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The new analytical solution of the 3D Navier-Stokes equation for compressible medium clarifies the sixth Millennium Prize problem

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Abstract

The limitations of capabilities of the existing mathematical weather prediction (including forecasting for weather-sensitive individuals) cannot be duly realized nowadays due to the fact that till now there is no proof of the existence and uniqueness of smooth solutions of the three-dimensional (3D) Navier-Stokes equation (in any finite period of time).

We have obtained a new analytical solution of the Cauchy problem of this equation in an unbounded space, which has finite energy for any values of time.

Keywords

Hydrodynamics, Compressibility, Viscosity, Turbulence, Vortex waves

Imprint

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Introduction

1. The proper understanding of many processes in nature and engineering systems is closely connected with the existence of the fundamental and applied problem of turbulence, which remains unsolved for more than a century due to the absence of exact analytical nonstationary smooth vortex solutions of the Navier – Stokes (NS) equation. The development of the statistical approach to its solution gave a lot of interesting results, but at the same time led to a new, still unsolved, problem of the closure in the description of different moments of the vortex field, an approximated solution of which was proposed by Kolmogorov A.N., Geizenberg V. et al. [1].

In order to solve the problem of turbulency, in its turn, it is required to properly understand the mechanism of the appearance of randomness due to instability of a deterministic continual dynamic system described by the NS equation. In this case, the problem of the appearance and development of turbulence is linked to the problem of self-organization of coherent structures emerging from chaos and to the associated issues of nonrandom randomness in an individual life of a human and in life spans of living species treated in the context of the Sinai billiards [2].

However, till the present, an analytically smooth on the whole time axis nonstationary solution of the three-dimensional (3D) NS equation has not been found and even the corresponding theorem of the existence and uniqueness of such solution has not yet been proven [1].

Actually, up to date in hydrodynamics only a few exact solutions are well known, but, however, none of them is nonstationary and at the same time is defined in an unbounded (or with periodic boundary conditions) space [1–4]. Only weak nonstationary solutions describing, for example, dynamics and interactions between singular vortex objects in the two-dimensional (2D) and three-dimensional (3D) ideal incompressible medium are known [3, 5, 6]. At the same time, for the 3D ideal medium flows there are some conceptual ideas on a possibility of the existence of nonstationary solutions of the Euler – Helmholtz (EH) equation only on an unbounded interval of time $0 \le t < t_0$ (see [1, 3, 6, 7] and the references given therein). This time value for incompressible medium is determined exclusively by the 3D effect of vortex filament stretching, which may lead to an explosive unbounded growth of the enstrophy (the integral of the squared vorticity over space) in finite time t_0 [1, 3, 6, 7]. On the other hand, known are the exact stationary modes of flows of the viscous incompressible medium in the form of the Burgers and Sullivan [3] vortices for which this, potentially dangerous with respect to appearance of singularity, effect of the vortex filament stretching is accurately compensated by the effect of the viscosity. For these solutions, however, a convergent integral of energy over the entire unbounded space does not exist.

2. As a result, for almost two hundred years (since 1827–1845), open remains the issue on the existence of smooth nonstationary divergent and divergent-free solutions of the 3D NS equation in an unbounded (or with periodic boundary condition) space and on an unbounded interval of time [8–12]. And the significance of this problem is determined not only by mathematical, but also by practical interest, owing to both the fundamental and applied problem of predictability in hydrometeorology and other related fields that might be the case with the applications of the methods utilized for the NS equation computational solution [9, 10].

Therefore, in 2000 the problem of the existence of a smooth nonstationary vortex solution of the 3D NS equation on an unbounded interval of time was included by Clay Mathematics Institute into the list of the seven fundamental Millennium Prize problems under number six [8, 9, 11, 12]. However, at the same time, in [8] it is proposed to consider this problem solution not for the full NS equation [4], but only for the equation, derived from it in assumption that the divergence of incompressible medium velocity field is equal to zero. Evidently, such a definition a priori assumes that for divergent flows (having a nonzero velocity field divergence) the full NS equation obviously cannot have smooth solutions on an unbounded time interval. Actually, in [12] written is the following: "The Millennium Prize problem refers to incompressible flows, as it is known that the compressible ones behave disgustingly". Thereupon, an example of appearance of the shock wave in compressible medium when an object moves therein with a velocity higher than the velocity of sound in this medium is given in [12]. However, it is clear that the viscosity forces do not allow for real singularity for any flow characteristics, that, as a result, does not exclude a possibility of the existence of smooth divergent solutions of the full NS equation.

3. Up to date, as we know, a direct proof of impossibility of the existence of smooth divergent solutions of the full NS equation has not been obtained yet, and therefore the problem formulation in [8] allows in full a generalization for the case of divergent compressible medium flows that is the matter under consideration in this paper.

Actually, in the present paper on the basis of the theory [13] found is a new analytical nonstationary vortex solution of the full 3D NS equation which because just to the finiteness of the viscosity forces (which are modeled by adding of the velocity field of the random Gaussian delta correlated in time to the velocity field [10]) remains smooth for any arbitrary large periods of time. At the same time, the NS equation solution may be extended in Sobolev space $H^q(R^3)$ for any $q \geq 1$ and $t \geq t_0$, where t_0 – is a minimum time of singularity (collapse) appearance for the corresponding exact solution of the EH and Riemann – Hopf (RH) equations in case of zero viscosity. The norm in Sobolev space $H^q(R^3)$ is determined in the form [14]:

$$\|\vec{u}\|_{H^q(R^3)} = \left(\sum_{\beta \leq q} \int d^3x (\nabla^\beta \vec{u})^2 \right)^{1/2} \quad \text{(B.1)}$$

Let us note that in [14] formulated is a local theorem of the existence of a 3D EH equation solution of the divergent-free ideal incompressible fluid flow. According to this theorem, a smooth EH equation solution exists if the initial velocity field $\overrightarrow{u_0}$ belongs to the Sobolev space $H^q(R^3)$ when $q \geq 3$, and the very solution corresponds to the class

$$\vec{u} \in C\left(\left[0,t_*\right];H^q\right) \cap C^1\left(\left[0,t_*\right];H^{q-1}\right),$$

where the norm is determined in (B.1). At the same time, for the considered herein exact EH and RH equation solution in case of the divergent ideal compressible medium flow there exists the possibility for extension of this solution for times $t_* \geq t_0$ only in Sobolev space $H^0(R^3)$. And there is no possibility for its extension in Sobolev space $H^1(R^3)$ by time $t_* \geq t_0$, when $q = 1$ is instead of the condition $q \geq 3$ of the theorem in [14].

The finite value of the velocity field divergence corresponds to the obtained NS equation analytical solution, that indicates an inconsistency of the above "quasi evident" *a priori* assumption on the absence of smooth divergent 3D vortex solutions of the full NS equation.

The noted method for taking into account the viscosity is a particular example of turbulence modeling, when instead of a random force a random velocity field is entered [15]. In [15], however, treated is only the spatially inhomogeneous large-scale random velocity field and excluded is the drift part of this velocity which depends only on time. At the same time, just the averaging over the random velocity field, which depends only on time, provides the proper modeling of the effective viscosity (in assumption that this velocity is Gaussian and delta – correlated in time) in the present paper. Besides, it is important that this method for modeling the viscosity effect does not change the structure typical for viscosity force $\vec{F}_v$, which is entered into the NS equation and, as an example, for the incompressible medium, having the form

$$\vec{F}_v = \nu\Delta\vec{u} \quad [3].$$

Actually, it is well known [15], that the existence of a NS equation solution is proven in case if to the conventional viscosity force added is a member which is proportional to a higher derivative (of the velocity of flow $\vec{u}$) of the form

$$\Delta^{\alpha}\vec{u}, \alpha \geq \frac{5}{4}$$

(see [16,17]) and which is responsible for changes of the viscosity force structure typical for the initial NS equation.

Besides, it is shown that an elimination of the singularity of the solutions of the EH, RH and NS equations takes place even in case of an introduction of a sufficiently great coefficient of external friction μ, satisfying the condition (5.3) and corresponding to the substitution

$$\nu\Delta\vec{u} \to -\mu\vec{u}$$

in the NS equation.

The new solution of the 3D NS equation is found under the condition of the zero total balance of normal stresses caused by pressure and the viscosity of the compressible medium divergent flow that, as shown in paragraph 2 hereof, corresponds to the sufficient condition of positive definiteness of the integral entropy growth rate. It allows reducing the NS equation solution to the solution of the 3D analog of the Burgers equation, and then to the solution of the 3D RH equation and its generalization for the case of taking into account the viscosity force (the external friction or the above effective friction related to the random velocity field).

Let us also note that in general case the vortex solutions of the 3D RH equations coincide with the 3D EH equation solutions for describing the ideal compressible medium vortex flows with the nonzero velocity field divergence [10, 13].

