New linear theory of hydrodynamic instability of the Hagen -Poiseuille flow and the blood swirling flows formation

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Aims

This paper deals with solving of a century-old paradox of linear stability for the Hagen-Poiseuille flow. A new mechanism of dissipative hydrodynamic instability has been established herein, and a basis for the forming of helical structural organization of bloodstream and respective energy effectiveness of the cardiovascular system functioning has been defined by the authors.

Materials and methods

Theory of hydrodynamic instability, Galerkin's approximation.

Results

A new condition $Re > Re_{th_{min}} \approx 124$ of linear (exponential) instability of the Hagen-Poisseuille (HP) flow with respect to extremely small by magnitude axially-symmetric disturbances of the tangential component of the velocity field is obtained. The disturbances necessarily shall have quasi-periodic longitudinal variability along the pipe axis that corresponds to the observed data.

Conclusion

We show that the obtained estimate of value of $Re_{th_{min}}$ corresponds to the condition of independence of the main result (on the linear instability of the HP flow when $Re > Re_{th_{min}}$) from the procedure of averaging used in the Galerkin approximation. Thus, we obtain the possible natural mechanism for the blood swirling flows formations observed in the aorta and the large blood vessels.

Keywords

Instability • Helicity • Structure • Bloodstream

Imprint

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Introduction

The problem of defining mechanisms of hydrodynamic instability for the Hagen-Poiseuille (HP) flow is of great fundamental and application significance, starting from the famous Hagen's researching of blood flows in the pipe[1].

So, currently, it is decided [1-4] that the HP flow is exponentially stable with respect to extremely small by magnitude disturbances for any large Reynolds number $\mathrm{Re} = \frac{V_{\max} R}{\nu}$, where $V_{\max}$ - maximal (near axis) velocity of the HP flow in the pipe of radius R, and ν - coefficient of kinematic viscosity. Such a conclusion of the linear theory of hydrodynamic stability is based on the traditional consideration of pure periodic spatial variability of disturbances along the pipe axis and contradicts to real experimental data and observations in different technical and biology systems.

In [5], it is shown that the conclusion on linear stability of the HP flow needs clarification since if instead of periodic to consider conditionally-periodic (quasi-periodic) disturbances, then already for finite Re it might happen linear (exponential, not algebraic [6, 7]) instability of the HP flow.

In the present paper, we further develop the representation [5] within the framework of the new theory of linear instability of the HP flow. Meanwhile, contrary to [5] in particular we show a possibility of getting a threshold by Re condition of linear instability of the HP flow which does not depend on the procedure of averaging when using Galerkin approximation (necessary because of consideration of longitudinal quasi-periodicity of disturbances).

Materials and methods

Let's consider evolution in time of axially symmetric extremely small by magnitude hydrodynamic disturbances of the tangential component of the velocity field V_{φ} in the cylindrical system of coordinates (z, r, φ):

$$\frac{\partial V_{\varphi}}{\partial t} + V_{0z}(r)\frac{\partial V_{\varphi}}{\partial z} = \nu\left(\Delta V_{\varphi} - \frac{V_{\varphi}}{r^2}\right) \quad (1)$$

[1] HP flow – laminar static flow of uniform viscous non-compressible fluid along static straight and infinite by length pipe with circular cross section.

Where: $V_{0z}(r) = V_{\max}(1 - \frac{r^2}{R^2})$, $V_{0r} = V_{0\varphi} = 0$ - main (undisturbed) HP flow along the pipe of radius R having in it a constant longitudinal pressure gradient $\frac{\partial p}{\partial z} = const$, when $V_{\max} = \frac{R^2}{4\rho\nu} \cdot \frac{\partial p_0}{\partial z}$ with constant density ρ of the uniform fluid. In (1) Δ is Laplace operator. Due to assumed axial symmetry of the extremely small disturbances V_φ (i.e. since $\frac{\partial V_\varphi}{\partial \varphi} = 0$, there is no derivative $\frac{\partial p}{\partial \varphi}$ for small disturbance of the pressure field p in the right-hand side of (1)). Meanwhile, (1) allows closed description of evolution of pure tangential disturbances of the HP flow.

Let's find solution of equation (1) in the following form

$$V_\varphi = V_{\max} \sum_{n=1}^{N} A_n(z,t) J_1(\gamma_{1,n} \frac{r}{R}) \quad (2)$$

which automatically meets boundary conditions of finiteness of V_φ for $r = 0$ and non-slipping $V_\varphi = 0$ for $r = R$ on the hard pipe boundary since J_1 is the Bessel function of the first order, and $\gamma_{1,n}$ are zeroes of that function ($n = 1,2,..$).

