Multiplicative excellent family of type E_6

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Abstract: We show that the equation (1) in the text defines a multiplicative excellent family of elliptic surfaces (or of cubic surfaces) with Galois group isomorphic to the Weyl group of type E_6. The main properties of the family are formulated as Theorems 1 and 2 in §3.

Key words: Weyl group; cubic surface; Mordell-Weil lattices.

1. Set-up. Let us consider the Weierstrass equation

$$(1)\; y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2)x + q_0 + q_1 t + q_2 t^2 + t^3$$

with the parameter $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$ and a variable t over $k_0 = \mathbf{Q}(\lambda) = \mathbf{Q}(p_0, p_1, p_2, q_0, q_1, q_2)$. We denote by k an algebraic closure of k_0.

The equation (1) defines the family, parametrized by λ, of three closely related objects:

- an elliptic curve E_λ over $k_0(t)$
- an elliptic surface $\pi : S_\lambda \to \mathbf{P}^1$ (t-line) defined over k_0, and
- a cubic surface V_λ in $\mathbf{P}^3$ (with inhomogeneous coordinates x, y, t) defined over k_0.

In this note, we show that S_λ (or V_λ) defines a *multiplicative excellent family* of elliptic surfaces (or of cubic surfaces) with Galois group isomorphic to the Weyl group $W(E_6)$. Roughly speaking, the parameter $\lambda = (p_i, q_j)$ forms a fundamental system of $W(E_6)$-invariants in the covering space for the splitting field of the Mordell-Weil lattice of S_λ (or the Néron-Severi lattice of V_λ). More precise formulation will be given as Theorems 1 and 2 in §3. Details and applications will appear elsewhere.

First we consider the Mordell-Weil lattice M_λ of S_λ over k. It is the group of sections of the elliptic surface S_λ over $\mathbf{P}^1$, which is identified with the group $E_\lambda(k(t))$ of $k(t)$-rational points of E_λ, equipped with a natural height pairing. Since S_λ is a rational elliptic surface with a singular fibre of Kodaira type I_3 at $t = \infty$, M_λ is isomorphic to E_6^*, the dual lattice of the root lattice E_6 under the assumption ($*$) that S_λ has no other reducible fibres (cf. [4,7]).

There are 54 minimal sections of height 4/3 in $M_\lambda \simeq E_6^*$, and the height formula ([7]) shows that half of them are defined by linear equations:

$$(2)\qquad P : \begin{cases} x = a\,t + b \\ y = d\,t + e \end{cases} \qquad (a, b, d, e \in k).$$

We call such P a *linear section*.

Obviously the linear sections correspond to the 27 lines on the cubic surface V_λ, and the results obtained below for the elliptic surface can be directly translated to the results for the cubic surface.

2. Algebraic equation of degree 27. By substituting (2) into (1), we get 4 polynomial relations among a, b, d, e over k_0:

$$(3)\qquad ad = a^3 + ap_2 + 1,$$

$$(4)\qquad ae = (3a^2 - d + p_2)b - (ap_1 + q_2 - d^2),$$

$$(5)\qquad 0 = 3ab^2 - be - 2de + ap_0 + bp_1 + q_1,$$

$$(6)\qquad 0 = b^3 - e^2 + bp_0 + q_0.$$

The first two relations imply that

$$(7)\qquad a \neq 0, \quad d, e \in \mathbf{Q}[\lambda][a, a^{-1}, b]$$

and then the remaining relations give two equations of b with coefficients in $\mathbf{Q}[\lambda][a, a^{-1}]$ of the form:

$$(8)\qquad b^3 + \cdots = 0, \quad (a^3 + 1)b^2 + \cdots = 0.$$

This implies first that, for λ generic, b is a rational function of a with coefficients in $k_0 = \mathbf{Q}(\lambda)$, and hence we have

$$(9)\qquad k_0(P) := k_0(a, b, d, e) = k_0(a).$$

On the other hand, taking the resultant of the

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equations (8) with respect to b, we obtain a monic algebraic equation of degree 27 of a with coefficients in $\mathbf{Z}[\lambda] = \mathbf{Z}[p_i, q_j]$:

$$\Phi(a) = a^{27} + ({p_2}^2 - q_2)a^{26} + \cdots + (6p_2)a + 1. \tag{10}$$

With the help of computer, the essential coefficients of the polynomial $\Phi(X) = \Phi(X, \lambda)$ in $\mathbf{Z}[\lambda][X]$ are given as follows:

(11)

$$\begin{aligned}\Phi(X) &= X^{27} + ({p_2}^2 - q_2)X^{26} + (-2p_1p_2 + 6p_2 + q_1)X^{25} \\ &\quad + (8{p_2}^3 + 2p_0p_2 + {p_1}^2 - 6p_1 - q_0 + 9)X^{24} \\ &\quad + \cdots \\ &\quad + (8{p_2}^3 + 2p_0p_2 + {p_1}^2 - 6p_1 - q_0 + 9)X^3 \\ &\quad + (13{p_2}^2 + p_0 - q_2)X^2 + 6p_2X + 1.\end{aligned}$$

3. Main results. Now we look at the Galois representation on the Mordell-Weil lattice:

$$\varrho_\lambda : \mathrm{Gal}(k/k_0) \longrightarrow \mathrm{Aut}(M_\lambda) \simeq \mathrm{Aut}(E_6^*). \tag{12}$$

Note that $\mathrm{Aut}(E_6^*) = \mathrm{Aut}(E_6) = W(E_6) \cdot \{\pm 1\}$, where $W(E_6)$ is the Weyl group of type E_6 ([1], [3, Ch.8.3], [11, Th.7].)

The *splitting field* of M_λ is the extension $\mathcal{K}_\lambda/k_0$ which corresponds to the kernel $\mathrm{Ker}(\varrho_\lambda)$ under the Galois correspondence. We have by definition

$$\mathrm{Gal}(\mathcal{K}_\lambda/k_0) \simeq Im(\varrho_\lambda). \tag{13}$$

The splitting field $\mathcal{K}_\lambda$ is equal to the minimal splitting field of the polynomial $\Phi(X, \lambda)$ over k_0, since the Mordell-Weil group $M_\lambda = E_\lambda(k(t))$ is generated by the 27 linear sections $P_i = (a_it + b_i, d_i + e_i)$ and we have

$$\mathcal{K}_\lambda = k_0(P_1, \ldots, P_{27}) = k_0(a_1, \ldots, a_{27}). \tag{14}$$

by (9).

Theorem 1. *Assume that λ is generic over $\mathbf{Q}$, i.e. p_i, q_j are algebraically independent over $\mathbf{Q}$. Then (i) ϱ_λ induces an isomorphism*:

$$\mathrm{Gal}(\mathcal{K}_\lambda/k_0) \simeq W(E_6). \tag{15}$$

Equivalently, $\Phi(X, \lambda)$ is an irreducible polynomial over $k_0 = \mathbf{Q}(\lambda)$ with Galois group $W(E_6)$.

(ii) The splitting field $\mathcal{K}_\lambda$ is a purely transcendental extension of $\mathbf{Q}$ which is isomorphic to the function field $\mathbf{Q}(Y)$ of the toric hypersurface $Y \subset \mathbf{G}_m^7$ defined by

$$s_1 \cdots s_6 = r^3. \tag{16}$$

Y has a $W(E_6)$-action such that

$$\mathbf{Q}(Y)^{W(E_6)} = \mathcal{K}_\lambda^{W(E_6)} = k_0. \tag{17}$$

(iii) The ring of $W(E_6)$-invariants in the affine coordinate ring $\mathbf{Q}[Y] = \mathbf{Q}[s_i, 1/s_i, r, 1/r]$ is equal to the polynomial ring $\mathbf{Q}[\lambda]$:

$$\mathbf{Q}[Y]^{W(E_6)} = \mathbf{Q}[\lambda] = \mathbf{Q}[p_0, p_1, p_2, q_0, q_1, q_2]. \tag{18}$$

To state the next result which is a refinement of Theorem 1(iii), we fix some notation. Let

$$s'_i := \frac{s_i}{r} \, (1 \le i \le 6), \quad s''_{ij} := \frac{r}{s_i s_j} \ (i < j) \tag{19}$$

and

$$\begin{aligned}\Omega &:= \{s_i,\ s_{6+i} := s'_i \ (i \le 6), s_{12+k} := s''_{ij} \ (i < j)\} \\ &= \{s_1, \ldots, s_{27}\}\end{aligned} \tag{20}$$

with suitable ordering. The Weyl group $W(E_6)$ acts on Ω as permutations and it is a transitive action.