In fact, all real media are more or less compressible, and their flows should be described just by the divergent solutions of the full NS equation. On the other hand, the divergent flows for a conditionally incompressible medium may also correspond to the presence of distributed sources and drains, modeling of which is successfully used in relativistic and non-relativistic hydrodynamics [18–21].

4. Let us notice that in [22] obtained is also an exact solution of the 3D RH equation, which describes, however, only in terms of the Lagrangian variables, an explosive evolution with time for the matrix of the first derivatives of the velocity field. It does not provide a possibility for obtaining on its basis an exact solution of the 3D EH equation for the vortex field, as it has been performed in [13] in the Eulerian representation of the solution. At the same time, the present paper shows that the obtained in [13] exact solution of the 3D RH equation for the velocity field (see formula (3.7) below) in the Lagrangian representation gives for the evolution of the matrix of first derivatives of the velocity field an expression (3.14), which exactly coincides with the formula given in [22] (see formula (30) in [22]).

Also found are new analytical solutions for the evolution of vortex intensities and helicity of the Lagrangian fluid particle in the 1D and 3D cases. In [23] considered is the similar in structure form of the EH equation solution (see formula (23) in [23]) on the basis of an application of a combination of the Eulerian and Lagrangian descriptions in the representation of the vortex lines. However, it does not permit to explicitly describe the peculiarities (including the enstrophy singularity) of the evolution with time for vorticity. The discovered herein description of the evolution vorticity in the Lagrangian representation for the 2D and 3D case (see (4.4) и (4.5)) may be considered as a concretization of the obtained in [23] form of the EH equation solution for the case of the inertial fluid particles motion.

Besides, herein specified is a new necessary and sufficient criterion for the realization of the explosive singularity (collapse) in a finite time (see (3.11), (3.12)) for the nonviscous RH and EH equation solutions in the 1D, 2D and 3D cases. At the same time, in [22] given is an integral criterion in the form of (3.13) (see formula (38) in [21]), which determines only the sufficient condition for the realization of a solution collapse. Besides, for example, for the case of the initial divergent-free velocity field, the collapse is possible only according to the necessary and sufficient criterion (3.12), but it cannot be established from the criterion (3.13). At the same time, from the completed in [22] consideration of the explosive mode for the 3D RH equation solution made is a conclusion about impossibility of extension of this solution by an infinite time in the Sobolev space $H^2(R^3)$, that differs from the above mentioned result indicated herein.

In the 2D case we have an exact correspondence between the criterion (3.11) and the similar criterion given in [24] (see formula (9) in [24]) in connection with the solution of the problem of flame front propagation (generated by self-sustained exothermic chemical reaction) on the basis of a simplified version of the Sivashinsky equation [25]:

$$\frac{\partial f}{\partial t}-\frac{1}{2}U_s\left(\vec{\nabla}f\right)^2=\gamma_0 f \qquad \text{(B.2)}$$

In the equation (B.2), the function

$$x_3=f(x_1,x_2,t)$$

determines the flame front representing the boundary between combustion agent ($x_3 > 0$) and combustion products ($x_3 < 0$), where U_s and γ_0 are constant positive values characterizing the front propagation velocity and the combustion intensity, respectively. With $\gamma_0 = 0$ the equation (B.2) coincides with the Hamilton – Jacobi equation for a free non-relativistic particle. The proposed herein exact RH equation solution (3.7) in the 2D case (to be more exact, in its modification, taking into account the external friction with the coefficient μ and the formal equality $\mu = -\gamma_0$) gives the exact equation solution (B.2). At the same time, the solution (3.7) describes the potential flow of the form

$$\vec{u}=-U_s\vec{\nabla}f\ .$$

5. An important result of the present paper is obtaining of the closed description of the with-time evolution of enstrophy and any higher vortex field moments, as well as the velocity field in the 2D and 3D cases. It is achieved on the basis of the corresponding analytical solution of the EH, RH and NS equations both for the case of the zero viscosity and the case with taking into account the external friction or the effective viscosity. As a result, not approximately, as usual, but exactly solved has been the problem of the closure in the theory of turbulence, which remained unsolved for a long time, despite multiple attempts for searching at least for its approximate solution [1]. Herein we have succeeded in finding the solution due to establishing a relatively simple and clear dependence on the initial condition for the obtained exact EH and RH equation solution for the velocity field (3.7) and the vortex field ((4.1) and (4.2)), which is absent, for example, in the well known exact Burgers equation solution, obtained with the use of the nonlinear Cole – Hopf transformations.

In particular, due to this fact, based on the exact solution (4.2), obtained can be an estimation for the integrals of the vorticity field in the 3D case close to the moment of solution singularity:

$$\Omega_{3(2m)}=\int d^3x\vec{\omega}^{2m}\cong O\left(\frac{1}{(t_0-t)^{2m-1}}\right)\ \text{and}$$

$$\Omega_{3(m)}=\int d^3x\vec{\omega}^{m}\cong O\left(\frac{1}{(t_0-t)^{m-1}}\right),$$

when $m = 1, 2, 3\ldots$. Thus, the following inequality is evident:

$$\Omega_{3(2m)}/\Omega^2_{3(m)}\cong O\left(\frac{1}{t_0-t}\right)>>1; \qquad \text{(B.3)}$$
$$t\to t_0$$

It demonstrates a strong intermittency of the vortex field in the vicinity of singularity.

Let us note, that usually the inequality

$$\Omega_{3(2m)} > \ \Omega^2_{3(m)},$$

actually, is regarded to be true under a strong vortex intermittency [15], but in the past it was impossible to derive it from the exact solution of the closure problem in the theory of turbulence, as it was done, when obtaining the estimation (B.3).

6. In conclusion hereof, based on an analysis of the exact closed solution of the enstrophy balance equation (5.6) and the rate of integral kinetic energy change in (6.1)–(6.4), discussed is a possibility of the existence of not only divergent, but also smooth divergent-free NS equation solutions on an unbounded time interval.

1. The Navier-Stokes (NS) and Euler-Helmholtz (EH) equations

The equation of the motion of the compressible medium may be written as follows [4]:

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=$$
$$=\frac{\eta}{\rho}\Delta u_i-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p-\left(\zeta+\frac{\eta}{3}\right)\left(\frac{\partial u_k}{\partial x_k}\right)\right);$$
$$\Delta=\frac{\partial^2}{\partial x_k\partial x_k} \qquad \text{(1.1)}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho u_k)=0 \qquad \text{(1.2)}$$

It follows from the form of the second member in the right side (1.1) that for the viscous compressible divergent flow the normal stresses are determined not only by pressure, but also by the velocity field divergence. In (1.1.1), (1.2) u_i – is a velocity of medium; in repeated indices implied is a summation from 1 to n (where n – is dimensionality of space, and later treated will be the cases, when $n = 1, 2, 3$), a p, ρ, η, ζ – is a pressure, a density, the constant coefficient of the viscosity and the constant coefficient of the second viscosity of medium, respectively [4].

For the incompressible medium with a constant density $\rho = \rho_0$ from equation (1.1), in the 3D case (when $n=3$) after curl operation from left and right sides, the following Euler – Helmholtz (EH) equation is obtained:

$$\frac{\partial \omega_i}{\partial t} + u_k \frac{\partial \omega_i}{\partial x_k} = \\ = \omega_k \frac{\partial u_i}{\partial x_k} - \omega_i div\bar{u} + \nu \Delta \omega_i \qquad (1.3)$$

In (1.3) $\vec{\omega} = rot\vec{u}$, and $\nu=\eta/\rho_0=const$ – is a coefficient of molecular kinetic viscosity.

For the case of the compressible medium, the equation (1.3) also takes place, but only if $\eta/\rho = const$ and the curl from the second member in the right side (1.1) is equal to zero. In particular, it occurs in the case if the second member in the right side is equal to zero (1.1) that corresponds to the zero total balance of the normal stresses produced by pressure and viscosity of the divergent medium flow.

In [13] obtained is an exact vortex solution of the 3D Riemann-Hopf (RH) equation (which coincides with (1.1) when vanishing the right side (1.1)) in case of arbitrary, smooth, vanishing at infinity, initial conditions. It coincides with the exact EH equation solution (1.3) for the compressible nonviscous medium (when in (1.3) the last member in the right side is equal to zero). At the same time, in [13], in particular, it is demonstrated that the obtained smooth solution may exist only on a bounded interval of time $0 \leq t < t_0$, (where the value t_0 is determined further from equation (3.11)).

Further (in paragraph 3 herein) it is demonstrated that for any arbitrarily small value of the effective viscosity (introduced instead of the last member in the right side (1.3)) obtained can be an exact solution of the equations (1.1) – (1.3) which exists even on an unbounded time interval.