Using the feature of orthonormality of Bessel functions and a standard averaging procedure in the Galerkin approximation (see [2]), one gets in the dimensionless from (1), (2) the following closed system of equations for the functions $A_n(z,t)$:

$$\frac{\partial A_m}{\partial \tau} + \gamma_{1,m}^2 A_m - \frac{\partial^2 A_m}{\partial x^2} + \mathrm{Re} \sum_{n=1}^{N} p_{nm} \frac{\partial A_n}{\partial x} = 0 \quad (3)$$

In (3), $x = \frac{z}{R}$, $\tau = \frac{t\nu}{R^2}$, $m = 1,2,..,N$, and coefficients p_{nm} are as follows:

$$p_{nm} = \delta_{nm} - \frac{2}{J_2^2(\gamma_{1,m})} \int_0^1 dy y^3 J_1(\gamma_{1,n} y) J_1(\gamma_{1,m} y) \quad (4)$$

where J_2 is a Bessel function of the second order, , δ_{nm} is Kronekker's symbol ($\delta_{nm} = 1$ for $n = m$ and $\delta_{nm} = 0$ if $n \neq m$). Obviously that $p_{12} \neq p_{21}$ in (4) due to the presence of a factor before the integral in (4) (since $J_2(\gamma_{1,1}) \neq J_2(\gamma_{1,2})$).

Let's limit ourselves by the case of $N=2$. For amplitudes A_1 and A_2, corresponding to different modes of radial variability V_φ in (2), we consider to have different periods of variability along the pipe axis when the next presentation takes place:

$$A_1(x,t)=A_{10}e^{\lambda\tau+i2\pi\alpha x},\ A_2(x,t)=A_{20}e^{\lambda\tau+i2\pi\beta x} \quad (5)$$

where $\alpha\neq\beta$ contrary to the usual (see [2]) consideration of the problem of stability of the HP flow in the linear approximation by amplitudes of disturbances. Meanwhile, the value of $\lambda=\lambda_1+i\lambda_2$ in (5) assumes and defines the same (synchronous) character of dependency of functions A_1 and A_2 on time. Substituting (5) in (3) (for $N=2$) leads to the following system

$$(\lambda+\gamma_{1,1}^2+4\pi^2\alpha^2+i\operatorname{Re}p_{11}\cdot 2\pi\alpha)A_{10}e^{i2\pi\alpha x}+i\operatorname{Re}p_{21}2\pi\beta A_{20}e^{i2\pi\beta x}=0 \quad (6)$$

$$i\operatorname{Re}p_{12}2\pi\alpha A_{10}e^{i2\pi\alpha x}+(\lambda+\gamma_{1,2}^2+4\pi^2\beta^2+i\operatorname{Re}p_{22}\cdot 2\pi\beta)A_{20}e^{i2\pi\beta x}=0 \quad (7)$$

System (6), (7) admits exact solution for constant coefficients A_{10} and A_{20} only in the case when $\alpha=\beta$ and in (5) functions A_1 and A_2 have the same pure periodic character of variability along the pipe axis. It is not difficult to check that from the condition of solvability of uniform system (6), (7) there may be obtained the well-known conclusion [1-4] on linear stability of the HP flow since in that case it is found that for any Re, $\lambda_1<0$.

Results

Considering the quasi-periodic variability of V_φ along the pipe axis for $\alpha\neq\beta$ in (5), we use Galerkin approximation to solve the system (6), (7). Meanwhile, let's average (6) multiplying (6) by the function $e^{-i2\pi\gamma_1 x}$ and integrating over x in the limits from 0 to $1/\gamma_1$ (i.e. applying to (6) an operation of $|\gamma_1|\int_0^{1/|\gamma_1|}dx$, where $|\gamma_1|$ is the modulus of γ_1). The equation (7) is averaged applying to (7) the same as in (6) operation of averaging but with replaced in it γ_1 by γ_2 , where in the general case $\gamma_1\neq\gamma_2$.

The solvability condition of the system of the equations obtained from (6), (7) after the specified above averaging is the following dispersion equation for λ:

$$\lambda^2+\lambda(a+b)+ab+gc=0 \quad (8)$$

where complex value $\lambda = \lambda_1 + i\lambda_2$ is uniquely defined by the following coefficients: $a = \gamma_{1,1}^2 + 4\pi^2\alpha^2 + i\,\mathrm{Re}\,2\pi\alpha p_{11} \equiv a_1 + ia_2$, $b = \gamma_{1,2}^2 + 4\pi^2\beta^2 + i\,\mathrm{Re}\,2\pi\beta p_{22} \equiv b_1 + ib_2$, $c = 4\pi^2\alpha\beta\,\mathrm{Re}^2\, p_{12}p_{21}$ and $g = \dfrac{I_{2\alpha}I_{1\beta}}{I_{2\beta}I_{1\alpha}}$.

Here, in g, we have values of elementary integrals: $I_{m\alpha} = |\gamma_m| \int_0^{1/|\gamma_m|} dx e^{i2\pi x(\alpha-\gamma_m)}$ and $I_{m\beta} = |\gamma_m| \int_0^{1/|\gamma_m|} dx e^{i2\pi x(\beta-\gamma_m)}$, where $m = 1,2$. From (8), one can obtain (see[5]) the condition of the linear instability of the HP flow when in (8) $\lambda_1 > 0$ for some $\mathrm{Re} > \mathrm{Re}_{th}$. A result in that case is significantly depending on the value of g, defined by the way of averaging of the system (6), (7) on the base of Galerkin approximation.

