S}_Y$, $E \subset C$ and A be a nonempty convex set. Suppose that $F(\cdot, y)$ is C -concave for all $y \in A$ and $M \neq \emptyset$, for all $x \in A$, $F(x, \cdot)$ is nearly E -subconvexlike on A . Then:

- (i) If $0_Y \in F(x, x)$ for all $x \in A$, then $We(F, A; E)$ is a connected set.
- (ii) If C has a base B , then $He(F, A, B; E)$ is a connected set.
- (iii) If C has a compact base B , then $Be(F, A; E)$ is a connected set.

Proof (i) Firstly, we show that $V_\varphi(F, A; E)$ is a connected set for all $\varphi \in E^+ \setminus \{0_{Y^*}\}$. Actually, for any $\varphi \in E^+ \setminus \{0_{Y^*}\}$ and for any fixed $y \in A$, let $x_1, x_2 \in V_\varphi(F, A; E)$ and $t \in [0, 1]$, we have $tx_1 + (1 - t)x_2 \in A$, $\varphi(F(x_1, y)) + \sigma_E(\varphi) \subseteq \mathbb{R}_+$ and $\varphi(F(x_2, y)) + \sigma_E(\varphi) \subseteq \mathbb{R}_+$. Therefore,

$$t\varphi(F(x_1, y)) + t\sigma_E(\varphi) \subseteq \mathbb{R}_+, \quad (1 - t)\varphi(F(x_2, y)) + (1 - t)\sigma_E(\varphi) \subseteq \mathbb{R}_+.$$

Because $F(\cdot, y)$ is C -concave for all $y \in A$, then, for the above $x_1, x_2 \in A$ and $t \in [0, 1]$ and fixed $y \in A$,

$$F(tx_1 + (1 - t)x_2, y) \subseteq tF(x_1, y) + (1 - t)F(x_2, y) + C.$$

From $\varphi \in E^+ \setminus \{0_{Y^*}\}$, $E \subset C$, it follows that $\sigma_E(\varphi) \geq 0$ and $\varphi(C) \geq 0$.

Hence

$$\varphi F(tx_1 + (1 - t)x_2, y) + \sigma_E(\varphi) \subseteq \mathbb{R}_+,$$

and so $tx_1 + (1 - t)x_2 \in V_\varphi(F, A; E)$. This shows that $V_\varphi(F, A; E)$ is a convex set. Hence, $V_\varphi(F, A; E)$ is a connected set for any $\varphi \in E^+ \setminus \{0_{Y^*}\}$.

Next, we show that $We(F, A; E)$ is a connected set. In fact, for any $\varphi \in E^+ \setminus \{0_{Y^*}\}$, it is easy to see that $M \subseteq V_\varphi(F, A; E)$ and so

$$\emptyset \neq M \subseteq \bigcap_{\varphi \in E^+ \setminus \{0_{Y^*}\}} V_\varphi(F, A; E).$$

From the connectedness of $V_\varphi(F, A; E)$, Lemma 4.1 and Theorem 3.1, it follows that $We(F, A; E)$ is a connected set.

The proofs of (ii) and (iii) are similar to (i). □

Remark 4.4

- (i) If Y is a n.v.s. and $E = C \setminus \{0_Y\}$, Theorem 4.3(i) reduces to Theorem 4.1(iii) in [24]. Theorem 4.3(ii) reduces to Theorem 4.1(ii) in [24].
- (ii) If Y is a n.v.s. $\varepsilon \geq 0$, $e \in \text{int } C$ and $E = \varepsilon e + C$, Theorem 4.3(i) reduces to Theorem 3.4 in [25].

5 Conclusions

In this paper, under the assumption of nearly E -subconvexlikeness of the binary function in real locally convex Hausdorff topological vector spaces, we obtain the linear scalarization of weak efficient solutions, Benson proper efficient solutions, Heing efficient solutions for (USVEP). By means of the scalarization results, we investigate the connectedness of the sets of weak efficient solutions, Benson proper efficient solutions, Heing efficient solutions for (USVEP). However, the connectedness of efficient solutions sets of for (USVEP) have been not established, it may be of great interest for us to discuss.

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Competing interests

The author declares that there is no conflict of interests. The authors declare that they have no competing interests.

Authors' contributions

ZW conceived and designed the research. HL wrote the manuscript, LZ reviewed it and has given substantive suggestions about English language of the revised manuscript. All authors read and approved the final copy of the manuscript.

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