2. Energy and entropy balance equations

1. Usually, when considering the system of four equations (1.1), (1.2) for five unknown functions, introduced is an additional condition of a relation (an equation of the medium state) between density and pressure in order to make equal the number of the equations and the number of the unknown functions. The representation of the equation of state for a nonequilibrium vortex flow needs to be specified. Instead of this, for the closure of the system (1.1), (1.2) usually utilized is an approximation of the velocity zero divergence for the incompressible medium, that is reasonable, in particular, in case of relatively lower (if to compare with sound velocity) medium motion velocities.

Let us derive a similar equation, which closes the system (1.1), (1.2) for the compressible medium divergent flow and which will substitute the condition of equality to zero of the velocity field divergence for the incompressible fluid flow.

For this purpose, we obtain the energy and entropy balance equations which follow from (1.1), (1.2) as well as from the conventional thermodynamic relations [26]. In case of a single-component medium, these relations have the following form [26] (see (14.3), (15.6) and (15.7) in [26]):

$$\varepsilon = Ts - \frac{p}{\rho} + \Phi \qquad (2.1)$$

$$-sdT + \frac{dp}{\rho} = d\Phi \qquad (2.2)$$

$$d\varepsilon = Tds + \frac{p}{\rho^2} d\rho \qquad (2.3)$$

In (2.1) – (2.3) T – is a temperature, a ε, s, Φ – are internal energy, entropy and thermodynamic potential or the Gibbs free energy (units of medium mass), respectively [26]. At the same time, the equation (2.3) immediately follows from the equation (14.3) in [26], and it exactly coincides with the equation (2.1) and (2.2) (which coincides with the equation (15.7) and (15.6) in [26], respectively) at any Φ. For the considered single-component medium under condition of the constant amount of the particles therein, we assume below that in (2.1) and (2.2) $d\Phi = 0$ или $\Phi = \Phi_0 = const$.

The equation (2.3) is further used in the following form (see also [4] on page 272):

$$\frac{\partial \varepsilon}{\partial t} = T\frac{\partial s}{\partial t} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} \qquad (2.4)$$

2. Based on the equations (1.1), (1.2), we may obtain the equation of the balance of the integral kinetic energy

$E = \frac{1}{2}\int d^n x \rho u^2$ as follows:

$$\frac{dE}{dt} = -\eta \int d^n x (\frac{\partial u_i}{\partial x_k})^2 + \\ + \int d^n x \left[p - (\zeta + \frac{\eta}{3}) div\bar{u} \right] div\bar{u} \qquad (2.5)$$

For the incompressible viscous medium, the divergent-free flow formula (2.5) exactly coincides with the formula (16.3) in [4], and it serves as its generalization for the case of the compressible viscous medium flow. To derive the equation (2.5) it is enough to scalarly multiply the equation (1.1) by the vector $\rho\, u_i$, multiply the equation (1.2) by the scalar $\vec{u}^2/2$, add the obtained expression and integrate over the entire space.

Let us notice, that in case of an ideal (nonviscous) medium, from (2.5) it follows that integral kinetic energy is an invariant only for the divergent-free flows, and for the divergent flows as an invariant should be only the total integral energy

$$E_h = \int d^3x(\rho\frac{\vec{u}^2}{2} + \rho\varepsilon),$$

conservation of which is assumed to be for the viscous medium, too [4].

Let us derive an equation of the total energy balance for the viscous compressible medium and the corresponding equation of entropy balance, on the basis of the equations (1.1), (1.2), (2.1) and (2.4). As opposed to the derivation given in [4], let us immediately use the equation (2.1) written taking into account the above equality $\Phi = \Phi_0 = const$. As a result, considering (2.1), we have the following from (2.4):

$$\frac{\partial}{\partial t}(\rho\varepsilon) = T\frac{\partial}{\partial t}(\rho s) + \Phi_0\frac{\partial\rho}{\partial t} \qquad (2.6)$$

In the equation (2.6), the second member in the right side, taking into account (1.2), is convenient to represent in the form

$$\Phi_0\frac{\partial\rho}{\partial t} = -div(\Phi_0\rho\vec{u}).$$

At the same time, from (1.1), (1.2) and (2.6) we obtain the following total energy balance equation:

$$\frac{\partial}{\partial t}(\rho\frac{\vec{u}^2}{2} + \rho\varepsilon) =$$
$$= -\frac{\partial}{\partial x_k}\begin{bmatrix} u_k(\rho(\frac{\vec{u}^2}{2} + \Phi_0) + p - \\ -(\zeta + \frac{\eta}{3})div\vec{u}) - \eta\frac{\partial}{\partial x_k}(\frac{\vec{u}^2}{2}) \end{bmatrix} +$$
$$+T(\frac{\partial}{\partial t}(\rho s) - \frac{B}{T}),$$
$$B = \eta(\frac{\partial u_i}{\partial x_k})^2 - \left[p - (\zeta + \frac{\eta}{3})div\vec{u}\right]div\vec{u} \qquad (2.7)$$

As in [4], from (2.7), taking into account the requirement of equality to zero of the time-related derivative of the integral total energy

$$\frac{d}{dt}E_h = 0,$$

we can write the following entropy balance equation:

$$\frac{\partial}{\partial t}(\rho s) = \frac{B}{T}, \qquad (2.8)$$

where expression B is given in (2.7).

The energy and entropy balance equations (2.7), (2.8) do not coincide with the equations given in [4] in the formula (49.3) and (49.4), respectively. However, from the balance equation (2.7) we may obtain exactly these equations (49.3), (49.4) and the given in [4] integral entropy balance equation (49.6) as well. For this purpose, in (2.7) we should use instead of the equation (2.6) its equivalent representation

$$\frac{\partial}{\partial t}(\rho\varepsilon) = (\varepsilon + \frac{p}{\rho})\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}$$

(applied in [4] without taking into account (2.1), but assuming the equality $\Phi = \Phi_0 = const$). It is more significant that, in addition thereto, to provide the coincidence of (2.7) with (49.3) in [4], the pressure gradient in (2.7), according to [4], should be expressed in the form of

$$\frac{\partial p}{\partial x_k} = \rho\frac{\partial}{\partial x_k}(\varepsilon + \frac{p}{\rho}) - \rho T\frac{\partial s}{\partial x_k},$$

which follows from the thermodynamic relation (2.3) (if to add member dp/ρ). to the left and right side (2.3)). Such thermodynamic representation for the pressure gradient which enters into (2.7) (and in (1.1)), corresponds to the conventional representation of pressure, which completely describes normal stresses for the compressible and incompressible medium only in case of the zero viscosity. It does not correspond to that new representation of pressure, which appears just in case of description of the viscous compressible hydrodynamics in (1.1) due to appearance of additional normal stresses, proportional to the velocity field divergence (see [4] page 275).

This statement on incompletely adequate representation of the pressure gradient (in formulas (2.7) and (1.1)) on the basis of the application of the thermodynamic relation (2.3) is further confirmed by the obtained in next clause fundamental relation (2.10) between the rates of with-time change of the integral entropy and the integral kinetic energy. Actually, the relation (2.10) immediately follows from (2.5) and the integral entropy balance equation, written just in the form of (2.9) on the basis of (2.8). On the other hand, this relation (2.10) obviously cannot be obtained from (2.5) and the integral entropy balance equation in the form given in [4] (see (49.6) in [4]).

4. From the entropy balance equation (2.8), the integral entropy balance equation

$$S = \int d^3x\rho s$$

in the given below form follows (for simplicity's sake, herein as well as in (2.7) and (2.8) we do not use members, which describe flows generated by the temperature gradient):

$$\frac{d}{dt}S=\eta\int d^3x\frac{1}{T}(\frac{\partial u_i}{\partial x_k})^2-$$
$$-\int d^3x\frac{1}{T}div\vec{u}\left[p-(\zeta+\frac{\eta}{3})div\vec{u}\right] \quad (2.9)$$

The balance equation (2.9), as already noted in the previous clause hereof, significantly differs from the integral entropy balance equation given in [4] (see formula (49.6) in [4]).

From (2.9) and (2.5), in case of constant temperature $T = T_0$ in (2.9), it immediately follows that the given fundamental relation is exactly satisfied:

$$T_0\frac{dS}{dt}=-\frac{dE}{dt} \quad (2.10)$$

(it is also given in [4] page 422) between the rate of the mechanical energy change and the rate of the integral entropy growth.

The expression for the rate dE / dt, given in formula (79.1) in [4], is not derived immediately from (1.1), (1.2), as it is done for the equation (2.5), but entered only on the basis of the relation (2.10), resulting from the presented in [4] integral entropy balance equation (49.6). At the same time, it is clear, that it is just the formula (2.5) for value dE/dt that provides a generalization of the formula (16.3) in [4] for the case of the compressible medium divergent flows, and it is not the formula (79.1), as stated in [4] without substantiation of derivation (79.1) on the basis of the Navier – Stokes equation (1.1) and the continuity equation (1.2).