So, if we change the averaging procedure (applying operation $|\gamma_1| \int_0^{1/|\gamma_1|} d\chi e^{-i2\pi\gamma_1\chi}$ already to the equation (7), not to (6), as it was done above; and vice versa, we apply to (6) the operation of averaging applied above for averaging of the equation (7)), then, in the dispersion equation, value of g is replaced by $1/g$.

We require that the conclusion on the stability of the HP flow should not depend on the pointed difference in the averaging procedure conducting that is possible only when $g^2 = 1$. This equation for g has two roots $g = 1$ and $g = -1$. For $g = 1$ the conclusion on the stability of HP the flow exactly coincides with the case of the pure periodic disturbances when $\alpha = \beta$.

Let's consider the second case of $g = -1$, and show that meanwhile the linear instability of the HP flow is possible already for finite value of the threshold Reynolds number Re_{th}.

Actually, for $g = -1$ from (8) it follows that $\lambda_1 > 0$ when

$$\mathrm{Re} > \mathrm{Re}_{th} = \frac{F}{2\pi\sqrt{D}} \quad (9)$$

where $F = (a_1 + b_1)\sqrt{a_1 b_1}$, $D = (a_1 + b_1)^2\alpha\beta p_{12}p_{21} - a_1 b_1 (p_{11}\alpha - p_{22}\beta)^2$.

Obviously, for realization of the linear exponential instability of the HP flow $D > 0$ is necessary, that is trivially to be met when $(p_{11}\alpha - p_{22}\beta)^2 < 4\alpha\beta p_{12}p_{21}$.

If to introduce a parameter $p=\frac{\alpha}{\beta}$, defining the ratio of periods in (5), then from the pointed inequality providing positiveness of D in (9), it follows that the following holds

$$x_{-} < p < x_{+} \quad (10)$$

where $x_{\pm}=\frac{p_{11}p_{22}+2p_{12}p_{21}\pm 2\sqrt{p_{12}p_{21}(p_{11}p_{22}+p_{21}p_{12})}}{p_{11}^{2}}$, i.e. according to (4), $x_{+}=1.739..,x_{-}=0.588...$

On the other hand, from the condition $g=-1$ it follows that the following inequality holds

$$g=\frac{(1-e^{i\frac{2\pi\beta}{|\gamma_1|}})(1-e^{i\frac{2\pi\alpha}{|\gamma_2|}})(\alpha-\gamma_1)(\beta-\gamma_2)}{(1-e^{i\frac{2\pi\beta}{|\gamma_2|}})(1-e^{i\frac{2\pi\alpha}{|\gamma_1|}})(\beta-\gamma_1)(\alpha-\gamma_2)}=-1 \quad (11)$$

In particular, equation (11) is satisfied when $\beta=\alpha+|\gamma_1|n$ and $\beta=\alpha+|\gamma_2|m$, where m,n - are any integers having the same sign since with necessity then holds $\frac{|\gamma_1|}{|\gamma_2|}=\frac{n}{m}>0$. Meanwhile from (11), it follows that the following relations defining the value $p=\frac{\alpha}{\beta}$ depending on the values of m,n and signs of γ_1,γ_2 hold:

$$p=\frac{1}{1+\frac{2}{B}},\quad B=B_{\pm}=\frac{\gamma_1}{n|\gamma_1|}+\frac{\gamma_2}{m|\gamma_2|}-1\pm\sqrt{1+\frac{1}{m^2}+\frac{1}{n^2}} \quad (12)$$

Obviously, p from (12) shall meet inequality (10). In particular, for $m=n=1$ value of p (when $B=B_{-}$ and $\gamma_1>0,\gamma_2>0$) is $p\approx 1.58...$ Since Re_{th} in (9) is a function of β and p, for the pointed value of p, meeting inequality (10), from (9), we can get that minimal value $\mathrm{Re}_{th\min}\approx 124$ is reached in the proximity of $\beta\approx 0.5$.

Discussion and conclusions

Thus, it has been found the possibility of the linear (exponential) instability of the HP flow already for $\mathrm{Re}>\mathrm{Re}_{th\min}\approx 124$, that does not contradict to the well-known estimates of the guaranteed stability of the HP flow obtained from the energy considerations (see [1]) for Re < 81. Obviously, an exponential growth of V_{φ} after reaching of some finite values shall be replaced by a new non-linear mode of evolution in which all components of velocity and pressure are already mutually cross-linked. This growth also produces the spiral type of the resulting flow. Indeed, for the flow of blood, the arising of spiral structure is observed in the aorta in a wide range of Re number and not only for very high values of the latter [8, 9]. The results of this paper are also published in [10].

Statement on ethical issues

Research involving people and/or animals is in full compliance with current national and international ethical standards.

Conflict of interest

None declared.

Author contributions

All authors contributed equally to this work. S.G.C. read and met the ICMJE criteria for authorship. All authors read and approved the final manuscript.

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