Let

$$\epsilon_n \quad (\text{or} \quad \epsilon_{-n}) \tag{21}$$

denote the n-th elementary symmetric polynomial of $\{s_i | 1 \le i \le 27\}$ (or $\{1/s_i | 1 \le i \le 27\}$). Note that $\epsilon_{-n} = \epsilon_{27-n}$ since $\prod_{i=1}^{27} s_i = 1$. Further we let

$$\delta_1 = r + \frac{1}{r} + \sum_{i \ne j} \frac{s_i}{s_j} + \sum_{i<j<k} \left(\frac{r}{s_i s_j s_k} + \frac{s_i s_j s_k}{r} \right) \tag{22}$$

which corresponds to the sum of 72 roots of E_6. Thus we have defined some explicit $W(E_6)$-invariants of $\mathbf{Q}[s_i, 1/s_i, r, 1/r]$.

Theorem 2. *For λ generic over $\mathbf{Q}$, we have*

$$\mathbf{Q}[\delta_1, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}] = \mathbf{Q}[p_0, p_1, p_2, q_0, q_1, q_2]. \tag{23}$$

More precisely, we have

$$\begin{cases} \delta_1 = -2p_1 \\ \epsilon_1 = 6p_2 \\ \epsilon_{-1} = {p_2}^2 - q_2 \\ \epsilon_2 = 13{p_2}^2 + p_0 - q_2 \\ \epsilon_{-2} = -2p_1p_2 + 6p_2 + q_1 \\ \epsilon_3 = 8{p_2}^3 + 2p_0p_2 + {p_1}^2 - 6p_1 - q_0 + 9. \end{cases} \tag{24}$$

This can be uniquely solved in terms of p_i, q_j as follows:

$$(25)\quad \begin{cases} p_2 = \frac{1}{6}\,\epsilon_1 \\ p_1 = -\frac{1}{2}\,\delta_1 \\ p_0 = \epsilon_2 - \frac{1}{3}\epsilon_1^2 - \epsilon_{-1} \\ q_2 = -\epsilon_{-1} + \frac{1}{36}\,\epsilon_1^2 \\ q_1 = -\epsilon_1 + \epsilon_{-2} - \frac{1}{6}\delta_1\epsilon_1 \\ q_0 = 9 + 3\delta_1 + \frac{1}{4}\delta_1^2 - \frac{1}{3}\epsilon_{-1}\epsilon_1 - \frac{2}{27}\epsilon_1^3 \\ \qquad + \frac{1}{3}\epsilon_1\epsilon_2 - \epsilon_3. \end{cases}$$

In view of the above theorems, the family of elliptic surfaces S_λ defined by the equation (1) will be called a *multiplicative excellent family* with Galois group $W(E_6)$. [Note that $\mathbf{Q}$ can be replaced by any field of characterisitic $\neq 2, 3$ in Theorems 1 and 2.]

Remark 1. We obtained similar results in our previous papers ([8,9]) for type $E_r (r = 6, 7, 8)$, and proposed to call such a family with parameter λ an *excellent family* with Galois group $W(E_r)$ (cf. [10,14]). Actually we mainly studied the situation where the family of elliptic surfaces has an additive singular fibre. In that case, we have a stronger result that the parameters p_i, q_j of the family become the fundamental polynomial invariants of the Weyl group.

In particular, for type E_6, take the Weierstrass equation

$$(26)\quad \begin{aligned} &y^2 + 2t^2y \\ &\quad = x^3 + (p_0 + p_1t + p_2t^2)x + q_0 + q_1t + q_2t^2. \end{aligned}$$

It has a singular fibre of type IV at $t = \infty$, and it defines an *additive excellent family* of type E_6. Namely, Theorem 1 above holds true verbatim provided that Y in the statement (ii) is replaced by the affine 6-space $\mathbf{A}^6$, and "Theorem 2" corresponds to the explicit formula of p_i, q_j as the fundamental polynomial invariants in the polynomial ring $\mathbf{Q}[Y] = \mathbf{Q}[a_1, \ldots, a_6]$ (see [8, p.679, (2.15)]).