Thus, it is evident from (2.5) and (2.9), that the negative definiteness of the integral kinetic energy dissipation rate and the corresponding positive definiteness of the integral entropy growth rate are possible in the compressible medium divergent flows only under condition of vanishing the second member in the right side (2.5) and (2.9), when the following relation is satisfied:

$$p=(\zeta+\frac{\eta}{3})div\vec{u} \quad (2.11)$$

The equation (2.11) demonstrates that the rate of decrease in the divergent flows integral kinetic energy in (2.5) is determined by only viscous dissipation, as it is the case with the divergent-free flows (see (16.3) in [4]).

When satisfying the equation (2.11), the positively determined value of the rate of the integral entropy growth in (2.9) is found to be significantly less than the growth rate of the integral entropy given in formula (49.6) in [4]. Actually, in (49.6) there is a member present, which is proportional to the second viscosity coefficient, and in (2.9) such a member is absent under condition (2.11). The relative decrease in the kinetic energy dissipation rate in (2.5), if to compare with the expression (79.1) in [4], corresponds to the said entropy growth rate decrease in (2.9) under condition (2.11). At the same time, at least a similarity to the minimum entropy production by I. Prigogine (see in [10]) takes place.

Thus, for the compressible medium divergent flows formulated is an additional equation (2.11), which closes this system (1.1), (1.2), based on the requirement of the positive definiteness of the integral entropy growth rate in (2.9) and the negative definiteness of the integral kinetic energy dissipation rate in (2.5). Therefore, the equation (2.11) for the compressible medium divergent flows must substitute the condition of the nondivergency, usually applied for the closure of the system (1.1), (1.2) in case of the incompressible medium approaching.

3. A new divergent solution of the NS equation

1. The condition (2.11) defines an exact mutual compensation between the normal stresses of pressure and the normal viscous stresses of the compressible divergent flow. As a result of such compensation, vanishing is the second member in the second side of equation (1.1). At the same time, the equation (1.1) exactly coincides with the n-dimensional generalization of the Burgers equation:

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=\frac{\eta}{\rho}\Delta u_i \quad (3.1)$$

In this case, the system (1.2), (3.1) is already closed and describes the evolution of the density and the medium inertial motion velocity field with decaying available, which should be attributed only to the action of the shearing viscous stresses, corresponding to the nonzero right side of the equation (3.1).

If in (3.1) the viscosity coefficient is equal to zero, from (3.1) we obtain the n-dimensional RH equation, for which in [13] obtained is an exact vortex solution, considered further and generalized for the case of taking into account the external friction or the effective viscosity. Let us notice that, as opposed to the given herein and in [13] consideration of the vortex solutions, previously studied was only a vortex-free solution of the equation (3.1), which corresponds to the potential flow and which is obtained when using the modification of the nonlinear Cole – Hopf transformation [27, 28].

Suppose that in (3.1) carried out is the substitution $u_i \to u_i + V_i(t)$, where $V_i(t)$ is a random Gaussian delta-correlated with-time velocity field, for which the following relations take place:

$$\langle V_i(t)V_j(\tau)\rangle = 2\nu\delta_{ij}\delta(t-\tau)$$
$$\langle V_i(t)\rangle = 0 \qquad (3.2)$$

In (3.2) δ_{ij} – is the Kronecker symbol, δ – is a delta function of Dirac – Heaviside, and the coefficient ν characterizes the viscosity force effect. In general case, it may depend on time, describing the effective turbulent viscosity, but also it might coincide with the constant coefficient of the kinematic molecular viscosity, when the considered random velocity field corresponds to molecular fluctuations. Let us consider only the case, when the said coefficient in (3.2) is constant, but at the same time remains sufficiently great in value, so that we have the following inequality which allows neglecting the member in the right side (3.1):

$$\nu >> \frac{\eta}{\min(\rho)} \qquad (3.3)$$

where *min (ρ)* – is an absolute minimum of the medium density value in space and time.

The substitution in (3.1) with an introduction of the random velocity field, as already mentioned in Introduction hereof, corresponds to the applied in [15] method for obtaining the stochastic NS equation not at the expense of the random force application, but by adding the random velocity to the velocity of that field, which enters into the conventional deterministic NS equation. Herein, as opposed to [15], considered is the case, when such random velocity field depends only on time (in [15] such velocity field is called a drift part of the large-scale inhomogeneous random field) and its accounting is equivalent to an introduction of the volume viscosity force, which coincides in its structure with the conventional friction force in the NS equation.

From the equation (3.1), upon averaging with due consideration of (3.2), under condition (3.3), we write the following equation:

$$\frac{\partial\langle u_i\rangle}{\partial t} + \left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle = \nu\Delta\langle u_i\rangle, \qquad (3.4)$$

where the angle brackets correspond to the operation of averaging over the random Gaussian field *$V_i(t)$*.

When deriving the equations (3.4) from (3.1), (3.2) except inequality (3.3) used is the following relation (which is a consequence of the Furutsu – Novikov formula [29-31]):

$$\left\langle V_k\frac{\partial u_i}{\partial x_k}\right\rangle = -\nu\Delta\langle u_i\rangle \qquad (3.5)$$

The equation (3.4), also without condition (3.3), may correspond to the equation (3.1) as well, if together with (3.5) the following equalities hold

$$\langle u_i\rangle = u_i,$$

$$\left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle = u_j\frac{\partial u_i}{\partial x_j}$$

(see [31]) and if in (3.1) the substitution

$$\frac{\eta}{\rho} \to \min(\frac{\eta}{\rho}) = \nu$$

is carried out before. Such disconnecting of the correlations is possible in case of an exact disintegration of the time scales related to the large-scale inertial motions and the motions with the typical scale of the viscous dissipation [23].

2. Instead of approximate solving (see [32]) the problem of closure when considering the equation (3.4) in order to find the average velocity field $\langle u_i\rangle$, let us use the initial equation, from which exactly evident is just the equation (3.4). This initial equation has the form of the RH n-dimensional equation [10, 13, 22]:

$$\frac{\partial u_i}{\partial t} + (u_j + V_j(t))\frac{\partial u_i}{\partial x_j} = 0 \qquad (3.6)$$

If to apply the curl operation to the left side of the equation (1.3.6), we obtain just the Helmholtz equation (1.3), where only the member $\nu\ \Delta\ \omega_i$ should be deleted and where the substitution $\vec{u} \to \vec{u} + \vec{V}(t)$ should be carried out.

The equation (3.6), as shown in [14], has the following exact solution for the case of an arbitrary dimensionality of the space (*n = 1, 2, 3,* etc.):

$$u_i(\vec{x},t) = \qquad (3.7)$$
$$= \int d^n\xi u_{0i}(\vec{\xi})\delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))\left|\det\hat{A}\right|$$

where

$$B_i(t) = \int_0^t d\tau V_i(\tau)$$

$$\hat{A} \equiv A_{nm} = \delta_{nm} + t\frac{\partial u_{0n}}{\partial \xi_m},$$

det $\hat{A}$ – are determinants of the matrix $\hat{A}$, and $u_{0i}(\vec{x})$ – is an arbitrary smooth initial velocity field. The solution (3.7) satisfies the equation (3.6) only at times for which with any spatial coordinate values the value of the matrix $\hat{A}$ determinant is positive, i.e.*det $\hat{A}$ > 0.* Therefore everywhere we shall take it into account, and, respectively, the modulus sign when writing *det $\hat{A}$*, will not be used, unless otherwise specified.

The solution (3.7) only in case of the initial velocity field potentiality is a potential vortex-free one and corresponds to the zero vortex field for all subsequent time moments. On the contrary, it is the vortex one and determines the vortex field evolution in case of a nonzero initial vortex field (see the next paragraph herein). Further, let us discuss the vortex solutions only (3.7). Let us note, however, that in [24] obtained is just the potential solution of the 2D RH equation (3.6) (at $\vec{B}$ *= 0* in (3.6)) in the Lagrangian representation, which exactly follows from (3.7) at $n = 2$, as already mentioned in Introduction hereof in connection with the possibility of description of the Sivashinsky equation (B.2) solution, using the potential solution (3.7). For the 1D case with $n=1$ we have

$$\det \hat{A} = 1 + t\frac{du_{01}}{d\xi_1}$$

in (3.7), and the solution (3.7) exactly coincides with the solutions given in [33, 34]. The solution (3.7) is found if applied has been an integral representation for the implicit solution of the equation (3.6) in the form of

$$u_k(\vec{x},t) = u_{0k}(\vec{x} - \vec{B}(t) - t\vec{u}(\vec{x},t))$$

with the use of the delta-function and the following identities [13]:

$$\delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}(\vec{x},t)) \equiv$$
$$\equiv \delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))\det \hat{A};$$
$$\frac{\partial\delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))}{\partial x_m} \equiv$$
$$\equiv -A^{-1}_{km}\frac{\partial\delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))}{\partial \xi_k}; \quad (3.8)$$

$$\delta(\vec{\xi}_1 - \vec{\xi} + t(\vec{u}_0(\vec{\xi}_1) - \vec{u}_0(\vec{\xi}))) \equiv$$
$$\equiv \frac{\delta(\vec{\xi}_1 - \vec{\xi})}{\det \hat{A}} \quad (3.9)$$

where A^{-1}_{km} is the matrix inverse to the matrix A_{km}.

After averaging over the random field $B_i(t)$ (with Gaussian density of probabilities distribution) from (3.7) we can obtain the following exact solution of equation (3.4) (and the equation (1.1) under the condition (2.11)):

$$\langle u_i \rangle = \int d^n\xi u_{0i}(\vec{\xi})\left|\det \hat{A}\right|\frac{1}{(2\sqrt{\pi\nu t})^n}\cdot$$
$$\cdot\exp\left[-\frac{(\vec{x} - \vec{\xi} - t\vec{u}_0(\vec{\xi}))^2}{4\nu t}\right] \quad (3.10)$$

The averaged solution (3.10), as opposed to (3.7), is already arbitrarily smooth in any unbounded span of time change, and not only just under the condition of positivity of the matrix $\hat{A}$ determinant.

3. Without taking into account the viscosity forces, when in (3.7) $\vec{B}(t) = 0$, the smooth solution (3.7), as already noted, is determined only under the condition $det\,\hat{A} > 0$ [13]. It corresponds to the bounded interval of timewhere the value of the bounded minimum time of the existence of the solution t_0 is computed from the solution of the following algebraic equation of order n (and further minimization of the obtained expression, which depends on spatial coordinates, by these coordinates):

$$\det \hat{A}(t) = 1 + t\frac{du_{01}(x_1)}{dx_1} = 0, n = 1$$

$$\det \hat{A}(t) = 1 + t div\vec{u}_0 + $$
$$+t^2 \det \hat{U}_{012} = 0, n = 2 \quad (3.11)$$

$$\det \hat{A}(t) = 1 + t div\vec{u}_0 +$$
$$+t^2(\det \hat{U}_{012} + \det \hat{U}_{013} + \det \hat{U}_{023}) +$$
$$+t^3 \det \hat{U}_0 = 0, n = 3$$

where $det\ \hat{U}_0$ – is the determinant of the 3D matrix $U_{0nm} = \partial u_{0n} / \partial x_m$, and

$$\det \hat{U}_{012} = \frac{\partial u_{01}}{\partial x_1}\frac{\partial u_{02}}{\partial x_2} - \frac{\partial u_{01}}{\partial x_2}\frac{\partial u_{02}}{\partial x_1}$$

is the determinant of the similar matrix in the 2D case for variables (x_1, x_2). At the same time $det\ \hat{U}_{013}$, $det\ \hat{U}_{023}$ – are the determinants of the matrices in the 2D case for the variables (x_1, x_3) and (x_2, x_3), respectively.

Let us notice, that in the 2D case (3.11) exactly coincides with the collapse condition, obtained in [24] in connection with the problem of front flame propagation, studied on the basis of the Sivashinsky equation (B.2). For an exact coincidence it is necessary to substitute

$$t \to b(t) = \frac{U_s(\exp(\gamma_0 t) - 1)}{\gamma_0}$$

in (3.11).

In the 1D case at $n = 1$ from (3.11) we have the minimum time for singularity appearing

$$t_0 = \frac{1}{\max\left|\frac{du_{01}(x_1)}{dx_1}\right|} > 0.$$

In particular, at the initial distribution

$$u_{01}(x_1) = a\exp(-\frac{x_1^2}{L^2}), a > 0$$

we obtain

$$t_0 = \frac{L}{a}\sqrt{\frac{e}{2}}$$

for the value

$$x_1 = x_{1\max} = \frac{L}{\sqrt{2}}.$$

At the same time, the singularity realization itself may take place only with positive values of the coordinate $x_1 > 0$, when the equation (3.11) has a positive solution for the time value.

It means that the singularity (collapse) of the smooth solution never occurs in case when the initial velocity field is other than zero only at negative values of the spatial coordinate $x_1 < 0$.

The value of the wave breaking time t_0 is computed similarly at $n > 1$, too. Thus, for (3.11) in the 2D case (with the initial velocity field under the zero divergence) for the initial function of the flow in the form

$$\psi_0(x_1, x_2) =$$
$$= a\sqrt{L_1 L_2}\exp(-\frac{x_1^2}{L_1^2} - \frac{x_2^2}{L_2^2}), a > 0,$$

the minimum value of the time of the existence of the smooth solution is equal to

$$t_0 = \frac{e\sqrt{L_1 L_2}}{2a}.$$

The indicated minimum time of the existence of the smooth solution in the treated example is realized for the spatial variable values corresponding to ellipse points

$$\frac{x_1^2}{L_1^2}+\frac{x_2^2}{L_2^2}=1 .$$

According to (3.11), the necessary condition of the singularity realization is the condition of the existence of the real positive solution of the quadratic (at $n = 2$) or cubic (at $n = 3$) equation in relation to time variable t. For example, in case of the 2D flow with the zero initial divergence of velocity field $div\vec{u}_0 = 0$ the necessary and sufficient condition for the singularity (collapse) solution realization in finite time according to (3.11) is the following:

$$\det U_{012} < 0 \qquad (3.12)$$

For the considered above example from (3.12), the inequality

$$\frac{x_1^2}{L_1^2}+\frac{x_1^2}{L_2^2}>\frac{1}{2},$$

follows, in case when it is satisfied for $n = 2$ there is a real positive solution of the quadratic equation in (3.11), for which found is the given above minimum value of the collapse time

$$t_0=\frac{e\sqrt{L_1L_2}}{2a}>0 .$$

On the contrary, if the initial velocity field is defined in the form of the finite function with a carrier in the domain

$$\frac{x_1^2}{L_1^2}+\frac{x_2^2}{L_2^2}\le\frac{1}{2},$$

the inequality (3.12) breaks down, and appearance of the singularity in a finite time becomes impossible, and the solution remains smooth for an unbounded time even without taking into account the viscosity effect.

The condition of the existence of the real positive solution of the equation (3.11) (see, for example, (3.12)) is necessary and sufficient for the realization of singularity (collapse) of the solution as opposed to the sufficient, but not necessary integral criterion, proposed in [22] (see formula (38) in [22]) and written in the following form:

$$(\frac{dI}{dt})_{t=0}=-\int d^3x div\vec{u}_0 \det{}^2\hat{U}_0>0;$$
$$I=\int d^3x \det{}^2\hat{U} \qquad (3.13)$$

Actually, according to this criterion, proposed in [22], the solution collapse is impossible for the case when the initial velocity field is divergent-free, i.e. $div\vec{u}_0 = 0$. At the same time, however, violation of the criterion (3.13) does not exclude a possibility of the solution collapse by virtue of the fact that criterion (3.13) does not determine the necessary condition for the collapse realization. Indeed, in the considered above example (when determining the minimum time of the collapse realization

$$t_0=\frac{e\sqrt{L_1L_2}}{2a})$$

for the 2D compressible flow, the initial condition corresponds just to the initial velocity field with $div\vec{u}_0 = 0$ in (3.11) at $n = 2$.

4. On the basis of the solution (3.7) it is possible, with the application of (3.8) and the Lagrangian variable $\vec{a}$ (where

$$\vec{x}=\vec{x}(t,\vec{a})=\vec{a}+t\vec{u}_0(\vec{a})),$$

to present the expression for the matrix of the first derivatives of the velocity field $\hat{U}_{im} = \partial u_i / \partial x_m$ in the following form:

$$\hat{U}_{im}(\vec{a},t)=\hat{U}_{0ik}(\vec{a})A^{-1}_{km}(\vec{a},t) \qquad (3.14)$$

At the same time, the expression (3.14) precisely coincides with the given in [21] formula (30) for the Lagrangian with-time evolution of the matrix of the first derivatives of velocity that satisfies the 3D RH equation (3.6) (in [22] the equation (3.6) is considered only at $\vec{B}(t)=0$). In particular, in the 1D case at $n = 1$ from (3.7) and (3.8) we obtain the following particular case of formula (3.14) in the Lagrangian representation:

$$(\frac{\partial u(x,t)}{\partial x})_{x=x(a,t)}=\frac{\frac{du_0(a)}{da}}{1+t\frac{du_0(a)}{da}}, \qquad (3.15)$$

where a – is the coordinate of a fluid particle at the initial moment of time $t=0$.

The solution (3.15) also coincides with the formula (14) in [22] and describes a catastrophic process of the collapse of a simple wave in a finite time t_0, the estimation of which is given herein above on the basis of the equation (3.11) solution when applying the Euler variables.