Remark 2. Theorem 2 gives an explicit description of the fundamental invariants of the Weyl group in the multiplicative case, i.e. in the ring of Laurent polynomials $\mathbf{Q}\langle s_1, \ldots, s_6, r\rangle$. The invariants $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}\}$ are essentially equal to the characters of the 5 fundamental representations of the simple algebraic group (or the simple Lie algebra) of type E_6:

$$(27)\quad \Lambda^m V \quad (m = 1, 2, 3) \text{ and } \Lambda^m V^* \quad (m = 1, 2)$$

where V denotes a 27-dimentional irreducible representation and V^* its dual representation (cf. [2, Ch.13]). In fact, this viewpoint inspired us to introduce the $W(E_6)$-invariant δ_1 as the one corresponding to the remaining fundamental representation, the adjoint representation.

See the Notes (added in proof) at the end of the paper.

4. Outline of proof. By [11, Th.7], we find six linear sections $\{P_i\ (1 \le i \le 6)\}$ such that

$$(28)\quad \langle P_i, P_j\rangle = \delta_{ij} + \frac{1}{3}$$

and a section R_0 of height 2 (a root of E_6) such that $3R_0 = P_1 + \cdots + P_6$. Let

$$(29)\quad P'_i := P_i - R_0, P''_{ij} := R_0 - P_i - P_j (i \neq j).$$

Then the 27 linear sections are given by

$$(30)\quad \{P_i, P'_i, P''_{ij} (i \neq j)\} = \{P_1, \ldots, P_{27}\}.$$

At the singular fibre of type I_3 at $t = \infty$:

$$(31)\quad \pi^{-1}(\infty) = \Theta_0 + \Theta_1 + \Theta_2,$$

we can define a specialization homomorphism:

$$(32)\quad sp_\infty : M_\lambda \to k^\times \times \mathbf{Z}/3\mathbf{Z}$$

such that the following lemma holds:

Lemma 3. *The map sp_∞ is a Galois-equivariant homomorphism such that, if P is a linear section defined by* (2)*, then*

$$(33)\quad sp_\infty(P) = \left(-\frac{1}{a}, [\Theta_1]\right).$$

(The 27 linear sections in (30) intersect one and the same component, which is named as Θ_1 above.) Let $sp'_\infty : M_\lambda \to k^\times$ be the projection to the first factor.

Proof of Theorem 2. We have

$$(34)\quad s_i := sp'_\infty(P_i) = -\frac{1}{a_i} \quad (1 \le i \le 27)$$

by Lemma 3. Therefore the polynomial $\Phi(X)$ in (11) is equal to

$$(35)\quad \begin{aligned} \Phi(X) &= \prod_{i=1}^{27}(X - a_i) \\ &= \prod_{i=1}^{27}\left(X + \frac{1}{s_i}\right) \\ &= X^{27} + \epsilon_{-1}X^{26} + \epsilon_{-2}X^{25} + \ldots \\ &\quad + \epsilon_4X^4 + \epsilon_3X^3 + \epsilon_2X^2 + \epsilon_1X + 1 \end{aligned}$$

where ϵ_{-n} (resp. ϵ_n) denotes the n-th elementary symmetric polynomial of $\{1/s_i\}$ (resp. $\{s_i\}$) as defined in (21).

By comparing the coefficients of the two expression of $\Phi(X)$, (11) and (35), we obtain equalities:

$$(36)\quad \begin{cases} \epsilon_1 = 6p_2 \\ \epsilon_{-1} = {p_2}^2 - q_2 \\ \epsilon_2 = 13{p_2}^2 + p_0 - q_2 \\ \epsilon_{-2} = -2p_1p_2 + 6p_2 + q_1 \\ \epsilon_3 = 8{p_2}^3 + 2p_0p_2 + {p_1}^2 - 6p_1 - q_0 + 9 \\ \epsilon_4 = -14{p_2}^4 - p_0{p_2}^2 + 16q_2{p_2}^2 + 4{p_1}^2p_2 \\ \quad -26p_1p_2 + 48p_2 - 2{q_2}^2 - 2p_1q_1 \\ \quad +7q_1 + p_0q_2. \end{cases}$$

This gives the formulas in (24) except that for δ_1. Note that (36) can be rationally solved in terms of p_i, q_j by allowing ϵ_{-2} in the denominator.