4. An exact solution of the EH and Riemann-Hopf (RH) equations

1. The velocity field (3.7) is in conformance with the exact solution for the vortex field having the form [13] in the 2D and 3D cases as follows:

$$\omega(\vec{x},t)= \qquad (4.1)$$
$$=\int d^2\xi\omega_0(\vec{\xi})\delta(\vec{\xi}-\vec{x}+\vec{B}(t)+t\vec{u}_0(\vec{\xi}))$$

$$\omega_i(\vec{x},t)=$$
$$=\int d^3\xi(\omega_{0i}(\vec{\xi})+t\omega_{0j}\frac{\partial u_{0i}(\vec{\xi})}{\partial\xi_j})\cdot \qquad (4.2)$$
$$\cdot\delta(\vec{\xi}-\vec{x}+\vec{B}(t)+t\vec{u}_0(\vec{\xi}))$$

where $\vec{\omega}_0 = rot\vec{u}_0$ in (4.2) and ω_0 is the initial distribution of vorticity in the 2D case in (4.1). The solution

(4.2), (3.7) corresponds to the following exact expression for helicity:

$$H = \omega_k u_k = \tag{4.3}$$

$$= \int d^3\xi (u_{0k}\omega_{0k} + t\omega_{0j}\frac{\partial}{\partial \xi_j}(\frac{\vec{u}_o^2}{2}))\cdot$$

$$\cdot\delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))$$

The representations for the 3D vortex (4.2) and velocity (3.7) fields exactly satisfy the 3D Helmholtz equation (1.3), where, as mentioned above, it is necessary to remove the last member in the right side (1.3) and enter the random velocity field $\vec{V}(t)$ for describing the viscosity forces. It may be verified by the direct substitution of the solution (4.2) and (3.7) into (1.3). For this purpose, when considering the nonlinear members, it is necessary to use the equality

$$\delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))\cdot$$
$$\cdot\delta(\vec{\xi}_1 - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}_1)) =$$
$$= \delta(\vec{\xi} - \vec{x} + \vec{B}(t) + t\vec{u}_0(\vec{\xi}))\cdot$$
$$\cdot\delta(\vec{\xi}_1 - \vec{\xi} + t(\vec{u}_0(\vec{\xi}_1) - \vec{u}_0(\vec{\xi}))),$$

and the following identities: (3.8), (3.9) and

$$\omega_{0m}(\vec{\xi}) = A_{mk}^{-1}(\omega_{0k} + t\omega_{0j}\frac{\partial u_{0k}}{\partial \xi_j}).$$

After averaging in (4.1) and (4.3) over the random Gaussian field $\vec{B}(t)$ taking into account (3.2), we obtain expressions where under integral sign in (4.1)–(4.3) the delta-function is substituted by an exponent with the normalizing multiplier as it is the case with (3.10). Only after the said averaging provided is the existence of not only the averaged vortex and helicity field values, but also the corresponding highest derivatives and higher moments in any time span. In particular, it takes place when for the enstrophy value (the integral of the vorticity square over the entire space) and higher moments of the vortex field, for which the explicit analytical expressions are obtained in an elementary way in the next paragraph without solving any closure problem.

2. In the Lagrangian variables, the expressions, which correspond to the Eulerian vortex (4.1), (4.2) and helicity (4.3) fields, may be presented in the following form (in case when $\vec{B}(t) = 0$):

$$\omega(\vec{a},t) = \frac{\omega_0(\vec{a})}{\det \hat{A}(\vec{a},t)} \tag{4.4}$$

$$\omega_i(\vec{a},t) = \frac{(\omega_{0i}(\vec{a}) + t\omega_{0m}(\vec{a})\frac{\partial u_{0i}(\vec{a})}{\partial a_m})}{\det \hat{A}(\vec{a},t)} \tag{4.5}$$

$$H(\vec{a},t) = \frac{(u_{0k}(\vec{a})\omega_{0k}(\vec{a}) + t\omega_{0k}(\vec{a})\frac{\partial}{\partial a_{0k}}(\frac{\vec{u}_0^2(\vec{a})}{2}))}{\det \hat{A}(\vec{a},t)} \tag{4.6}$$

where

$$\det \hat{A} = \det(\delta_{im} + t\frac{\partial u_{0i}(\vec{a})}{\partial a_m})$$

From (4.4) – (4.6) it follows that for the Lagrangian fluid particle the vortex value singularity in the 2D and 3D cases, as well as the helicity singularity, take place at $t \to t_0$, when $\det \hat{A}(\vec{a},t) \to 0$ and the finite time value of the existence of the corresponding smooth fields is determined by the Lagrangian analog of the condition (3.11). At the same time, it follows from (4.5) и (4.6) that the 3D effect of vortex filaments stretching leads to only not an explosive, but a weaker power-raise-related increase in the vortex and helicity values as opposed to the catastrophic process of the vortex wave collapse in a finite time t_0 just for the divergent flow of the compressible medium.

Let us notice that in [23] (see formula (23) in [23]) obtained is the representation of the EH equation solution (1.3) (at zero viscosity) in the following form:

$$\omega_i = \frac{\omega_{0k}(\vec{a})}{J}\frac{\partial R_i(\vec{a},t)}{\partial a_k} \tag{4.7}$$

In (4.7) $J = det\,(\partial x_n/\partial a_m)$ is the Jacobian transformation to the Lagrangian variables $\vec{a}$. At the same time, $\vec{\omega}_0(\vec{a})$ is a new Cauchy invariant (coinciding with the initial vorticity) which is characterized by the zero divergence

$$\frac{\partial \omega_{0k}(\vec{a})}{\partial a_k} = 0 \text{ and}$$

$$x_i = R_i(\vec{a},t) \text{ and}$$

$$\frac{d\vec{R}}{dt} = \vec{V}_n(\vec{R},t),$$

where $\vec{V}_n$ is the velocity component being normal to the vorticity vector so that for the component we have $div\vec{V}_n \neq 0$ [22].

As opposed to (4.1) and (4.2), the expression (4.7) does not give an explicit representation for the EH equation solution, since in (4.7) no definite relation for the Jacobian J and the vector $\vec{R}$ is provided. At the same time, there exists a structural correspondence between (4.7) and (4.1), (4.2), and for case of the Lagrangian fluid particles motion due to inertia, at least, for the Jacobian in (4.7) may be used the explicit representation $J = det\,\hat{A}$, where $det\,\hat{A}$ is determined from (3.11).

5. Equation of enstrophy balance and due consideration of external friction

1. Disregarding the viscosity force (i.e. without averaging in (4.1) and (4.2) over the random field $\vec{B}(t)$) from (4.1), (4.2) it follows that the enstrophy values conforming with them in the 2D and 3D cases allow for an exact closed description and take on the form [14]:

$$\Omega_2 \equiv \int d^2x\omega^2(\vec{x},t) = \tag{5.1}$$
$$= \int d^2\xi\omega_0^2(\vec{\xi}) / \det \hat{A}$$

$$\Omega_3 \equiv \int d^3x\omega_i^2(\vec{x},t) = \\ = \int d^3\xi(\omega_{0i} + t\omega_{0j}\frac{\partial u_{0i}}{\partial \xi_j})^2 / \det \hat{A} \tag{5.2}$$

To write the expressions indicated in (5.1) and (5.2) above, there has been no necessity to solve the closure problem which usually exists in the theory of turbulence. In our case, we succeed in avoiding this problem due to a comparatively simple representation of the exact solution of the Helmholtz nonlinear equation utilized for the description of the vortex flow of an ideal compressible medium.

The expressions (5.1) and (5.2) tend to approach infinity in a finite time t_0, determined from the solution of the algebraic equation (3.11) and subsequent minimization of this solution with the use of the space coordinates.

Using the exact solution of the EH equation in the form of (4.1) and (4.2), we can obtain a closed description of the with-time evolution not only for enstrophy, as it was the case with (5.1) and (5.2), but also for any other higher moments of a vortex field.

For example, in the 2D case, from (4.1), taking into account (3.8), we will obtain:

$$\Omega_{2(m)} = \int d^2x\omega^m = \\ = \int d^2\xi \frac{\omega_0^m(\vec{\xi})}{\det^{m-1}\hat{A}}; \\ \Omega_{2(2m)} = \int d^2x\omega^{2m} = \\ = \int d^2\xi \frac{\omega_0^{2m}(\vec{\xi})}{\det^{2m-1}\hat{A}}; \\ m = 1,2,3,...$$

In Introduction hereof, presented has been the estimation (B.3) for a relation of different moments in the 3D vortex field that was obtained on the basis of the expressions of the similar type from (4.2) and (3.8).

To obtain (B.3), utilized is also the estimation $\det \hat{A} \cong O(t_0 - t)$, which is realized in the limit $t \to t_0$. The quantity of the collapse minimum time t_0 is computed in this case on the basis of (3.11).

2. Let's take into account the external friction now. For this purpose, in the equation (1.3) we should replace $\nu\Delta\omega_i \to -\mu\omega_i$. In doing so, from the expressions (3.7),(4.1) and (4.2) we can obtain an exact solution, which is found from (3.7), (4.1) and (4.2) by carrying-out the substitution of the time variable t for a new variable

$$\tau = \frac{1-\exp(-t\mu)}{\mu} \text{ [13].}$$

Changes of the new time variable τ take place now within the finite ranges from $\tau = 0$ (for $t = 0$) to $\tau = 1/\mu$ (at $t \to \infty$). It leads to the fact that in case, when under the given initial conditions the following inequality

$$\mu > \frac{1}{t_0}, \tag{5.3}$$

holds, then the value $\det \hat{A} > 0$ in the denominator (5.1) and (5.2) cannot go to zero at any moment of time, since the necessary and sufficient condition for realization of the singularity is no longer met for any instant of time (3.11), where the substitution $t \to \tau(t)$ should be made.

Subject to the condition (5.3), the solution of the 3D EH equation is smooth in an unbounded span of time t. The corresponding analytical divergent vortex solution of the 3D NS equation (where the relation (2.11) for pressure should be taken into account and where carried out should be the substitution of the first term in the right side (1.1)

$$\frac{\eta}{\rho}\Delta u_i \to -\mu u_i; \mu = const)$$

also remains smooth at any $t \geq 0$ subject to the condition (5.3). We should also notice that when the values of parameters are formally coinciding $\mu = -\gamma_0$ (see the Sivashinsky equation (B.2)), the equality $\tau(t) = b(t)$ takes place subject to the condition that the singularity has been realized (3.11) with $n = 2$ and in accordance with the solution of the Sivashinsky equation in [24].

3. It should be noted that for flows of the nonviscous (ideal) incompressible fluid with the zero divergence of the velocity field an explosive growth of enstrophy is characteristic of the 3D flows only, and enstrophy for the 2D flows should be viewed as an invariant. A different situation arises with the divergent flows of the compressible medium under consideration herein.

Actually, for the divergent flows of the nonviscous medium, the equations of enstrophy balance in the 2D and 3D cases, which follow from the EH equation (1.3) (at $\nu = 0$ in (1.3)), hold true as given below:

$$\frac{d\Omega_2}{dt} = -\int d^2\xi\omega^2 div\bar{u} \\ \frac{d\Omega_3}{dt} = 2\int d^3\xi\omega_i\omega_k\frac{\partial u_i}{\partial \zeta_k} - \\ -\int d^3\xi\omega_i^2 div\vec{u} \tag{5.4}$$

It can be seen from (5.4) that in the 3D case the evolution of enstrophy Ω_3 with time is determined not only by the effect of stretching of vortex filaments (by the first term to the right), but also by the second term as well, determined by the finiteness of the value of the velocity field divergence. As to the 2D flow, the with-time evolution of enstrophy Ω_2 occurs only, if the flow velocity field divergence is other than zero.

In order to solve (3.7), the value of the divergence of the velocity field is of the form [13]:

$$\frac{\partial u_k}{\partial x_k}= \qquad (5.5)$$

$$=\int d^n\xi \frac{\partial \det \hat{A}}{\partial t}\delta(\vec{\xi}-\vec{x}+\vec{B}(t)+t\vec{u}_0(\vec{\xi})).$$

An integral over the entire unbounded space from the right side (5.5) is equal to zero by virtue of the fact that the identities (3.9) hold and subject to the condition of becoming zero at infinity for the initial velocity field. As a result, for the solution in question, the equality $\int d^n x div\vec{u}=0$ takes place, which is responsible for the fulfillment of the law of conservation of full mass of fluid and an exact mutual integral compensation of intensities of the distributed sources and drains.

For the 3D case in (5.4), based on (3.7), (3.8), (3.9), (4.2) and (5.5), we can formulate exact expressions for the first term and the second one in the right side (5.4), which describe the contribution to a growth rate of enstrophy due to the effect of stretching of vortex filaments and due to the nonzero divergence of the velocity field, accordingly. It is not difficult to see that the same expressions for the above two terms can be also obtained by direct differentiation of the expression for enstrophy in (5.2) that results in formulation of an equality as follows:

$$\frac{d\Omega_3}{dt}= \qquad (5.6)$$

$$=2\int d^3\xi(\omega_{0i}+t\omega_{0k}\frac{\partial u_{0i}}{\partial \xi_k})\omega_{0m}\frac{\partial u_{0i}}{\partial \xi_m}/\det\hat{A}-$$

$$-\int d^3\xi(\omega_{0i}+t\omega_{0k}\frac{\partial u_{0i}}{\partial \xi_k})^2\frac{\partial \det\hat{A}}{\partial t}/\det^2\hat{A}$$

In (5.6) the first and the second terms in the right side are exactly in conformance with the corresponding first and second terms in the right side (5.4). From (5.6) it follows that for the non-viscous case both of these terms tend to approach infinity at $t\to t_0$, when $\det\hat{A}\to 0$ according to (3.11). The first term in the right side (5.6) corresponds to the effect of stretching of the vortex filaments. Its expression under integral sign is proportional to the value $O(1/\det\hat{A})$. It is evident that it makes a relatively lesser contribution to the rate of an explosive growth of enstrophy as against the second term in (5.6), the expression of which under the integral sign is proportional to the value $O(1/\det^2\hat{A})$ and which exists only for the case with the divergent flows with a nonzero divergence of the velocity field.

Since, as noted above, taking into account the viscosity (in particular, under due consideration of the external friction, when the condition (5.3) is met) leads to a regularization even of divergent solutions of the NS equation, it might be expected that it is also possible for solutions with the zero-divergence. As for them, a similar regularization, probably, would be possible because of a comparatively weaker (in the above mentioned sense) effect of stretching of the vortex lines as against the process of a wave collapse in the divergent flow. This issue will be also treated in the next paragraph herein.

4. From (2.11) and (5.5), upon averaging the Gaussian probability distribution for random field $B_k(t)$ we obtain with due account of (3.2) the following representation for pressure

$$\langle p\rangle=(\zeta+\frac{\eta}{3})\cdot$$

$$\cdot\int d^n\xi\frac{\partial \det\hat{A}}{\partial t}\frac{1}{(2\sqrt{t\pi\nu})^n}\cdot \qquad (5.7)$$

$$\cdot\exp\left[-\frac{(\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi}))^2}{4\nu t}\right].$$

An expression for the density conforming with the equations (1.2) and (3.6) takes the form [12]:

$$\rho=\int d^n\xi\rho_0(\vec{\xi})\delta(\vec{\xi}-\vec{x}+\vec{B}(t)+t\vec{u}_0(\vec{\xi})). \qquad (5.8)$$

Upon averaging in (5.8) with due consideration of (3.2) we can write an expression which is smooth at any times for the medium density as follows

$$\langle\rho\rangle=\int\begin{array}{l} d^n\xi\rho_0(\vec{\xi})\frac{1}{(2\sqrt{t\pi\nu})^n}\cdot \\ \cdot\exp\left[-\frac{(\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi}))^2}{4t\nu}\right]\end{array} \qquad (5.9)$$

By replacing $\rho\to\omega$, $\rho_0\to\omega_0$ in (5.9) we obtain an expression for the 2D vortex field, since the expressions (5.8) and (4.1) have the same structure.

6. On the existence of divergent-free solutions of the NS equation

The found smooth divergent solution of the NS equation (1.1) in the form (3.10), (5.7), as stated above, by virtue of the fact that there is its analytical representation for arbitrary smooth initial conditions is just the proof that the solution of the NS equation really is existent and unique. It is of importance that in order to model the viscosity effect, it is precisely the random Gaussian delta-correlated with-time velocity field that has been introduced for that purpose, that leads to an effective viscosity force, which is structurally exactly in conformance with the viscosity force in the NS equation, as distinct from the derivatives treated in [16, 17], which are higher than the Laplacian, in computation of the viscosity force in the NS equation.

Let us perform a comparative analysis of integral values for the divergent and divergent-free flows which characterize the evolution of the integral kinetic energy with time, the finiteness of which in [8] is the major criterion supporting the evidence of the existence of a solution of the NS equation.

For this purpose, let us consider the equation of the balance of the integral kinetic energy (2.5) on the condition (2.11) that has replaced the assumption of the zero divergence of the velocity field and that has provides the closure of the system of the equations (1.1), (1.2) for the case of the divergent flows of the compressible medium. In doing so, from (2.5) we can write an expression as given below

$$\frac{dE}{dt} = -\eta F; \qquad F = \int d^3x \left(\frac{\partial u_i}{\partial x_k}\right)^2 \tag{6.1}$$

The equation of the balance (6.1) in its form is exactly coincides with the equation of the balance of the integral kinetic energy for the divergent-free flow of incompressible fluid, as indicated in [4] (please, see formula (16.3)). Unlike the formula (16.3) from [4], there is in the formula (6.1) just the divergent velocity field, which has the nonzero divergence and which describes the motion of the compressible medium. In this case, for the divergent flow, the functional F in (6.1) is connected with enstrophy

$$\Omega_3 = \int d^3x (rot\vec{u})^2$$

by the following relationship:

$$F = \Omega_3 + D_3; \qquad D_3 = \int d^3x (div\vec{u})^2 \tag{6.2}$$

As this takes place, for the divergent solution of the NS equation, the right side (6.2), with other conditions being equal, obviously exceeds the value of the functional $F = F_0 = \Omega_3$ for the solution with the zero divergence of the velocity field.