To complete the proof, it is enough to show that $\delta_1 = -2p_1$. By using the rational expression of p_1 in terms of $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_{-1}, \epsilon_{-2}\}$ just mentioned, this reduces to showing

$$(37)\quad \delta_1\epsilon_{-2} = -(7\epsilon_{-2} - \epsilon_{-1}^2 + \epsilon_1 - \epsilon_{-1}\epsilon_2 - \epsilon_4),$$

and the verification of this equality in the ring $\mathbf{Q}[s_i, 1/s_i, r, 1/r]$ is straightforward. □

Proof of Theorem 1. By (14) and (34), we have

$$(38)\quad \mathcal{K}_\lambda = k_0(s_1, \dots, s_{27}) = \mathbf{Q}(s_1, \dots, s_6, r),$$

because p_i, q_j are contained in $\mathbf{Q}(s_1, \dots, s_6, r)$ by (19), (20) and (25). Note that the relation (16) holds. Hence $(s_1, \dots, s_6, r)$ is a point of the toric hypersurface Y defined in (ii). The rest of proof will follow from Theorem 2 by elementary arguments using Galois theory. □

5. Examples and applications. Once a multiplicative excellent family is given, we have various applications of it to number theory and algebraic geometry. In particular, we can systematically construct explicit examples of semi-stable rational elliptic surfaces over $\mathbf{Q}$ with a property such as

- big Galois over $\mathbf{Q}$
- small Galois over $\mathbf{Q}$
- degeneration via "vanishing roots".

The correponding results using the additive excellent family (26) have been known (cf. [9]), but the multiplicative family is necessary for treating the semi-stable case.

5.1. Big Galois over Q. By Hilbert's irreducibility theorem (cf. [6]), Theorem 1(i) implies that for most choice of $\lambda^0 \in \mathbf{Q}^6$, $\Phi(X, \lambda^0) \in \mathbf{Q}[X]$ has the same Galois group $W(E_6)$ as the generic case. Conversely, the existence of a single example $\lambda^0 \in \mathbf{Q}^6$ with Galois group $W(E_6)$ will prove Theorem 1(i) for generic λ, in view of Proposition 2 of [6, 9.2].

Let us exhibit such an explicit example of λ^0:

Example 4. Take $\lambda^0 = (1, \dots, 1)$, i.e. $p_i = q_i = 1 (i = 0, 1, 2)$. Thus the elliptic surface S_{λ^0} (and the cubic surface V_{λ^0}) is defined by the Weierstrass equation:

$$y^2 + txy = x^3 + (1 + t + t^2)x + 1 + t + t^2 + t^3.$$

Then the algebraic equation of degree 27 becomes:

(39)

$$\begin{aligned} \Phi(X, \lambda^0) &= X^{27} + 5X^{25} + 13X^{24} - X^{23} + 76X^{22} \\ &+ 19X^{21} + 99X^{20} + 85X^{19} + 122X^{18} + 133X^{17} \\ &+ 222X^{16} + 232X^{15} + 450X^{14} + 340X^{13} + 546X^{12} \\ &+ 650X^{11} + 369X^{10} + 320X^9 + 287X^8 + 151X^7 \\ &+ 103X^6 + 82X^5 + 31X^4 + 13X^3 + 13X^2 + 6X + 1. \end{aligned}$$

This integral polynomial has Galois group $W(E_6)$.

Proof. We can use the same argument as in the additive case ([9, Ex. 7.4]). Look at the factorization of $\Phi(X, \lambda^0)$ mod p into irreducible factors in $\mathbf{F}_p[X]$, and check that it has cycle type $(9)^3$ for $p = 23$, and $(2)(5)^3(10)$ for $p = 43$. Then the claim follows from [9, Lemma 7.5]. □

While the above proof is the same as for the additive case, the resulting $W(E_6)$-extension $\mathcal{K} = \mathcal{K}_{\lambda^0}$ is given with a "multiplicative" structure that the 27 roots $\{a_i\}$ form a set of 27 units, stable under the Galois group $W(E_6)$. One could ask what the structure of the unit group of $\mathcal{K}$ will be as $W(E_6)$-module.