For the obtained exact solution, the expression for enstrophy Ω_3 вin the right side (6.2) takes the form (5.2), and for the integral of the square of the divergence from (5.5) and (3.8) we arrive at an expression as given below:

$$D_3 = \int d^3\xi \left(\frac{\partial \det \hat{A}}{\partial t}\right)^2 / \det \hat{A} \tag{6.3}$$

From comparison of (6.3) and (5.2) it follows that in the vicinity if the singularity of the solution at $t \to t_0$ (see (3.11)) the values of the first and the second terms in the right side (6.2) are of the same order of magnitude.

Besides, for functional F in (6.1) let us make upper estimate with the use of the Koshi-Bunyakovsky inequality as follows:

$$F^2 = \left[\int d^3x\vec{u}\Delta\vec{u}\right]^2 \le \int d^3x\vec{u}^2 \int d^3x(\Delta\vec{u})^2 = \\ = \int d^3x\vec{u}^2 \int d^3x\left[(rotrot\vec{u})^2 + (graddiv\vec{u})^2\right] \tag{6.4}$$

According to (6.2) – (6.4), the divergent flows, with other conditions being equal, demonstrate obviously a higher value of functional F as against the divergent-free flows, for which the addend in the square brackets in the right side is absent (6.4).

From the preceding, it is clear that a conclusion must be made that the smooth divergent-free solutions of the NS equation are existent because of the fact of the proven existence of the divergent smooth solutions of the NS equation on an unbounded time interval with due consideration of the effective viscosity or external friction subject to the condition (5.3).

Conclusions

Therefore, in (3.10), (5.7) and (5.9) represented is the analytical solution of the NS equation (1.1) and the equation of continuity (1.2) for the divergent flows which have a nonzero divergence of the velocity field (5.5). From (3.10), boundedeness of the energy integral in the 3D case

$$E_0 = \frac{1}{2}\int d^3\zeta \langle u \rangle^2$$

obviously follows, too, that meets the main requirement, when formulating the problem of the existence of a solution of the NS equation [8]. Besides, satisfied is the requirement specified in [8] for unbounded smoothness of solutions for any time intervals, when describing velocity and pressure fields.

We should notice that for the found solution of the equation (3.7) even without averaging (for example, in the case $\vec{B}(t) = 0$) the energy integral

$$E_{00} = \frac{1}{2}\int d^3x\vec{u}^2 = \\ = \frac{1}{2}\int d^3\xi\vec{u}_0^2 \det\hat{A} < \infty$$

remains finite for any finite moment of time, while at the limit $t \to \infty$ energy also tends to approach infinity in a power-raise manner as $O(t^3)$ (refer to (3.11)). In this case, the solution (3.7), (4.2) can be extended by any finite time $t_* \ge t_0$ in the Sobolev space $H^0(R^3)$. It means that for the case with the ideal (nonviscous) medium the flow energy meets the requirement specified in [8] to prove the existence of the solution of the NS equation.

At the same time, however, the integral of enstrophy in (5.2) demonstrates an explosive unbounded growth (in a finite time t_0, determined from (3.11)), when

$$\Omega_3 \cong O\left(\frac{1}{t_0 - t}\right)$$

in case with the unique positive real root of the equation (3.11). By this is meant that the obtained exact solution of the EH equation in the form (3.7) and (4.2) cannot be further extended in the Sobolev space $H^1(R^3)$ by a time $t_* \geq t_0$, i.e. even at $q = 1$ when defining the norm (B.1). Only considering the viscosity makes possible to avoid the said singular behavior of enstrophy and higher moments of a vortex field which is to say that there is a possibility of extending the solutions of the EH and NS equations for any $t_* \geq t_0$ in Sobolev space $H^q(R^3)$ even at any $q \geq 1$.

In [7], under formulating the problem of the existence of a solution of the 3D NS equation, it has been offered to impose restrictions to considering only cases of solutions with the zero divergence of the velocity field. Therewith in [7] noted is importance of treatment of those particular 3D flows, for which the effect of stretching of vortex filaments in a finite time may lead to a limitation on the existence of solutions of the NS equation in the small (im Kleinen) only.

The reached conclusion that there are smooth divergent solutions of the 3D NS equation at the expense of considering even small viscosity bears witness to an admissibility of a positive solution of the problem of the existence of smooth divergent-free solutions on an unbounded interval of time as well. Really, as it has been established in (5.6), the effect of stretching of vortex filaments makes a considerably lesser contribution to the realization of the singularity of the solution than the effect of collapse of a vortex wave in the divergent compressible flow. This possibility is suggested by the inequality (6.4) as well as the equality (6.2), which determine the value of rate of change in the integral kinetic energy.

It should be also stressed that the exact solution of the EH and RH equations found in [13] gives a closed description of the with-time evolution for enstrophy and any higher moments of the vortex, velocity, pressure and density fields. The possibility of a closed statistic description for modes of turbulence without pressure (modeled with the linear 3D RH equation (3.6)) was mentioned first in 1991 in [13]. We should note that the paper by A.M.Polyakov [35] published in 1995 develops for the 3D RH equation with a random white-noise type force (Delta-correlated with time) a general theoretical field approach to the theory of turbulence and establishes a relationship between the breakdown of the Galilean invariance and intermittency. With this, however, only for 1D case found was a concrete solution of the problem of the closure in the form (refer to formula (41) in [35]) of an explicit expression for distribution of probability $w(x, y)$ of the value of velocity difference u at points being at a distance y from each other.

This paper offers a fresh approach capable of considering also exactly pressure on the basis of which an analytical solution of the full NS equation for a flow of the viscous compressible fluid has been found. In doing so, actually, the main problem of the theory of turbulence has been resolved [1], when the precise representation for a joint characteristic functional of the velocity and density fields (the pressure field in this case is uniquely defined from (2.11)) is provided. In the past, it was generally accepted that the main problem of the theory of turbulence in case with the compressible fluid could not be solved, and in [1] in this connection it was stated: "Unfortunately, this general problem is too difficult, and at present an approach to finding a full solution thereof cannot be seen." (please, refer to [1] P.177).

Utilizing the proposed exact solutions of the EH, RH and NS equations, modeling of turbulent modes can be provided, including modeling performed on the basis of the method of randomization of integrated problems of hydrodynamics offered by E.A. Novikov [36] and developed in [10]. For this purpose, we must introduce a probability measure on ensemble of realization of the initial conditions, which should be treated in this case as random functions.

The possibility established herein that a solution of the NS equation is existent is based on the fresh nonstationary analytical solution of the said equation that was said in the past to be unreachable [9, 13]. Following this way, it has been discovered that for the existence of the solution in an unbounded span of time it is just the viscosity that is required to be taken into account for this purpose. On the other hand, the issue on stability of the obtained solution should be treated on the basis of the available results which bear witness to a possibility of some destabilizing effects of the viscosity which may lead to a dissipative instability [37–40].

By this means, detected has been the mechanism of the appearance of the limitation in predictability and forecasting of a wind velocity field and impurity fields (affecting human health under the variable climatic factors of the environment) that can be realized, for example, with the numerical solution of the NS equation (for the divergent flows of compressible medium).

This mechanism is connected the truncation λ, typical for the high wave numbers or small scales that corresponds to entering of the external friction with the coefficient

$$\mu = \frac{\nu}{\lambda^2}.$$

In this case, from the condition (5.3) it follows that only with selection of a sufficiently small scale of the truncation

$$\lambda < \lambda_{th} = \sqrt{t_0 \nu}$$

(where the value t_0 is computed in (3.11)) it is possible to avoid the explosive loss of the smoothness of the solution and the loss of predictability in a finite time t_0 even at the exactly defined initial data of the numerical forecasting based on the solution of the NS equation for compressible medium.

At the same time, actually the initial data are defined not accurately, but with a certain inevitable error. This may lead to breaking down the condition $\lambda < \lambda_{th}$ and loss of predictability in a finite time. In this regard, fascinating and intriguing is the relationship, as noted in [2], between the nonrandom randomness of the Sinai billiards, the problem of predictability based on the NS equation solution and another problem of relative longevity of biological species closely related by their initial physical and physiological parameters (raven and crow etc.) that has been known since the Sir Francis Bacon's time.

Conflict of interest

None declared.

Author contributions

The authors read the ICMJE criteria for authorship and approved the final manuscript.

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