5.2. Small Galois over Q. Next we consider the specialization "upstairs" $\xi \to \xi^0$ where $\xi = (s_1, \dots, s_6, r) \in Y$ (in contrast to the specialization "downstairs" $\lambda \to \lambda^0$ as in §5.1). Namely we choose some $\xi^0 \in Y(\mathbf{Q})$ to obtain a $\mathbf{Q}$-split example of a semi-stable rational elliptic surface $S = S_{\lambda^0}$ such that $E_{\lambda^0}(\mathbf{Q}(t))$ coincides with $E_{\lambda^0}(k(t)) \simeq E_6^*$ with explicit $\mathbf{Q}(t)$-rational generators P_i.

For example, take $s_i = i + 1 (i < 6)$, $s_6 = 7^3/6!, r = 7$ for ξ^0. The formula (25) gives $\lambda^0 = (p_i, q_j) \in \mathbf{Q}^6$, which defines E_{λ^0}, S_{λ^0} and V_{λ^0} by (1).

The MW group $E_{\lambda^0}(\mathbf{Q}(t))$ of rank 6 is generated by $P_i = (a_i t + b_i, d_i t + e_i)(i \leq 27)$, in which $a_i = -1/s_i$ has the prescribed values $-1/2, -1/3, \ldots$, etc.

As mentioned in §1, the 27 lines on the cubic surface V_λ are defined by: $x = a_i t + b_i, y = d_i t + e_i$. Thus all the lines are $\mathbf{Q}$-rational if $\xi \in Y(\mathbf{Q})$. Moreover if L_i, L_i' denote the lines corresponding to P_i, P_i' in (30), then the 6 lines $\{L_i\}$ and $\{L_i'\}$ form a double six of lines in Schläfli's sense by our construction.

5.3. Degeneration via "vanishing roots". By the same method as above, we can also study the degeneration of S_λ, V_λ under specialization of parameters. Here we drop the assumption $(*)$ in §1, and consider the case where there may be some new reducible fibres at $t \neq \infty$.

Let $\psi : Y \to \mathbf{A}^6$ be the surjective morphism defined by (25). If $\psi(\xi) = \lambda \in \mathbf{A}^6$, then we consider the elliptic surface $S_\xi := S_\lambda$ defined by (1). On the other hand, for $\xi = (s_1, \ldots, s_6, r) \in Y$, we let

$$(40) \quad \Pi = \{1/r, s_i/s_j (i < j), r/(s_i s_j s_k)(i < j < k)\}$$

be the set of 36 elements corresponding to the 36 positive roots of E_6 (cf. [11, Th.7 (iv)]). Further let $\nu = \nu(\xi)$ denote the number of times 1 occurs in Π, and call it the number of *vanishing roots*, as the idea behind is very close to the vanishing cycles in the deformation of singularities (cf. [12,13]).

Theorem 5. *S_ξ has new reducible fibres at $t \neq \infty$ iff $1 \in \Pi$, i.e. iff $\nu(\xi) > 0$. More generally, the number of roots in the root lattice T_{new} is equal to 2ν, where $T_{new} := \oplus_{v \neq \infty} T_v$ is the new part of the trivial lattice.*

Note that the condition $\nu = 0$ is equivalent to the smoothness of S_λ and of V_λ.

5.4. Numerical examples. As an illustration, we sketch how to prove the refined existence of every possible type of semi-stable rational elliptic surfaces (having I_3-fibre), by writing down an explicit $\mathbf{Q}$-split example.

For the classification of rational elliptic surfaces with a section, see Persson [5] and Oguiso-Shioda [4]. The list of [5] is finer than that of [4] as far as singular fibres are concerned, but [4] gives the structure of Mordell-Weil lattices M for each type.

There are exactly 21 OS-types such that the trivial lattice T contains A_2, and they are listed in the first three columns of Table I, together with the structure of T_{new} and M.

Table I.

OS	T_{new}	M	$\{s_1, \ldots, s_6\}$	ν
3	0	E_6^*	$2, 3, 4, 5, 6, 7^3/6!$	0
6	A_1	A_5^*	$2, 4, 8, 3, 3, 1/9$	1
11	A_2	$(A_2^*)^2$	$2, 4, 8, 3, 3, 3$	3
12	$2A_1$	$rk4(n.r.l)$	$8, 8, 27, 27, 5, 25$	2
19	A_3	$rk3(n.r.l)$	$2, 2, 2, 2, 1/2, 8$	6
20	$A_1 + A_2$	$A_2^* + \langle 1/6 \rangle$	$2, 2, 2, 8, 27, 27$	4
23	$3A_1$	$A_1^* + (rk2)$	$8, 8, 27, 27, 125, 125$	3
31	A_4	$(rk2)$	$2, 2, 2, 2, 2, 1/4$	10
32	D_4	$(rk2)$	$1, 1, 2, 2, 1/2, 1/2$	12
37	$A_1 + A_3$	$A_1^* + \langle 1/12 \rangle$	$8, 8, 8, 8, 27, 27$	7
39	$2A_2$	$A_2^* + \mathbf{Z}/3\mathbf{Z}$	$2, 2, 2, 3, 3, 3$	6
40	$2A_1 + A_2$	$(\langle 1/6 \rangle)^2$	$-1, -1, -1, 2, 2, 1/4$	5
41	$4A_1$	$(rk2) + \mathbf{Z}/2\mathbf{Z}$	$-1, -1, 2, 2, 1/2, 1/2$	4
50	D_5	$\langle 1/12 \rangle$	$2, 2, 2, 2, 1/4, 1/4$	20
51	A_5	$A_1^* + \mathbf{Z}/3\mathbf{Z}$	$2, 2, 2, 2, 2, 2$	15
56	$A_1 + A_4$	$\langle 1/30 \rangle$	$2, 2, 2, 2, 2, 1/32$	11
59	$2A_1 + A_3$	$\langle 1/12 \rangle + \mathbf{Z}/2\mathbf{Z}$	$2, 2, 2, 2, -1/4, -1/4$	8
61	$A_1 + 2A_2$	$\langle 1/6 \rangle + \mathbf{Z}/3Z$	$2, 2, 2, 1/2, 1/2, 1/2$	7
66	$A_1 + A_5$	$\mathbf{Z}/6\mathbf{Z}$	$-1, -1, -1, 1, 1, 1$	16
68	$3A_2$	$(\mathbf{Z}/3\mathbf{Z})^2$	$\omega, \omega, \omega, \omega', \omega', \omega'$	9
69	E_6	$\mathbf{Z}/3\mathbf{Z}$	$1, 1, 1, 1, 1, 1$	36

Table I should be read as follows: take the data $\{s_1, \ldots, s_6\}$ in the 4th column such that $\xi = (s_1, \ldots, s_6, r) \in Y(\mathbf{Q})$ for a unique $r \in \mathbf{Q}$. The 5th column computes the number of vanishing roots ν in Π.

Computing the discriminant and the j-invariant, one checks that the elliptic surface $S_\xi = S_\lambda$ has a required configuration of reducible fibres and T_{new} as in the 2nd column. S is semi-stable except for the cases No. 32, 50 or 69 (where semi-stability is impossible) and it is $\mathbf{Q}$-split except for No. 68 (which can never be $\mathbf{Q}$-split; it is $\mathbf{Q}(\omega)$-split with $\omega^3 = 1$).

Furthermore, our method gives the 27 linear sections (counted with multiplicity) P_i, P_j', P_{ij}'', which contain generators of the Mordell-Weil lattices, and which describe the lines on cubic surface. The multiplicities can be determined in the same way as in [12,13].

Remark 3. The equation (1) for each of the above give some explicit examples of affine surfaces S' in (x, y, t)-space which has precisely the ADE-singularities indicated by T_{new}. Note that all these singular points have coordinates with rational numbers and their resolution can be achieved by blowing up only $\mathbf{Q}$-rational points.

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Notes (Added in proof). Recently we came to notice that our Theorem 2 is essentially equivalent to the statement about "E_6-curve" (a special case of Seiberg-Witten curve) in the paper of Eguchi and Sakai [15]. Many interesting results in [15] are based on the mirror type arguments, and our result can be viewed as a purely mathematical proof for "E_6-curve". Finally, further multiplicative excellent families will be studied in the case of type E_7 and E_8, in a joint paper with Abhinav Kumar (in preparation).

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